Eigenvalue Statistics for Random Block Operators

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Contents

1 Introduction

There are many introductory texts about random Schrödinger operators. In the upcoming we will follow the review by Werner Kirsch [Kir08] which is an especially nice survey.

In quantum mechanics a particle moving through d-dimensional space is described by a vector ψ in the Hilbert space $L^2(\mathbb{R}^d)$. The time evolution of such a state ψ is described by a self-adjoint operator of the form

$$H = H_0 + V \tag{1.1}$$

acting on $L^2(\mathbb{R}^d)$, a so called *Schrödinger operator.* The operator H_0 represents the kinetic energy of the particle and is, in the absence of a magnetic field, given by the Laplacian

$$H_0 = -\frac{\hbar^2}{2m}\Delta \tag{1.2}$$

where we shall choose the physical units such that $\hbar^2(2m)^{-1} = 1$. The operator V is the multiplication operator with the function $V(x)$

$$(V\psi)(x) = V(x)\psi(x) \tag{1.3}$$

and it represents the potential and thus the physics behind the model.

However, this approach relies on the assumption that we know exactly how this potential looks, i.e. we know the exact placement and type of the atoms involved, which may not always be the case.
For example, this is the case for solids with an almost crystalline structure, that is a structure where each atom of the solid may deviate a little bit from a point in a periodic lattice. Another such example would be an unordered alloy, being a solid that consists out of several materials with the respective atoms located at lattice points.
Thus, to model such disordered solids, it is a reasonable approach to consider the potential to be a random variable. The resulting operators of the form

$$H_\omega = H_0 + V_\omega \tag{1.4}$$

are then called *random Schrödinger operators.*
As a simplification we will only consider random Schrödinger operators on the lattice $\mathbb{Z}^d$ and thus on the Hilbert space $l^2(\mathbb{Z}^d)$ instead of $L^2(\mathbb{R}^d)$. A consequence of the probabilistic approach is that we are now no longer interested in the spectral properties of an operator H_ω for one possible ω of the result space but we must ask ourselves what are the

"typical" properties of H_ω or in other words which properties hold true with probability one.
For example, we have the existence of an almost surely non-random spectrum, the existence of the integrated density of states which will give a notion of how many eigenvalues we can expect to find per unit volume and the exhibition of a localized regime in the spectrum near the band edges, i.e. there exists, with probability one, a regime near the edges of the spectrum of H_ω where we will only find dense pure point spectrum and the corresponding eigenfunctions decay exponentially fast (these standard results can be found for example in [Kir08, PF92, CL90]).
However, we shall mostly be interest in the correlation of the eigenvalues in the localized regime. Since we have in this part of the spectrum exponential decay of the corresponding eigenfunctions, it is natural to assume that there is no or only little correlation between the eigenvalues. This result, indeed that there is no correlation between the eigenvalues in the localized regime has first been proven in 1981 by Molchanov (see [Mol81]) in the case of one-dimensional random Schrödinger operators of the form

$$-\frac{d^2}{dt^2} + F(x_t(\omega)), \quad \text{for } t \in \mathbb{R}, \tag{1.5}$$

where $\{x_t(\omega)\}$ is a Brownian motion on a compact Riemannian manifold K and F denotes a smooth Morse function with $\min_K F = 0$. Molchanov showed in [Mol81] that if we consider the operator (1.5) restricted to a large interval there is no correlation between the eigenvalues as the interval gets larger. More precisely, he showed that the eigenvalues are locally distributed according to Poisson distribution law as the interval gets larger.
However, as commented by Minami in [Min96] Molchanov's method is strongly dependent on the considered dimension. Minami extended the result in 1996 to random Schrödinger operators, in particular to the discrete Anderson model H_ω, in arbitrary dimension (see [Min96]) by proving that the properly rescaled eigenvalues in the localized regime obey the Poisson distribution law.
A key idea of Minami's proof of the result was an estimate on the probability that we find at least two eigenvalues in an interval I in the localized regime, more precisely he proved that

$$\mathbb{P}(\operatorname{tr} \chi_I(H_\omega^\Lambda) \geq 2) \leq \mathbb{E}\left[\operatorname{tr} \chi_I(H_\omega^\Lambda)\left\{\operatorname{tr} \chi_I(H_\omega^\Lambda) - 1\right\}\right] \leq \mathit{const.}\big(|I|\,|\Lambda|\big)^2 \tag{1.6}$$

by estimating the average of a determinant of a 2×2 matrix whose entries are the imaginary parts of the resolvent of the considered random operator (cf. equation (2.52) and Lemma 2 in [Min96]). This kind of estimate, often referred to as *Minami estimate*, can also be used to gain other results such as the simplicity of the eigenvalues of the Anderson model in the localized region which was proven by Klein and Molchanov in 2006 (cf. [KM06]). However, it was unknown how to extend the methods used by Minami to prove a Minami estimate to the case of the continuum Anderson model until Combes, Germinet and Klein provided in [CGK09a] in 2009 a new approach to prove a Minami estimate for which it was clear how to extend it to the continuum case. In 2010 Klopp provided in [Klo] a structural statement for random Schrödinger operators H_ω saying that once we have

(A) spectral localization in an interval $I \subset \sigma(H_\omega)$,

(B) the independence of local Hamiltonians sufficiently far away from each other, i.e. there exists an $R_0 > 0$ such that for any two hypercubes Λ_1 and Λ_2 with $\mathrm{dist}(\Lambda_1, \Lambda_2) > R_0$ the random Hamiltonians $H_\omega^{\Lambda_1}$ and $H_\omega^{\Lambda_2}$ are stochastically independent,

(C) a Wegner estimate in I, i.e. there exists a constant $C > 0$ such that for all $J \subset I$ and all hypercubes Λ we have

$$\mathbb{E}\left[\mathrm{tr}\ \chi_J(H_\omega^\Lambda)\right] \leq C|J|\,|\Lambda|,$$

(D) and a Minami estimate in I, i.e. there exist constants $C > 0$ and $\rho > 0$ such that for all $J \subset I$ and all hypercubes Λ we have

$$\mathbb{E}\left[\mathrm{tr}\ \chi_J(H_\omega^\Lambda)\left\{\mathrm{tr}\ \chi_J(H_\omega^\Lambda) - 1\right\}\right] \leq C\big(|J|\,|\Lambda|\big)^{1+\rho},$$

then we can conclude Poisson statistics of the eigenvalues in the sense of Minami.

However, random Schrödinger operators are not the only type of random operator of interest for mathematical modelling of disordered system. In the mathematical modelling of mesoscopic disordered systems, such as dirty superconductors, random block operators of the form

$$\begin{pmatrix} H_\omega & B_\omega \\ B_\omega & -H_\omega \end{pmatrix}, \tag{1.7}$$

with both H_ω and B_ω self-adjoint, arise in the eigenvalue problem describing quasi-particle states in a mean-field approximation of BCS theory (cf. [KMM11, de 89]), also know as the Bogoliubov–de Gennes equations (for more details see for example [KMM11, AZ97, VSF00, de 89]). In the case of random block operators we can also retrieve many properties which are already known in case of the Anderson model such as the existence of an almost surely non-random closed spectrum, the existence of the integrated density of states (which were proven in [KMM11]) and the exhibition of a localized regime near the spectral gap (which was proven in [Geb11]).
However, since we have spectral localization near the spectral gap for random block operators, it is only natural to ask whether the eigenvalues in the localized regime are stochastically independent. Therefore, it is the main goal of this thesis to provide a structural statement for random block operators, similar to one Klopp provided for random Schrödinger operators in [Klo], presenting conditions which are sufficient to prove local Poisson statistics of the eigenvalues of random block operators and hence the stochastical independence of the eigenvalues in the localized regime of the spectrum. To this end, this thesis is organized as follows:
First we will to study the methods which where used by Minami to prove the afore mentioned result and in particular, we will study the method of Combes, Germinet and Klein (cf. [CGK09a]) for proving a Minami estimate. Therefore, in Chapter 2, we will introduce the Anderson model on the lattice $\mathbb{Z}^d$, give a Wegner estimate and review a

few prominent spectral properties such as the existence of an almost surely non-random spectrum, the existence of the integrated density of states and the exhibition of a localized regime in the spectrum near the band edges.
In Chapter 3 we shall analyze Minami's result for the energy level statistics and review the method of Combes, Germinet and Klein for proving a Minami estimate.
Since, we want to extend Minami's result to a certain kind of discrete random block operator, we will, in Chapter 4, introduce the mathematical concept of random block operators of the form (1.7) based on the analysis obtain in [KMM11, Tre08]. Furthermore, we will give a Wegner estimate and discuss some important properties such as the exhibition of a localized regime in the spectrum.
In Chapter 5 we will turn to the main goal of this thesis and provide a structural theorem saying that we can conclude the independence of the eigenvalues in the localized regime for random block operators given that we have a Wegner and a Minami estimate and that we have exponentially decaying fractional moments of the Green's function (similar to the work of Klopp in [Klo] for the random Schrödinger case). We will also analyze the assumptions demanded in our structural theorem and give results that might lead to proving them for random block operators. In particular, we will work on adapting the method of [CGK09a], for proving a Minami estimate, to the case of random block operators.

2 The Anderson model

The goal of this chapter is to introduce the Anderson model on the lattice $\mathbb{Z}^d$, i.e. the Hilbert space $L^2(\mathbb{R}^d)$ is replaced by the space of sequences $l^2(\mathbb{Z}^d)$. Furthermore, we will review some of its properties. Most definitions and notations were taken from [Kir08, KMM11, Klo].

Definition 2.1. For $d \in \mathbb{N}$ define

$$\begin{aligned}\mathcal{H} := l^2(\mathbb{Z}^d) &:= \left\{ \psi : \mathbb{Z}^d \longrightarrow \mathbb{C} \;\middle|\; \sum_{n\in\mathbb{Z}^d} |\psi(n)|^2 < \infty \right\} \\ &= \left\{ (\psi(n))_{n\in\mathbb{Z}^d} \;\middle|\; \sum_{n\in\mathbb{Z}^d} |\psi(n)|^2 < \infty \right\}.\end{aligned}$$

We define on $\mathcal{H}$ the scalar product

$$\langle \psi, \phi \rangle := \sum_{n\in\mathbb{Z}^d} \overline{\psi(n)}\phi(n) \quad \text{for all } \psi, \phi \in \mathcal{H}$$

and the induced norm

$$\|\psi\| := \sum_{n\in\mathbb{Z}^d} |\psi(n)|^2 \quad \text{for all } \psi \in \mathcal{H},$$

whereas the norm we use on $\mathbb{Z}^d$ shall be defined by

$$|n|_1 := \sum_{\nu=1}^{d} |n_\nu|$$

for all $n \in \mathbb{Z}^d$. Furthemore, we will define for $n \in \mathbb{Z}^d$

$$\delta_n : \mathbb{Z}^d \longrightarrow \mathbb{C} : i \longmapsto \delta_n(i) = \begin{cases} 0, & \text{for } i \neq n \\ 1, & \text{for } i = n. \end{cases}$$

Remark 2.2. Equipped with this scalar product $\mathcal{H}$ is a Hilbert space and the set $\{\delta_n\}_{n\in\mathbb{Z}^d}$ forms an orthonormal basis of $\mathcal{H}$.

Definition 2.3. The operator $\Delta : \mathcal{H} \to \mathcal{H} : \ \psi \mapsto \Delta\psi$ with

$$(\Delta\psi)(n) := \sum_{|n-m|_1=1} \psi(m) \quad \text{for } \psi \in \mathcal{H}$$

is called the *centered discrete Laplace operator*.

Remark 2.4. Via direct calculation it is easy to see that Δ is a bounded and symmetric operator and thus self-adjoint.

Now we will turn to the definition of a random potential. For this purpose, we will construct the so-called *canonical probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ as following:

Let $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu)$ be a probability space where $\mathfrak{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$ and μ a probability measure on $\mathbb{R}$ with no atoms. Let $\Omega := \mathbb{R}^{\mathbb{Z}^d}$ be the sample space and the corresponding σ-algebra $\mathcal{F}$ is generated via cylinder sets of the form

$$\{\omega \mid \omega_{i_1} \in B_1, \ldots, \omega_{i_n} \in B_n,\ B_1, \ldots, B_n \in \mathfrak{B}(\mathbb{R}),\ n \in \mathbb{N}\}. \tag{2.1}$$

Furthermore, let $\mathbb{P} := \mu^{\otimes \mathbb{Z}^d}$ be the infinite product measure on $(\Omega, \mathcal{F})$ that is induced by the probability measure μ on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$. More details regarding this strategy for constructing a probability space by cylinder sets can be found in [Kir08] and in general from a probabilistic point of view in, for example, [Kle08].

Definition 2.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the canonical probability space defined above. Let $\{T_j\}_{j \in \mathbb{Z}^d}$ be an ergodic group of measure-preserving transformations on Ω such that we have for all $\omega \in \Omega$, all $j \in \mathbb{Z}^d$ and all $n \in \mathbb{Z}^d$

$$(T_j \omega)(n) = \omega(n - j).$$

For $n \in \mathbb{Z}^d$ let $\omega_n : \Omega \to \mathbb{R} :\ \omega \mapsto \omega(n)$ be the projection on the n-the component of ω and let the family $(\omega_n)_{n \in \mathbb{Z}^d}$ be an ergodic (w.r.t. to $\{T_j\}_{j \in \mathbb{Z}^d}$) stochastic process which is independent and identically distributed with law μ. Then we call the induced multiplication operator V_ω with the function ω_n, i.e.

$$(V_\omega \psi)_n = (V_\omega \psi)(n) = \omega(n)\psi(n),$$

a *random potential*.

Definition 2.6. (*Anderson model*) Let $L(\mathcal{H})$ denote the space of linear operators on $\mathcal{H}$. The random operator $H : \Omega \to L(\mathcal{H})$

$$\begin{aligned} H :\ \Omega &\longrightarrow L(\mathcal{H}) \\ \omega &\longmapsto H_\omega = H_0 + V_\omega \end{aligned}$$

is, for $H_0 = \Delta$, called the *Anderson model*. In the slightly more general case where H_0 is a self-adjoint operator on $\mathcal{H}$, we call H_ω, as in [CGK09a], the *generalized Anderson model*.

Definition 2.7. We define the *kernel* for a bounded self-adjoint operator A on $\mathcal{H}$ by

$$A(n, m) := \langle \delta_n, A\delta_m \rangle,$$

for all $n, m \in \mathbb{Z}^d$ and thus we have for all $\psi \in \mathcal{H}$ and all $n \in \mathbb{Z}^d$

$$A(\psi)(n) = \sum_{m \in \mathbb{Z}^d} A(n, m)\psi(m).$$

2.1 Boundary Conditions

We will often want to consider an operator restricted to some subset of its domain which leaves various choices for the behavior of the operator on the boundary of this subset. The aim of the following is to equip us with a precise notion of *Dirichlet*, *Neumann* and *simple* boundary conditions.

Definition 2.8. Let $\Lambda \subset \mathbb{Z}^d$, then we define the *boundary* $\partial\Lambda$ of Λ by

$$\partial\Lambda := \big\{(n,m) \in \mathbb{Z}^d \times \mathbb{Z}^d \ : \ |n-m|_1 = 1 \text{ and } \{n \in \Lambda, m \notin \Lambda\} \text{ or } \{n \notin \Lambda, m \in \Lambda\}\big\}.$$

Furthermore, we define the *inner boundary* of D by

$$\partial^-\Lambda := \big\{n \in \mathbb{Z}^d \ : \ n \in \Lambda,\ \exists m \notin \Lambda \text{ such that } (n,m) \in \partial\Lambda\big\}$$

and the *outer boundary* of D by

$$\partial^+\Lambda := \big\{m \in \mathbb{Z}^d \ : \ m \notin \Lambda,\ \exists n \in \Lambda \text{ such that } (n,m) \in \partial\Lambda\big\}.$$

For $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$ we define the *relative boundary* by

$$\begin{aligned}\partial_{\Lambda_2}\Lambda_1 = \big\{&(n,m) \in \mathbb{Z}^d \times \mathbb{Z}^d \ : \\ &|n-m|_1 = 1 \text{ and } \{n \in \Lambda_1, m \in \Lambda_2\backslash\Lambda_1\} \text{ or } \{n \in \Lambda_2\backslash\Lambda_1, m \in \Lambda_1\}\big\}.\end{aligned}$$

Definition 2.9. Let H_0 be a self-adjoint operator on $\mathcal{H} = l^2(\mathbb{Z}^d)$ and $\Lambda \subset \mathbb{Z}^d$. Then we define:

(i) $H_{0,\Lambda}$ with *simple boundary conditions* on Λ as an operator on $l^2(\Lambda)$ such that

$$H_{0,\Lambda} = \chi_\Lambda H_0 \chi_\Lambda$$

holds, where χ_Λ denotes the characteristic function of the set Λ.

(ii) $H^N_{0,\Lambda}$ with *Neumann boundary conditions* on Λ as an operator on $l^2(\Lambda)$ such that

$$H^N_{0,\Lambda} = \chi_\Lambda H_0 \chi_\Lambda - (2d - n_\Lambda)$$

holds, where $2d$ denotes a multiple of the identity and n_Λ denotes the multiplication operators with the function $n_\Lambda(i) := |\{j \in D \ : \ |i-j|_1 = 1\}|$ for all $i \in \Lambda$.

(iii) $H^D_{0,\Lambda}$ with *Dirichlet boundary conditions* on Λ as an operator on $l^2(\Lambda)$ such that

$$H^D_{0,\Lambda} = \chi_\Lambda H_0 \chi_\Lambda + (2d - n_\Lambda)$$

holds, where $2d$ and n_D are as above.

Remark 2.10. As mentioned in [Kir08, Sec. 5.2] we would apriori think that the boundary conditions defined as simple were a good candidate for Dirichlet boundary conditions since all we do is just cut away everything not belonging to Λ. However, we want our boundary conditions, in analogy to the continuum, to obey the Dirichlet-Neumann bracketing, i.e. for two disjoint open sets M_1, M_2 and $M = \left(\overline{M_1 \cup M_2}\right)^\circ$ we demand

$$H^N_{0,M_1} \oplus H^N_{0,M_2} \leq H^N_{0,M} \leq H^D_{0,M} \leq H^D_{0,M_1} \oplus H^D_{0,M_2}. \tag{2.2}$$

The simple boundary conditions as a candidate for Dirichlet boundary conditions do, however, not satisfy (2.2). Hence the above definition for Dirichlet boundary conditions.

Definition 2.11. Let H_0 be a self-adjoint operator and V a multiplication operator (with the function V) on $\mathcal{H}$ and let $H := H_0 + V$, then we define for $\Lambda \subset \mathbb{Z}^d$

$$H^X_\Lambda := H^X_{0,\Lambda} + V,$$

where V stands for the canonical restriction of the multiplication operator V to Λ and X represents either Dirichlet, Neumann or simple boundary conditions.

Proposition 2.12. *Let $H = H_0 + V$ be the Anderson model. Then we have for $\Lambda \subset \mathbb{Z}^d$*

$$H^N_\Lambda \oplus H^N_{\mathbb{Z}^d \setminus \Lambda} \leq H \leq H^D_\Lambda \oplus H^D_{\mathbb{Z}^d \setminus \Lambda}$$

and in case of simple boundary conditions the following splitting formula for $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$

$$H_{\Lambda_2} = H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} + \Gamma^{\Lambda_2}_{\Lambda_1}$$

with

$$\Gamma^{\Lambda_2}_{\Lambda_1}(n,m) := \begin{cases} -1 & \text{if } (n,m) \in \partial_{\Lambda_2}\Lambda_1 \\ 0 & \text{else.} \end{cases}$$

Proof. The statements follow by direct calculation, see [Kir08, Sec. 5.2] for more details. □

Remark 2.13. Whenever practical, we are going to use the following notation for the resolvent of an operator H on $\mathcal{H}$

$$(H - z)^{-1}(n,m) = G(z; n, m) \tag{2.3}$$

for all $z \notin \sigma(H)$ and all $n, m \in \mathbb{Z}^d$. If we restrict the operator H to a subset $\Lambda \subset \mathbb{Z}^d$ then we will write

$$(H^\Lambda - z)^{-1}(n,m) = G^\Lambda(z; n, m) \tag{2.4}$$

where $z \notin \sigma(H^\Lambda)$ and $n, m \in \mathbb{Z}^d$.

Proposition 2.14 (Geometric resolvent equation)**.** *Let $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$, $x \in \Lambda_1$, H be the Anderson Model and let H^{Λ_i} for $i = 1, 2$ be its restriction under simple boundary conditions to Λ_i. Let $z \notin \left(\sigma(H^{\Lambda_1}) \cup \sigma(H^{\Lambda_2}) \cup \sigma(H^{\Lambda_2 \setminus \Lambda_1})\right)$, then we have*

$$G^{\Lambda_2}(z; x, x) = G^{\Lambda_1}(z; x, x) + \sum_{\substack{(y,y') \in \partial\Lambda_1 \\ y \in \Lambda_1, y' \in \Lambda_2}} G^{\Lambda_1}(z; x, y) G^{\Lambda_2}(z; y', x).$$

Remark 2.15. The geometric resolvent equation can occur in different forms, we present the version in Proposition 2.14 since we will need it later on. Further details can be found in [Kir08, Sec. 5.3].

Proof of Proposition 2.14. The proof of this statement can be found in [Kir08, Sec. 5.3]. □

2.2 Ergodic properties of the Anderson Model

In the following we will have a short review of some important properties of the Anderson model. For this purpose, I will follow [Kir08].

Definition 2.16. Let $\{T_j\}_{j\in\mathbb{Z}^d}$ be an ergodic group of measure-preserving transformations on Ω such that we have for all $\omega \in \Omega$, all $j \in \mathbb{Z}^d$ and all $n \in \mathbb{Z}^d$

$$(T_j\omega)_n = \omega_{n-j}.$$

Furthermore, let for all $j \in \mathbb{Z}^d$ U_j be the unitary translation operator induced by the ergodic group $\{T_j\}_{j\in\mathbb{Z}^d}$, i.e. for $\psi \in \mathcal{H}$ we have for all $n \in \mathbb{Z}^d$

$$(U_j\psi)(n) := \psi(n-j).$$

Then we call H *ergodic* w.r.t. $\mathbb{Z}^d$-translations if there exists a $\{T_j\}_{j\in\mathbb{Z}^d}$ as above such that

$$U_j H_\omega U_j^* = H_{T_j\omega}$$

holds for every $\omega \in \Omega$ and every $j \in \mathbb{Z}^d$.

The next result guarantees us that the spectrum of H_ω is, almost surely, a closed set and in particular a non-random quantity.

Theorem 2.17. *For $\mathbb{P}$-almost all ω we have $\sigma(H_\omega) = \sigma(H_0) + \mathrm{supp}\mu$.*

Proof. The proof uses a standard argument via the Weyl criterion and is analogous to the proof given in [Kir08, Sec. 3]. □

Remark 2.18. Another important result is the existence of a quantity that measures the density of spectral values of an ergodic random operator. It thus provides us with a notion of how many states there are per unit volume in a certain energy regime. This quantity is generally called *density of states* which we shall make more precise in the following.

Definition 2.19. For $L \in \mathbb{N}$ we define the finite-volume hypercube

$$\Lambda_L := \Lambda := [-L, L]^d \subset \mathbb{Z}^d$$

and its cardinality by $|\Lambda| := (2L+1)^d$.

Proposition 2.20. *For every $A \in \mathfrak{B}(\mathbb{R})$ and $\mathbb{P}$-almost surely we have*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\left[\mathrm{tr}\{\chi_{\Lambda_L}\chi_A(H_\omega)\}\right] = \mathbb{E}[< \delta_0, \chi_A(H_w)\delta_0 >].$$

Proof. The proof applies Birkhoff's ergodic theorem and can be found in [Kir08, Sec. 5.1]. □

With the Proposition 2.20 the following definition is justified.

Definition 2.21. Let H_ω be the Anderson model and χ_{Λ_L} denotes the multiplication operator with the indicator function of Λ_L on $\mathcal{H}$. The measure ν, defined by

$$\nu(A) := \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\left[\mathrm{tr}\{\chi_{\Lambda_L}\chi_A(H_\omega)\}\right] \quad \text{for all } A \in \mathfrak{B}(R),$$

is called the *density of states* of H_ω. The non-decreasing function $N : \mathbb{R} \to [0,1]$, defined via

$$N(E) := \nu(]-\infty, E]) \quad \text{for all } E \in \mathbb{R},$$

is called *integrated density of states* of H_ω.

Definition 2.22. Let H_ω be the Anderson model and let X represent either D for the Dirichlet, N for the Neumann or S for simple boundary conditions, then we define the *finite-volume eigenvalue counting function*

$$N^{\Lambda}_{\omega,X}(E) := \frac{1}{|\Lambda|}\mathrm{tr}\left[\chi_{]-\infty,E]}(H^X_{\omega,\Lambda})\right],$$

for all $E \in \mathbb{R}$.

Proposition 2.23. *Let $E \in \mathbb{R}$ such that N exists at E. Then there exists a set $\Omega_0 \subset \Omega$ of full probability, i.e. $\mathbb{P}(\Omega_0) = 1$, such that*

$$N(E) = \lim_{L\to\infty} N^{\Lambda}_{\omega,X}(E)$$

for all $\omega \in \Omega_0$ and every boundary condition X.

Proof. The proof can be found in [Kir08]. □

Remark 2.24. 1. Proposition 2.23 tells us that we can, indeed, interpret the integrated density of state N as an eigenvalue counting function.

2. In the following we will often consider an operator H_ω restricted to the finite volume box Λ with some kind of boundary conditions and then we will let Λ become very large. Therefore, the notion $|\Lambda| \to \infty$ is a short for considering the hypercube $\Lambda = \Lambda_L$ in the limit $L \to \infty$.

2.3 Wegner estimate

In this section we want to establish a Wegner estimate. It can be used to prove that the integrated density of states N is absolutely continuous with respect to the Lebesgue measure and that it has a bounded density.

Theorem 2.25 (Wegner estimate)**.** *Let H_ω be the generalized Anderson model. Consider a probability measure μ with no atoms, let $S_\mu(s) := \sup_{a\in\mathbb{R}} \mu([a, a+s])$ be the concentration function of μ, and let*

$$Q_\mu(s) := \begin{cases} \|g\|_\infty s & \text{if } \mu \text{ has a bounded Lebesgue density } g \\ 8S_\mu(s) & \text{otherwise.} \end{cases}$$

Let $I \subset \mathbb{R}$ be a bounded interval, then

$$\mathbb{E}\left[\operatorname{tr} \chi_I(H_\omega^\Lambda)\right] \leq Q_\mu(|I|)|\Lambda|.$$

Remark 2.26. We can find many kinds of Wegner-type estimates (see e.g.[Kir08, CGK09a, AM93]). The version above is from [CGK09a]. The proof can be found in [CGK09a] and is stated, for the readers' convenience, in Appendix A.1.

Corollary 2.27. *Suppose the probability measure μ has a bounded density g, then the integrated density of states is absolutely continuous with a bounded density $n(E)$ which shall be called* density of states*, in particular we have for Lebesgue-almost all $E \in \mathbb{R}$*

$$n(E) \leq \|g\|_\infty.$$

Proof. The statement follows immediately from theorem (2.25). □

2.4 Anderson localization

A very important result about disordered systems is the fact that there exists a regime in the spectrum near the band edges where the spectrum is almost surely pure point and the corresponding eigenfunctions decay exponentially. This phenomenon is called *Anderson Localization*. It has first been proven in 1977 by Gol'dshtein, Molchanov and Pastur in [GMP77] for the discrete one-dimensional random Schrödinger operator. This result was then extended to the case of discrete random Schrödinger operators in arbitrary dimensions (cf. [FS83, FMSS85, vK89]). It was then proven in 1994 by Combes and Hislop in [CH94] that we have Anderson localization even in the case of continuum random Schrödinger operators. In the following we will state a result about the spectral localization which can be found in [Kir08].

Definition 2.28. The random operator H_ω exhibits spectral localization in an interval I with $I \cap \sigma(H_\omega) \neq \emptyset$ if for $\mathbb{P}$-almost all ω

$$I \cap \sigma(H_\omega) = I \cap \sigma_{pp}(H_\omega),$$

where $\sigma_{pp}(H_\omega)$ denotes the pure point spectrum of H_ω. In particular, the spectrum inside I is pure point almost surely.

Theorem 2.29. *There exists $E_1 > E_0 := \inf(\sigma(H_\omega))$ such that the spectrum of H_ω exhibits spectral localization in the interval $I = [E_0, E_1]$. More strongly, the corresponding eigenfunctions decay exponentially.*

Proof. There are two major ways to prove this result. One of them, using multiscale analysis, can be found in [Kir08, Sec. 9-11]. The other, using the decay of the fractional moments, goes back to Aizenman and Molchanov and relies on [AM93]. □

Let us have a quick look at the results of multiscale analysis which can be found in [Kir08, Sec. 9]. We consider an interval $I = [E_1, E_2]$ close to the bottom of the spectrum.

Proposition 2.30. *1. Multiscale analysis - weak form: There exists an $\alpha > 1$, $p > 2d$ and a $\gamma > 0$ such that for all $E \in I$*

$$\mathbb{P}\left[\forall n \in \Lambda_{L^{1/2}},\ m \in \partial^- \Lambda_L :\ |G^{\Lambda_L}(E; n, m)| > \mathrm{e}^{-\gamma L}\right] \leq \frac{1}{L^p} \tag{2.5}$$

holds.

2. Multiscale analysis - strong form: There exists a $p > 2d$, an α with $1 < \alpha < \frac{2p}{p+2d}$ and a $\gamma > 0$ such that for any two disjoint cubes $\Lambda_1 := \Lambda_L(n)$ and $\Lambda_2 := \Lambda_L(m)$

$$\mathbb{P}\Big[\exists E \in I :\ \forall n \in \Lambda_{L^{1/2}}, m \in \partial^- \Lambda_L : \\ |G^{\Lambda_1}(E; n, m)| > \mathrm{e}^{-\gamma L} \text{ and } |G^{\Lambda_2}(E; n, m)| > \mathrm{e}^{-\gamma L}\Big] \leq \frac{1}{L^{2p}} \tag{2.6}$$

holds.

Proof of Proposition 2.30. One way of proving these results is to use induction over the cube side length L and can be found in [Kir08, Sec. 10 - 11]. □

As stated above a different approach to proving spectral localization relies on proving exponential decay of the so-called *fractional moments* (cf. Definition 2.31) of the Green's function and goes back to Aizenman and Molchanov [AM93].

Definition 2.31. Let H_ω be as above, $0 < s < 1$, $z \in \mathbb{C} \backslash \mathbb{R}$ and $x, y \in \mathbb{Z}^d$ then we call

$$|\langle x, (H_\omega^\Lambda - z)^{-1} y \rangle|^s = |G^\Lambda(z; x, y)|^s$$

the *fractional moment* of the Green's function. Furthermore, let $E \in \mathbb{R}$, we then say that the fractional moment decays exponentially fast if there exists an $s \in]0, 1[$, a $C(s) > 0$, an $m(s) > 0$ and a $r(s) > 0$ such that for all $\Lambda \subset \mathbb{Z}^d$

$$\mathbb{E}\left[|G^\Lambda(z; x, y)|^s\right] \leq C(s) \mathrm{e}^{-m(s)|x-y|_1}$$

with $x \in \Lambda$ and $y \in \partial\Lambda$ and $z \in \{z \in \mathbb{C}\ :\ \Im z > 0,\ |z - E| < r(s)\}$.

Remark 2.32. The Conditions under which exponential decay of the fractional moments holds can be found for example in [AM93, ASFH01, Gra94]. As shown in [ASFH01], it is possible to conclude the exponential decay of the fractional moments from the results of multiscale analysis. However, the proof for this fact given in [ASFH01] requires the boundedness of the fractional moments. More precisely:

Proposition 2.33. *Let V_ω be a random potential of a random self-adjoint operator H_ω, as in Definition (2.6), satisfying the following regularity condition:*
For $n \in \mathbb{Z}^d$ let $\mu(dx)$ be the probability distribution of $V_\omega(n)$. There exist a $\tau \in]0,1]$ and a $C < \infty$ such that for all $\epsilon > 0$

$$\mu(a-\epsilon, a+\epsilon) \leq C\epsilon^s$$

with $a \in \mathbb{R}$.
Then for all $\Lambda \in \mathbb{Z}^d$ and all $s < \tau$

$$\mathbb{E}\left[|G^\Lambda(z;x,y)|^s\right] \leq C(s)$$

holds for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $x, y \in \mathbb{Z}^d$.

Proof. The statement of the Proposition and its proof can be found in [ASFH01]; a key method used in the proof is a rank-2-perturbation of the Green's function. □

Remark 2.34. Similar statements upon the decay and boundedness of the fractional moments can be found in [AM93, Gra94] but with somewhat less general demands on the potential than given in Proposition 2.33.

3 Eigenvalue statistics for the Anderson model

In this chapter we will focus on understanding an important result about the statistical distribution of the eigenvalues in the localized regime of the spectrum. Our aim is to show that the eigenvalues in the the localized regime are stochastically independent. As mentioned in the introduction this has first been proven by Molchanov for the one-dimensional random Schrödinger operator (cf. [Mol81]).
However, the result we are going to analyze has been proven by Nariyuki Minami in 1996 [Min96] and holds for discrete random Schrödinger operators in arbitrary dimension. Furthermore, we will discuss some improvements on Minami's proof by Combes, Germinet and Klein [CGK09a] with the aim in mind, that we want to extend Minami's result to discrete random block operators (see Chapters 4 and 5).

3.1 Point processes

The following is a brief introduction to the theory of point processes, mostly taken and summarized from [Kle08, DVJ08, Min96], more details can be found in there.

Definition 3.1. Let $\mathcal{M}(\mathbb{R})$ denote the set of all non-negative Radon measures on $\mathbb{R}$ and let $\mathcal{C}_c^+(\mathbb{R})$ denote all non-negative continuous function on $\mathbb{R}$ with compact support. A sequence $(\mu_n)_{n\mathbb{N}} \subset \mathcal{M}(\mathbb{R})$ converges *vaguely* to a $\mu \in \mathcal{M}(\mathbb{R})$ if

$$\lim_{n\to\infty} \int \phi(x) d\mu_n(x) = \int \phi(x) d\mu(x) \quad \forall \phi \in \mathcal{C}_c^+(\mathbb{R}).$$

Remark 3.2. The concept of vague convergence defines a topology on $\mathcal{M}(\mathbb{R})$ which is called the *vague topology*. More details can be found in [Kle08]. Furthermore, with respect to this topology, we will get a Borel σ-algebra.

Definition 3.3. Let $\mathcal{M}_p(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$ be the space of all integer valued Radon measures. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Furthermore, let $\mathfrak{B}(\mathcal{M}_p(\mathbb{R}))$ denote the trace-σ-algebra induced by the Borel σ-algebra given by the vague topology on $\mathcal{M}(\mathbb{R})$. A random variable ξ^ω, defined by

$$\xi^\omega : \Omega \longrightarrow \mathcal{M}_p(\mathbb{R}) : \omega \longmapsto \xi^\omega$$

is called a *point process*. The measure μ, defined by

$$\mu(A) := \mathbb{E}[\xi^\omega(A)] \quad \forall A \in \mathfrak{B}(\mathbb{R}),$$

is called the *intensity measure of* ξ^ω.

Remark 3.4. The set $\mathcal{M}_p(\mathbb{R})$ is closed in $\mathcal{M}(\mathbb{R})$ with respect to the vague topology. By Fubini's theorem and the fact that ξ^ω is a Radon measure it can easily be checked that the intensity measure μ is indeed a measure, thus well defined.

Definition 3.5. A point process ξ^ω with intensity measure μ is called a *Poisson point process* if the following two conditions hold:

i) for all Borel sets $A \in \mathfrak{B}(\mathbb{R})$ and all $k \neq 0$:

$$\mathbb{P}(\xi^\omega(A) = k) = \mathrm{e}^{-\mu(A)} \frac{\mu(A)^k}{k!}$$

ii) for all $A_1, \ldots, A_n$ disjoint Borel sets:

$$\xi^\omega(A_1), \ldots, \xi^\omega(A_n) \text{ are independent random variables.}$$

Definition 3.6. Let $(\xi_n^\omega)_{n\in\mathbb{N}}$ be a sequence of point processes defined on a probability space $(\Omega, \mathcal{F}, P)$. We say that this sequence converges *weakly* to a point process $\xi^{\tilde{\omega}}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ if and only if

$$\lim_{n\to\infty} \int \phi(\xi_n^\omega) dP(\omega) = \int \phi(\xi^{\tilde{\omega}}) d\tilde{P}(\tilde{\omega})$$

holds for all bounded continuous functions ϕ on $\mathcal{M}(\mathbb{R})$. As an abbreviation for the above we will simply write

$$\xi_n^\omega \xrightarrow{w} \xi^{\tilde{\omega}}, \quad \text{for } n \to \infty.$$

Lemma 3.7. *The statement $\xi_n^\omega \xrightarrow{w} \xi^{\tilde{\omega}}$ for $n \to \infty$, as above, is equivalent to*

$$\lim_{n\to\infty} \mathbb{E}_P \left[\mathrm{e}^{-\xi_n^\omega(\phi)}\right] = \mathbb{E}_{\tilde{P}} \left[\mathrm{e}^{-\xi^{\tilde{\omega}}(\phi)}\right] \quad \textit{for all } \phi \in \mathcal{C}_c^+(\mathbb{R}),$$

where we have set

$$\xi_n^\omega(\phi) := \int_{\mathbb{R}} \phi(x) d\xi_n^\omega(x)$$

and for ξ accordingly.

Proof. A proof for this lemma can be found in [DVJ08, chapter 11]. □

Remark 3.8. Let δ_y denote the Dirac measure concentrated at the point y. We can write each $\xi \in \mathcal{M}_p(\mathbb{R})$ in the form

$$\xi(A) = \sum_{j\in\mathbb{N}} \delta_{x_j}(A) \quad \text{for all Borel sets } A,$$

where $(x_j)_{j\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ with no finite accumulation point.

3.2 Eigenvalue statistics

Now that we are equipped with the notion of a point process, defined in the previous section, we can turn to the theorem proven by Minami [Min96]. In the following we will stick closely to the ideas and notations of [Min96]. The statement of the theorem is, roughly speaking, as following:
Let H^Λ be the restriction of the Anderson model to the finite volume Λ and let E be an energy which lies in the regime of the spectrum where we have Anderson localization. Then we find that there is no correlation between the eigenvalues of H^Λ in a neighborhood of E as Λ gets large.
Let us further note that, as afore mentioned in the Introduction, Klopp provided in [Klo] a generalization of the above statement saying that we only need four properties (cf. [Klo] and Chapter 1) to hold for a random Schrödinger operator such that we can conclude local Poisson statictics for the eigenvalues in the localized spectral regime and hence their stochastic independence.

Remark 3.9. (1) In this section we shall make the following assumptions:
Let $H_\omega = H$ be the Anderson model with the independent and identically distributed (iid) ergodic stochastic process $V = \{V_j : \ j \in \mathbb{Z}^d\}$ as random potential and let their common distribution μ have a bounded Lebesgue density g such that $\|g\|_\infty < \infty$. Furthermore, let $\Lambda \subset \mathbb{Z}^d$ be a hypercube, as in Section 2, H^Λ the restriction of H to Λ with simple boundary conditions (cf. Section 2.1) and let the set $\{E_j(\Lambda)\}_{1\leq j\leq|\Lambda|}$ be the eigenvalues of H^Λ ordered by magnitude and repeated according to multiplicity, i.e.

$$E_1(\Lambda) \leq \cdots \leq E_{|\Lambda|}(\Lambda).$$

(2) To gain knowledge about the spacing of the eigenvalues of H^Λ as $|\Lambda| \to \infty$ we will have to rescale the eigenvalues according to their average density. Otherwise, they would just move closer and closer together as $|\Lambda|$ gets large. Fortunately, we know by the Wegner estimate (cf. Theorem 2.25) that the average spacing behaves like $|\Lambda|^{-1}$ as $|\Lambda|$ gets large. Therefore, we will consider the eigenvalues of H^Λ shifted by E and rescaled with $|\Lambda|$ to gain information about the local structure of the eigenvalues in the vicinity of E. More precisely we consider the set $\{\xi_n(\Lambda, E)\}_{1\leq n\leq|\Lambda|}$ with $\xi_n(\Lambda, E) := |\Lambda|(E_j(\Lambda) - E)$.

Definition 3.10. Let $E \in \mathbb{R}$ be in the localized regime of $\sigma(H)$ such that the density of states $n(E)$ exists at E. We define the family of point processes $\{\xi(\Lambda, E)\}_\Lambda$ by

$$\xi(\Lambda, E) := \sum_{j=1}^{|\Lambda|} \delta_{|\Lambda|(E_j(\Lambda)-E)}.$$

Now let us state the main result of Minami's theorem:

Theorem 3.11 (Minami 1996, [Min96]). *Suppose that the density of states $n(E)$ exists at E and is positive and the fractional moment of the Green function decays exponentially* *fast as in Definition 2.31. Then the point process $\xi(\Lambda, E)$ converges weakly, as $L \to \infty$, to the Poisson point process ξ with intensity measure $n(E)dx$.*

Remark 3.12. The key ingredients to the proof of this theorem (cf. [Min96]) are the following results:

(1) *Wegner estimate*, i.e. for all $I \subset \mathbb{R}$ bounded intervals and all $\Lambda \subset \mathbb{Z}^d$ finite-volume hypercubes we have

$$\mathbb{E}\left[\operatorname{tr} \chi_I(H_\omega^\Lambda)\right] \leq \|g\|_\infty |I|\ |\Lambda| \tag{3.1}$$

(see Section 2.3), and in particular that we have for all $j \in \mathbb{Z}^d$

$$\mathbb{E}\left[\langle \delta_j, \chi_I(H_\omega^\Lambda)\delta_j\rangle\right] \leq \|g\|_\infty |I|. \tag{3.2}$$

(2) *Boundedness of the fractional moments*, i.e. for all $z \in \mathbb{C}$, all $x, y \in \mathbb{Z}^d$, all $\Lambda \subset \mathbb{Z}^d$ and $0 < s < 1$ there exists a constant $C(s) > 0$ such that

$$\mathbb{E}\left[|G^\Lambda(z; x, y)|^s\right] \leq C(s) \tag{3.3}$$

holds.

(3) *Exponential decay of the fractional moments*, i.e. there exists an $s \in]0, 1[$, a $C(s) > 0$, an $m(s) > 0$ and a $r(s) > 0$ such that for all $\Lambda \subset \mathbb{Z}^d$

$$\mathbb{E}\left[|G^\Lambda(z; x, y)|^s\right] \leq C(s)\mathrm{e}^{-m(s)|x-y|} \tag{3.4}$$

with $x \in \Lambda$ and $y \in \partial\Lambda$ and $z \in \{z \in \mathbb{C}\ :\ \Im z > 0,\ |z - E| < r(s)\}$.

Proofs for the last two results can be found for example in [AM93, ASFH01, Gra94]. The proof for Minami's theorem, as can be found in [Min96], essentially relies on proving the following two assertions:

(A) *Asymptotic negligibility*, i.e. we split the finite volume cube Λ into small cubes C_p and then prove that we can approximate the point process $\xi(\Lambda, E)$, induced by the spectrum of H^Λ near E, by a superposition of independent point processes $\eta(C_p; E)$ induced by the spectrum of H^{C_p} near E, more precisely

$$\eta(C_p; E) = \sum_{j=1}^{|C_p|} \delta_{|\Lambda|(E_j(C_p)-E)}. \tag{3.5}$$

(B) *Minami estimate*, i.e. that we have for all finite $\Lambda \subset \mathbb{Z}^d$ and any bounded interval $I \subset \mathbb{R}$

$$\mathbb{E}\left[\operatorname{tr} \chi_I(H_\omega^\Lambda)\left\{\operatorname{tr} \chi_I(H_\omega^\Lambda) - 1\right\}\right] \leq \left(\|g\|_\infty |I|\ |\Lambda|\right)^2. \tag{3.6}$$

In the original proof of Minami the *Minami estimate*, was proven by ingeniously estimating the average of determinant of a 2×2 matrix whose entries are given by the imaginary part of the resolvents of H^Λ (cf. [Min96, Lemma 2]). A key ingredient of this strategy is using a rank-2-perturbation which is also referred to as Krein's formula (cf. [AM93, Appendix I]).
The other approach, based on Theorem 3.14, was proposed by Combes, Germinet and Klein in [CGK09a] which has the advantage that this method allows to prove a Minami estimate for the continuum Anderson Hamiltonian.

We will omit the explicit proof for Minami's theorem (it can be found in [Min96]) as we are going to present a modified version of it in Section 5.

3.3 Minami estimate

Remark 3.13. The statement of the following Theorem can be found in [CGK09a] and is an essential ingredient, along with a Wegner estimate, for the proof of Minami's Theorem. However, these two estimates are essential for further results about eigenvalue statistics in the localized regime, in particular, level spacing statistics (cf. [Klo]).
As explained in the Introduction, the aim of this thesis is not only to achieve a structural theorem providing us with local Poisson statistics of the eigenvalues in the sense of Minami but also to work on proving a Minami estimate for random block operators (see Chapters 4 and 5). Hence, we will present in this section the strategy to achieve the Minami estimate for the Anderson model quite explicitly to fully understand it and thus prepare us for the task of extending it to discrete random block operators.

Theorem 3.14 (Minami estimate)**.** *Let H_ω be the generalized Anderson model with the same assumptions and notations as in Theorem 2.25 (Wegner estimate) and let $\Lambda \subset \mathbb{Z}^d$ be a finite volume. Then we have for any bounded interval $I \subset \mathbb{R}$*

$$\mathbb{E}\left[\mathrm{tr}\ \chi_I(H_\omega^\Lambda)\left\{\mathrm{tr}\ \chi_I(H_\omega^\Lambda) - 1\right\}\right] \leq \left(Q(|I|)|\Lambda|\right)^2.$$

Remark 3.15. We will prove this Theorem here under the assumption that the common probability distribution $\mu = \mu_n$ of the iid stochastic process $\{V_\omega(n)\}_{n\in\mathbb{Z}^d}$ has compact support. Since we have $\mathrm{tr}\ \chi_I(H_\omega^\Lambda) \leq |\Lambda|$ for any interval I, assuming that μ has compact support is indeed sufficient for proving the estimate in full generality by the approximation argument given in [CGK09a, Appendix B].
However, before we can prove the Minami estimate we have to introduce the following results which are essential to the proof and are based on the ideas and analysis obtained in [CGK09a, CGK09b].

Lemma 3.16. *Let H_0 and W be self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $W \geq 0$ be bounded. Consider for $s \geq 0$ the self-adjoint operator $H_s = H_0 + sW$ and suppose that for all $c \in \mathbb{R}$ $\mathrm{tr}\ \chi_{]-\infty,c]}(H_s) < \infty$. Then for all $a, b \in \mathbb{R}$ with $a < b$ we have for $0 \leq s \leq t$*

$$\mathrm{tr}\ \chi_{]a,b]}(H_s) \leq \mathrm{tr}\ \chi_{]a,b]}(H_t) + \left\{\mathrm{tr}\ \chi_{]-\infty,b]}(H_0) - \mathrm{tr}\ \chi_{]-\infty,b]}(H_t)\right\}.$$

Proof. Let $a < b \in \mathbb{R}$ and $0 \leq s \leq t$. Then, since $W \leq 0$ we get

$$\text{tr } \chi_{]-\infty,b]}(H_t) \leq \text{tr } \chi_{]-\infty,b]}(H_s). \tag{3.7}$$

This implies

$$\begin{aligned}\text{tr } \chi_{]a,b]}(H_s) &= \text{tr } \chi_{]-\infty,b]}(H_s) - \text{tr } \chi_{]-\infty,a]}(H_s)\\ &\leq \text{tr } \chi_{]-\infty,b]}(H_0) - \text{tr } \chi_{]-\infty,a]}(H_t)\\ &= \left\{\text{tr } \chi_{]-\infty,b]}(H_0) - \text{tr } \chi_{]-\infty,b]}(H_t)\right\} - \text{tr } \chi_{]a,b]}(H_t). \end{aligned}\tag{3.8}$$

□

Corollary 3.17. *Consider a Hilbert space $\mathcal{H}$, $\phi \in \mathcal{H}$ with $\|\phi\| = 1$ and let P_ϕ denote the orthogonal projection onto the subspace spanned by π. Let H_0 be a self-adjoint operator on $\mathcal{H}$ bounded from below and consider for $s \in \mathbb{R}$ the self-adjoint operator $H_s = H_0 + sP_\phi$ and suppose that for all $c \in \mathbb{R}$ tr $\chi_{]-\infty,c]}(H_s) < \infty$. Then for all $a, b \in \mathbb{R}$ with $a < b$ we have for $0 \leq s \leq t$*

$$\text{tr } \chi_{]a,b]}(H_s) \leq 1 + \text{tr } \chi_{]a,b]}(H_t).$$

Proof. By analytic perturbation theory (cf. [Kat76]) the statement is indeed an immediate consequence of Lemma 3.16. However, for later references, we will calculate the result explicitly:
Let $\{E_n(s)\}_n$ denote the eigenvalues of H_s ordered from least to greatest including multiplicity and let $\{E_n\}_n$ denote the eigenvalues of H_0 accordingly. Since $P_\phi \geq 0$ and $s \geq 0$ we have $sP_\phi \geq 0$ and thus $H_0 \leq H_s$. Therefore, we have for all n and all $s \geq 0$

$$E_n \leq E_n(s). \tag{3.9}$$

Furthermore, by the min-max principle (cf. [Kir08, RS78]) we have

$$\begin{aligned}E_n(s) &= \sup_{\psi_1,\ldots,\psi_{n-1}} \inf_{\substack{\rho\perp\psi_1,\ldots,\psi_{n-1}\\ \|\rho\|=1}} \langle \rho, H_0\rho\rangle + s|\langle \rho, \phi\rangle|^2\\ &\leq \sup_{\psi_1,\ldots,\psi_{n-1}} \inf_{\substack{\rho\perp\psi_1,\ldots,\psi_{n-1},\phi\\ \|\rho\|=1}} \langle \rho, H_0\rho\rangle + s|\langle \rho, \phi\rangle|^2\\ &= \sup_{\psi_1,\ldots,\psi_{n-1}} \inf_{\substack{\rho\perp\psi_1,\ldots,\psi_{n-1},\phi\\ \|\rho\|=1}} \langle \rho, H_0\rho\rangle\\ &\leq \sup_{\psi_1,\ldots,\psi_{n}} \inf_{\substack{\rho\perp\psi_1,\ldots,\psi_{n}\\ \|\rho\|=1}} \langle \rho, H_0\rho\rangle\\ &= E_{n+1}. \end{aligned}\tag{3.10}$$

Hence, we have for $0 \leq s \leq t$

$$0 \leq \text{tr } \chi_{]-\infty,b]}(H_0) - \text{tr } \chi_{]-\infty,b]}(H_t) \leq 1, \tag{3.11}$$

where the first inequality is an immediate consequence of equation (3.7). The statement then follows from Lemma 3.16. □

Proof of Theorem 3.14. Let us first recall that we have assumed that the probability distribution μ_n of $V_\omega(n) = \omega_n$ has compact support and no atoms. In particular this implies that we have for all $c \in \mathbb{R}$

$$\mathbb{E}\left[\mathrm{tr}\ \chi_{\{c\}}(H_\omega^\Lambda)\right] = 0. \tag{3.12}$$

Therefore, we can assume the interval I to be of the form $]a, b]$ with $a < b \in \mathbb{R}$ and thus we can apply Corollary 3.17 to the following construction:
We will procede similarly to the proof of the Wegner estimate (cf. Theorem 2.25) and consider, for $j \in \Lambda$, H_ω^Λ as a rank-1-perturbation of the form

$$H_\omega^\Lambda = H^\Lambda + \omega_j P_j, \tag{3.13}$$

with $H^\Lambda := H_\omega^\Lambda - \omega_j P_j$ which is independent of ω_j. Furthermore, recall the notation introduced in the proof of Theorem 2.25:

$$\omega = (\omega_j^\perp, \omega_j) \quad \text{for } j \in \Lambda. \tag{3.14}$$

Building on this, by writing $\chi_I(H_{(\omega_j^\perp, s)})$ for $s \in \mathbb{R}$ we mean that we have, in accordance with (3.13), increased or decreased ω_j to s.
Thus by using Corollary 3.17 we get for $\tau_j \geq \omega_j$

$$\begin{aligned}\mathrm{tr}\ \chi_I(H_\omega^\Lambda)\left\{\mathrm{tr}\ \chi_I(H_\omega^\Lambda) - 1\right\} &= \sum_{j\in\Lambda}\left\{\langle\delta_j, \chi_I(H_\omega^\Lambda)\delta_j\rangle\big(\mathrm{tr}\ \chi_I(H_\omega^\Lambda) - 1\big)\right\} \\ &\leq \sum_{j\in\Lambda}\left\{\langle\delta_j, \chi_I(H^\Lambda_{(\omega_j^\perp,\omega_j)})\delta_j\rangle\big(\mathrm{tr}\ \chi_I(H^\Lambda_{(\omega_j^\perp,\tau_j)})\big)\right\}.\end{aligned} \tag{3.15}$$

The grand scheme behind this estimate is that now the expectation factorizes due to the fact that only one of the factors in (3.15) is dependent on ω_j. We may now take $\tau_j \geq \max \mathrm{supp}\mu_j$ for all $j \in \Lambda$ and then perform the expectation for the random vector $\{\omega_j\}_{j\in\mathbb{Z}^d}$ to get together with (3.15) and the Wegner estimate (cf. Theorem 2.25)

$$\begin{aligned}&\mathbb{E}_\omega\Big[\mathrm{tr}\ \chi_I(H_\omega^\Lambda)\Big\{\mathrm{tr}\ \chi_I(H_\omega^\Lambda) - 1\Big\}\Big] \\ &\qquad\leq \sum_{j\in\Lambda}\mathbb{E}_{\omega_j^\perp}\left[\Big\{\mathrm{tr}\ \chi_I(H^\Lambda_{(\omega_j^\perp,\tau_j)})\Big\}\Big\{\mathbb{E}_\omega\langle\delta_j, \chi_I(H^\Lambda_{(\omega_j^\perp,\omega_j)})\delta_j\rangle\Big\}\right] \\ &\qquad\leq Q_\mu(|I|)\sum_{j\in\Lambda}\mathbb{E}_{\omega_j^\perp}\Big[\mathrm{tr}\ \chi_I(H^\Lambda_{(\omega_j^\perp,\tau_j)})\Big].\end{aligned} \tag{3.16}$$

This estimate holds for all $\tau_j \geq \max \mathrm{supp}\mu_j$ and $j \in \Lambda$ Therefore, we may choose $\tau_j := \max \mathrm{supp}\mu_j + \tilde{\omega}_j$, where $\tilde{\omega} = \{\tilde{\omega}_j\}_{j\in\mathbb{Z}^d}$ is a new family of random variables which have the same distribution as $\{\omega_j\}_{j\in\mathbb{Z}^d}$, more precisely $\{\tilde{\omega}_j\}_{j\in\mathbb{Z}^d}$ and $\{\omega_j\}_{j\in\mathbb{Z}^d}$ are two independent and identically distributed families of random variables. Averaging over these new random variables yields

$$\begin{aligned}\mathbb{E}_\omega\Big[\mathrm{tr}\ \chi_I(H_\omega^\Lambda)\Big\{\mathrm{tr}\ \chi_I(H_\omega^\Lambda)-1\Big\}\Big] &= \mathbb{E}_{\tilde\omega}\Big\{\mathbb{E}_\omega\Big[\mathrm{tr}\ \chi_I(H_\omega^\Lambda)\Big\{\mathrm{tr}\ \chi_I(H_\omega^\Lambda)-1\Big\}\Big]\Big\}\\ &\le Q_\mu(|I|)\sum_{j\in\Lambda}\mathbb{E}_{(\omega_j^\perp,\tilde\omega_j}\Big[\mathrm{tr}\ \chi_I(H^\Lambda_{(\omega_j^\perp,\tau_j)})\Big]\\ &\le \Big(Q_\mu(|I|)|\Lambda|\Big)^2, \qquad (3.17)\end{aligned}$$

where we have again used the Wegner estimate. $\square$

4 Random block operators

As was mentioned in the introduction random block operators of the type considered in this chapter arise in the modelling of mesoscopic disordered systems such as dirty superconductors. Our main goal in this section is to introduce the concept of certain block operators, study their extension to random block operators and analyze some basic properties. For this chapter we will mostly follow the analysis obtained in [KMM11, Tre08].

4.1 Structural properties of block operators

Let $(\mathcal{H}; \langle , \rangle)$ denote a Hilbert space which is for now arbitrary, although later we are going to be interested mainly in the case where $\mathcal{H} = l^2(\mathbb{Z}^d)$ as in Section 1. Hence, consider the Hilbert space $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$ equipped with the scalar product

$$\langle\!\langle \Psi, \Phi \rangle\!\rangle := \langle \psi_1, \phi_1 \rangle + \langle \psi_2, \phi_2 \rangle, \tag{4.1}$$

with

$$\Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \Phi := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H}^2. \tag{4.2}$$

We shall denote the norm induced by this scalar product as following

$$|\!|\!| \Psi |\!|\!| := \sqrt{\|\psi_1\|^2 + \|\psi_2\|^2}. \tag{4.3}$$

On this Hilbert space we shall consider operators of the form

$$\mathbb{H} := \begin{pmatrix} H & B \\ B & -H \end{pmatrix}, \tag{4.4}$$

where H and B are self-adjoint operators on $\mathcal{H}$. As shown in [KMM11, Tre08], we can conclude self-adjointness for $\mathbb{H}$ under certain conditions, more precisely

Proposition 4.1. *Let H,B be self-adjoint operators on $\mathcal{H}$ and assume that* $\mathrm{dom}(B) \cap \mathrm{dom}(H)$ *is a core for H and that* $\mathrm{dom}(|H|^{1/2}) \subset \mathrm{dom}(B)$. *Then $\mathbb{H}$ is essentially self-adjoint on* $\{\mathrm{dom}(B) \cap \mathrm{dom}(H)\} \oplus \{\mathrm{dom}(B) \cap \mathrm{dom}(H)\}$.

Proof. As mentioned in [KMM11], the statement is a consequence of Proposition 2.3.6 in [Tre08]. □

Remark 4.2. From here on we will always assume that the conditions of Proposition 4.1 are satisfied. The following results and proofs can be found in [KMM11].

Lemma 4.3. *Let $\mathbb{H}$ be defined as above. Then the spectrum of $\mathbb{H}$, denoted by $\sigma(\mathbb{H})$, is symmetric around 0; more precisely we have*

$$\sigma(\mathbb{H}) = \sigma(-\mathbb{H})$$

and in particular, if we have $\mathbb{H}\Psi = E\Psi$ for some $E \in \mathbb{R}$ and $\Psi = (\psi_1, \psi_2)^T \in \mathcal{H}^2$ then

$$\mathbb{H}\tilde{\Psi} = -E\tilde{\Psi},$$

where $\tilde{\Psi} = (\psi_2, -\psi_1)^T$.

Proof. Consider the unitary transformation

$$\mathbb{U} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.5}$$

on $\mathcal{H}^2$ which satisfies $\mathbb{U}^{-1} = -\mathbb{U}$. Thus we have

$$\mathbb{U}\mathbb{H}\mathbb{U}^* = -\mathbb{H} \tag{4.6}$$

which implies the statement. $\square$

Lemma 4.4. *The operator $\mathbb{H}^2$ is given by*

$$\mathbb{H}^2 = \begin{pmatrix} H^2 + B^2 & [H, B] \\ -[H, B] & H^2 + B^2 \end{pmatrix},$$

where $[-, -]$ denotes the commutator.

Proof. The statement follows by direct computation

$$\mathbb{H}^2 := \begin{pmatrix} H & B \\ B & -H \end{pmatrix} \begin{pmatrix} H & B \\ B & -H \end{pmatrix} = \begin{pmatrix} H^2 + B^2 & HB - BH \\ BH - HB & H^2 + B^2 \end{pmatrix}. \tag{4.7}$$

$\square$

Corollary 4.5. *The spectrum of the operator $\mathbb{H}^2$ has multiplicity at least 2 with exception possibly at 0. Furthermore, we have*

$$\sigma(\mathbb{H}) = \left\{ E \in \mathbb{R} \ : \ E^2 \in \sigma(\mathbb{H}^2) \right\}.$$

Proof. The statement follows immediately form Lemma 4.3. $\square$

However, we can infer more information about the structure of the spectrum of $\mathbb{H}$, in particular we can prove the occurrence of a spectral gap around 0 which arises due to the block structure of $\mathbb{H}$.

Proposition 4.6. *Let $\mathbb{H}$ be defined as above. Then*

(i) $\sigma(\mathbb{H}) \subset [-\|H\| - \|B\|, \|H\| + \|B\|]$.

(ii) If there exists a $\lambda \geq 0$ *such that* $H \geq \lambda \mathbb{1}$*, then we have*

$$\sigma(\mathbb{H}) \ \cap \ ]-\lambda, \lambda[= \emptyset.$$

(iii) Assume that there exists a $\beta \geq 0$ *such that* $B \geq \beta \mathbb{1}$*, then*

$$\sigma(\mathbb{H}) \ \cap \ ]-\beta, \beta[= \emptyset.$$

(iv) Assume that there exist $\lambda, \beta \geq 0$ *such that* $H \geq \lambda 1$ *and* $B \geq \beta 1$*, then we have*

$$\sigma(\mathbb{H}) \ \cap \ \Big]-\sqrt{\lambda^2+\beta^2}, \sqrt{\lambda^2+\beta^2}\Big[= \emptyset.$$

Proof. The proof can be found in [KMM11]. □

4.2 Ergodic properties of random block operators

Similarly to the Anderson Model on $l^2(\mathbb{Z}^d)$, let us consider the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where we set $\Omega := \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$. For the construction of $\mathcal{F}$, consider the two probability spaces $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu_1)$ and $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu_2)$. Then, let $\mathcal{F}$ be the σ-algebra induced by the cylinder sets of the form

$$\Big\{\omega = (\omega^1, \omega^2) \in \Omega \mid \omega^1_{i_1} \in B_1, \dots, \omega^1_{i_n} \in B_n,\ B_1, \dots, B_n \in \mathfrak{B}(\mathbb{R}),\ n \in \mathbb{N} \\ \text{and } \omega^2_{i_1} \in B_1, \dots, \omega^2_{i_m} \in B_m,\ B_1, \dots, B_m \in \mathfrak{B}(\mathbb{R}),\ m \in \mathbb{N}\Big\}. \tag{4.8}$$

Finally, let $\mathbb{P}$ be the infinite product measure on $(\Omega, \mathcal{F})$ induced by $\mu_1 \otimes \mu_2$.

Remark 4.7. From this point on we shall, unless explicitly noted, only consider the discrete Hilbert space $\mathcal{H} := l^2(\mathbb{Z}^d)$ as in Section 2, and accordingly $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$ with the structure induced by $\mathcal{H}$. Furthermore, we note that the set

$$\left\{ \begin{pmatrix} \delta_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \delta_j \end{pmatrix} \right\}_{j \in \mathbb{Z}^d} =: \{\delta_j \oplus 0, 0 \oplus \delta_j\}_{j \in \mathbb{Z}^d}, \tag{4.9}$$

where δ_j is as in Definition 2.1, forms an orthonormal basis of $\mathcal{H}^2$.

Definition 4.8. Let $L(\mathcal{H}^2)$ denote the set of linear operators on the Hilbert space $\mathcal{H}^2$. Let H_ω, B_ω be self-adjoint operators such that Proposition 4.1 is fulfilled. Then the block-operator-valued random variable $\mathbb{H}$, defined by

$$\begin{aligned} \mathbb{H}:\ \Omega &\longrightarrow L(\mathcal{H}^2) \\ \omega &\longmapsto \mathbb{H}_\omega := \begin{pmatrix} H_\omega & B_\omega \\ B_\omega & H_\omega \end{pmatrix} \end{aligned}$$

where $\mathbb{H}_\omega$ is densely defined on $\mathcal{H}^2$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, will be called *random block operator*.

We will now introduce a notation similar to Definition 2.7, as is used in [Geb11], to make the block structure more explicit.

Definition 4.9. Let $\Psi \in \mathcal{H}^2$, then we define for all $n \in \mathbb{Z}^d$

$$\Psi(n) := \begin{pmatrix} \langle\!\langle \delta_n \oplus 0, \Psi \rangle\!\rangle \\ \langle\!\langle 0 \oplus \delta_n, \Psi \rangle\!\rangle \end{pmatrix} =: \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix}.$$

Let A be a bounded self-adjoint operator on $\mathcal{H}^2$ then we define the *kernel* of A by

$$A(n,m) := \begin{pmatrix} \langle\!\langle \delta_n \oplus 0, A(\delta_m \oplus 0) \rangle\!\rangle & \langle\!\langle \delta_n \oplus 0, A(0 \oplus \delta_m) \rangle\!\rangle \\ \langle\!\langle 0 \oplus \delta_n, A(\delta_m \oplus 0) \rangle\!\rangle & \langle\!\langle 0 \oplus \delta_n, A(0 \oplus \delta_m) \rangle\!\rangle \end{pmatrix}$$

for all $n, m \in \mathbb{Z}^d$ such that we have for all $\Psi \in \mathcal{H}^2$ and all $n \in \mathbb{Z}^d$

$$(A\Psi)(n) = \sum_{m \in \mathbb{Z}^d} A(n,m)\Psi(m).$$

Definition 4.10. Let $p \in \mathbb{N}^d$, $p\mathbb{Z}^d := \bigoplus_{k=1}^d (p_k\mathbb{Z})$ and $\{T_j\}_{j \in p\mathbb{Z}^d}$ an ergodic group of measure-preserving transformations on Ω such that we have for all $\omega \in \Omega$, all $j \in p\mathbb{Z}^d$ and all $n, m \in \mathbb{Z}^d$

$$(T_j\omega)_{(n,m)} = (\omega^1_{n-j}, \omega^2_{m-j}).$$

Furthermore, let for all $j \in p\mathbb{Z}^d$ U_j and in particular $\mathbb{U}_j$ be the unitary translation operator induced by the ergodic group $\{T_j\}_{j \in p\mathbb{Z}^d}$, i.e. for $\Psi \in \mathcal{H}^2$ we have

$$\mathbb{U}_j \Psi := \begin{pmatrix} U_j & 0 \\ 0 & U_j \end{pmatrix} \Psi = \begin{pmatrix} U_j\psi_1 \\ U_j\psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1(\cdot - j) \\ \psi_2(\cdot - j) \end{pmatrix}.$$

Then we call $\mathbb{H}$ *ergodic* w.r.t. $p\mathbb{Z}^d$-translations if there exists a $p \in \mathbb{N}^d$ and a $\{T_j\}_{j \in p\mathbb{Z}^d}$ as above such that

$$\mathbb{U}_j \mathbb{H}_\omega \mathbb{U}_j^* = \mathbb{H}_{T_j\omega}$$

holds for every $\omega \in \Omega$ and every $j \in p\mathbb{Z}^d$.

With this setting we can proceed quite similar to the case of the Anderson model and gain analogous results such as the integrated density of states (cf. Section 2) and an almost surely non-random and closed spectrum.

Proposition 4.11. *For $\mathbb{P}$-almost all ω we have $\sigma(\mathbb{H}_\omega) = \Sigma$ for $\Sigma \subset \mathbb{R}$ a non-random closed set.*

Proof. The proof uses a standard argument via the Weyl criterion and proceeds along analogous arguments as can be found in [Kir08]. □

Definition 4.12. Let tr denotes the trace on $\mathcal{H}^2$ and $\mathbb{E}$ denote the expectation on Ω. Let $\Lambda_0 := \{j \in \mathbb{Z}^d \ : \ 0 \leq j < p_k \ \forall k \in \{1, \dots, d\}\}$ denote the elementary cell, then we call the right-continuous, non-decreasing function $\mathbb{N} : \mathbb{R} \to [0,1]$, defined by

$$\mathbb{N}(E) := \frac{1}{2|\Lambda_0|}\mathbb{E}\left[\operatorname{tr}\left\{(\chi_{\Lambda_0} \oplus \chi_{\Lambda_0})\chi_{]-\infty,E]}(\mathbb{H}_\omega)\right\}\right],$$

integrated density of states of $\mathbb{H}$.

Remark 4.13. In the following we will write instead of $\chi_{\Lambda_0} \oplus \chi_{\Lambda_0}$ simply χ_{Λ_0} to abbreviate the notation.

Lemma 4.14. *Let $\mathbb{H}$ be a random block operator defined as above. Then we have $\mathbb{P}$-almost surely for every $E \in \mathbb{R}$ and all $j \in \mathbb{Z}^d$*

$$\mathbb{N}(E) = \lim_{L\to\infty} \frac{1}{2|\Lambda|}\operatorname{tr}\left[\chi_{\Lambda(j)}\chi_{]-\infty,E]}(\mathbb{H}_\omega)\right].$$

Proof. The proof is analogous to the case of the Anderson model and can be found, with obvious adaptions, in [Kir08, Sec. 5.1]. □

From this point on we will focus on random block operators with the following properties: Let H_ω be the Anderson model as in Section 2 but with the slight difference that we consider the common probability distribution μ_V of the iid real-valued stochastic process $V = \{V_j : \ j \in \mathbb{Z}^d\}$ to have a Lebesgue density ϕ_V of bounded variation. As before V gives rise to the multiplication operator V_ω on $\mathcal{H}$, more precisely we have for all $\psi \in \mathcal{H}$ and $n \in \mathbb{Z}^d$

$$(V_\omega\psi)(n) = V_\omega(n)\psi(n). \tag{4.10}$$

Furthermore, consider the iid ergodic real-valued stochastic process $b = \{b_j : \ j \in \mathbb{Z}^d\}$ such that

$$b_j : \ \Omega \longrightarrow \mathbb{R} : \ \omega \longmapsto b_\omega(j) \tag{4.11}$$

with the common probability distribution μ_b with a Lebesgue density ϕ_b of bounded variation. Thus, let $B_\omega = b_\omega$ be the multiplication operator on $\mathcal{H}$ induced by b, more precisely we have for all $\psi \in \mathcal{H}$ and $n \in \mathbb{Z}^d$

$$(b_\omega\psi)(n) = b_\omega(n)\psi(n). \tag{4.12}$$

Remark 4.15. Both processes V and b are ergodic with respect to $p\mathbb{Z}^d$-translations. We will also, in analogy with Remark 2.3, assume that μ_V and μ_b have compact support.

Hence, from now on we shall consider the random block operator of the form

$$\mathbb{H} : \omega \longmapsto \mathbb{H}_\omega := \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix} \tag{4.13}$$

which is self-adjoint according to Proposition 4.1.

4.3 Almost sure spectrum of random block operators

Remark 4.16. In the following we want to make the notion of the almost sure spectrum of $\mathbb{H}$ more precise. For this purpose we will again follow the ideas and methods which can be found in [KMM11]. However, we already know from Section 2 that we have

$$\sigma(H_\omega) = \sigma(H_0) + \operatorname{supp}(\mu_V) \tag{4.14}$$

$\mathbb{P}$-almost surely and, by using an analogous argument,

$$\sigma(b_\omega) = \operatorname{supp}(\mu_b) \tag{4.15}$$

$\mathbb{P}$-almost surely.

Proposition 4.17. *Consider the random block operator* $\mathbb{H}$ *defined in (4.13) and define*

$$r := \sup_{E\in\sigma(H_\omega)} |E| + \sup_{\beta\in\operatorname{supp}(\mu_b)} |\beta|.$$

Then we have $\mathbb{P}$*-almost surely*

$$\left\{\pm\sqrt{E^2+\beta^2} \;:\; E \in \sigma(H_\omega), \beta \in \operatorname{supp}(\mu_b)\right\} \subset \sigma(\mathbb{H}_\omega) \subset [-r, r].$$

If $\inf\sigma(H_\omega) \geq 0$ *and* $\inf\operatorname{supp}(\mu_b) \geq 0$ *and if*

$$\lambda_\pm := \pm\sqrt{[\inf\sigma(H_\omega)]^2 + [\inf\operatorname{supp}(\mu_b)]^2},$$

then λ_+ *and* λ_- *are the endpoints of the open gap interval which separates the positive and the negative parts of the symmetric almost sure spectrum of* $\mathbb{H}_\omega$*.*

Proof. The proof uses a Borel–Cantelli and a standard Weyl sequence argument and can be found in [KMM11, Lemma 4.3, Corollary 4.5] □

4.4 Boundary conditions

In this section we will give a precise notion of boundary conditions for random block operators, analogous to Section 2.1.

Definition 4.18. Let $\mathbb{H}$ be the random block operator defined in (4.13) and let $\Lambda \subset \mathbb{Z}^d$ be a finite-volume hypercube. Then we define the following boundary conditions for $\mathbb{H}$ restricted to the $2|\Lambda|$-dimensional Hilbert space $\mathcal{H}^2_\Lambda := l^2(\Lambda) \oplus l^2(\Lambda)$:

(i) We define *Dirichlet boundary conditions* by

$$\mathbb{H}^\Lambda_D := \begin{pmatrix} H^\Lambda_D & b \\ b & -H^\Lambda_D \end{pmatrix},$$

(ii) *Neumann boundary conditions* by

$$\mathbb{H}_N^\Lambda := \begin{pmatrix} H_N^\Lambda & b \\ b & -H_N^\Lambda \end{pmatrix},$$

(iii) and *simple boundary conditions* by

$$\mathbb{H}^\Lambda := \begin{pmatrix} H^\Lambda & b \\ b & -H^\Lambda \end{pmatrix},$$

where H_D^Λ, H_N^Λ and H^Λ denote the restrictions of H to the Hilbert space $l^2(\Lambda)$ with Dirichlet, Neumann and simple boundary conditions, as we introduced them in Section 2.1. Furthermore, we consider all multiplication operators to have canonical restrictions. We also define the *Dirichlet-bracketing* and *Neumann-bracketing* restrictions by

$$\mathbb{H}_+^\Lambda := \begin{pmatrix} H_D^\Lambda & b \\ b & -H_N^\Lambda \end{pmatrix} \quad \text{and} \quad \mathbb{H}_-^\Lambda := \begin{pmatrix} H_N^\Lambda & b \\ b & -H_D^\Lambda \end{pmatrix}.$$

Proposition 4.19. *Let $\mathbb{H}$ be the random block operator defined in (4.13) and let $\Lambda \subset \mathbb{Z}^d$ be a finite-volume hypercube. Then we have*

$$\mathbb{H}_-^\Lambda \leq \mathbb{H}_N^\Lambda \leq \mathbb{H}_+^\Lambda \quad \textit{and} \quad \mathbb{H}_-^\Lambda \leq \mathbb{H}_D^\Lambda \leq \mathbb{H}_+^\Lambda.$$

Furthermore, in case of simple boundary conditions, we have the following splitting formula for $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$

$$\begin{aligned}
\mathbb{H}_{\Lambda_2} &= \mathbb{H}_{\Lambda_1} \oplus \mathbb{H}_{\Lambda_2 \setminus \Lambda_1} + \Gamma_{\Lambda_1}^{\Lambda_2} \oplus \big(-\Gamma_{\Lambda_1}^{\Lambda_2} \big) \\
&:= \begin{pmatrix} H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} & b \\ b & -H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} \end{pmatrix} + \begin{pmatrix} \Gamma_{\Lambda_1}^{\Lambda_2} & 0 \\ 0 & -\Gamma_{\Lambda_1}^{\Lambda_2} \end{pmatrix}
\end{aligned}$$

with

$$\Gamma_{\Lambda_1}^{\Lambda_2}(n,m) := \begin{cases} -1 & \textit{if } (n,m) \in \partial_{\Lambda_2} \Lambda_1 \\ 0 & \textit{else.} \end{cases}$$

Proof. Details for the proof of the first statement can be found in [KMM11]. The proof for the second statement proceeds along the same reasoning as for Proposition 2.12 and can be found, with slight modifications, in [Kir08, Sec. 5.2]. □

4.5 Wegner estimate

In this section we ultimately want to proof a Wegner estimate for $\mathbb{H}$, analogous as for the Anderson Model. However, since we have lost the monotonicity in the growth of the eigenvalues of $\mathbb{H}_\omega$ due to the operator $-H_\omega$ in $\mathbb{H}$ we will need a couple of additional lemmata. The ideas and methods in this section are again from [KMM11]. However, first we introduce the following notations:

Definition 4.20. Let $\mathbb{H}$ denote the random block operator defined in (4.13), let $\Lambda \subset \mathbb{Z}^d$ be a finite-volume hypercube and $E \in \mathbb{R}$. Let, furthermore, X denote any self-adjoint restriction such that

$$\mathbb{H}_\omega^{\Lambda,-} \leq \mathbb{H}_\omega^{\Lambda,X} \leq \mathbb{H}_\omega^{\Lambda,+}$$

holds. Then we define the random, finite-volume eigenvalue counting function by

$$\mathbb{N}_\omega^{\Lambda,X}(E) := \frac{1}{2|\Lambda|}\operatorname{tr}\left[\chi_{]-\infty,E]}(\mathbb{H}_\omega^{\Lambda,X})\right].$$

Lemma 4.21. *Given the above definition, there exists a set Ω_0 of full probability, i.e. $\mathbb{P}[\Omega_0] = 1$, such that*

$$\mathbb{N}(E) = \lim_{|\Lambda|\to\infty} \mathbb{N}_\omega^{\Lambda,X}(E)$$

for any self-adjoint restriction X as in Definition 4.20, every $\omega \in \Omega_0$ and every continuity point $E \in \mathbb{R}$ of $\mathbb{N}$.

Proof. The proof proceeds along similar lines as in the case of the Anderson model (cf. Section 2), for more details see [Kir08, KMM11]. □

Definition 4.22. For a complex-valued function ϕ with compact support in $\mathbb{R}$ the *total variation* norm is defined by

$$\|\phi\|_{BV} := \sup_{\mathcal{P}} \sum_i |\phi(x_{i+1}) - \phi(x_i)|,$$

where $\mathcal{P} := \{(x_1, \dots, x_p) \ : \ \inf\operatorname{supp}(\phi) \leq x_1 \leq \dots \leq x_p \leq \sup\operatorname{supp}(\phi), \ p \in \mathbb{N}\}$ denotes the set of all partitions of the support of ϕ. ϕ is said to be of *bounded variation* if

$$\|\phi\|_{BV} < \infty.$$

Remark 4.23. We will now state the result of the Wegner estimate, as it can be found in [KMM11] with a slight deviation in the constants of the estimates. The reason for this difference will become apparent during the proof.

Theorem 4.24 (Wegner estimate)**.** *Consider the random block operator*

$$\mathbb{H} : \ \omega \longmapsto \mathbb{H}_\omega = \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix}$$

defined as in (4.13). Assume that at least one of the following two conditions is satisfied:

(i) there exists a $\lambda > 0$ such that $H \geq \lambda 1$ holds $\mathbb{P}$-almost surely and μ_V is absolutely continuous with a piecewise continuous Lebesgue density ϕ_V of bounded variation and compact support,

(ii) there exists a $\beta > 0$ such that $b \geq \beta 1$ holds $\mathbb{P}$-almost surely and μ_b is absolutely continuous with a piecewise continuous Lebesgue density ϕ_b of bounded variation and compact support.

Then we have for all $E \in \mathbb{R}$ for $\epsilon \in]0, \min\{\lambda, 1\}/3[$ that

$$\mathbb{E}\big\{\mathrm{tr}\chi_{]E-\epsilon,E+\epsilon]}(\mathbb{H}^\Lambda)\big\} \leq 4\epsilon|\Lambda|\frac{E+1}{\lambda}\|\phi_V\|_{BV}.$$

Remark 4.25. 1. Consider the unitary transformation

$$\mathbb{U} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.16}$$

It then follows by direct calculation that the two operators $\mathbb{H}_\omega$ and $\mathbb{H}'_\omega$ of the form

$$\mathbb{H}_\omega = \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix} \quad \text{and} \quad \mathbb{H}'_\omega = \begin{pmatrix} b_\omega & H_\omega \\ H_\omega & -b_\omega \end{pmatrix} \tag{4.17}$$

are unitary equivalent, i.e. $\mathbb{H}_\omega = \mathbb{U}\mathbb{H}'_\omega\mathbb{U}^*$. Hence, we see that the roles of the operators H_ω and b_ω are interchangeable and thus it suffices to prove Theorem 4.24 only for one of the two conditions. Hence, we will choose to prove the Theorem under condition *(i)* wherefore we will suppress the b_ω-dependence of the Eigenvalues and simply write $E(V_\omega)$ to have an easier notation. Furthermore, for convenience, we will write $V = V_\omega$.

2. As indicated at the beginning of this section, we will have to face additional difficulties if we want to proof the Wegner estimate. Due to the form of the block operator, in particular due to the operator $-H$, we cannot use the ideas of the standard methods, such as in [Kir08, Sec. 5.5] or the method of [CGK09a] we used in Section 2.3, since both of them rely on the monotonic growth of the eigenvalues which we do not have in case of the considered block operator. However, as can be found in [KMM11], there is a way around this problem wherefore we need the following lemmata.

Lemma 4.26. *Let $E(V)$ be an eigenvalue of*

$$\begin{pmatrix} H^\Lambda(V) & b \\ b & -H^\Lambda(V) \end{pmatrix}.$$

Then we have

$$E(V)\sum_{j\in\Lambda}\frac{\partial E(V)}{\partial V_j} \geq \inf\mathrm{supp}\big(H^\Lambda(V)\big).$$

Proof. This statement can be derived as a consequence of the Feynman-Hellmann theorem and can be found in [KMM11]. □

Lemma 4.27. *Let $\phi: \mathbb{R} \to \mathbb{C}$ be a piecewise continuous function of bounded variation with compact support. Let $F \in \mathcal{C}^1(\mathbb{R})$ and assume that there exists a constant $a > 0$ such that $|F(x) - F(y)| \leq a$ holds for all $x, y \in \mathbb{R}$. Then we have*

$$\left|\int_{\mathbb{R}} F'(x)\phi(x)dx\right| \leq a\|\phi\|_{BV}.$$

Proof. This statement can be shown by explicitly calculating the statement for step functions approximating ϕ and then using a density argument to conclude it for all ϕ of bounded variation. The explicit proof can be found in [KMM11]. □

Remark 4.28. Let us note at this point that in the standard random Schrödinger case it proved to be extremely helpful to consider the random operator as a rank-1-perturbation of the form

$$H_\omega^\Lambda = H^\Lambda + \omega_k P_k, \tag{4.18}$$

where P_k, $k \in \Lambda$, denotes the orthogonal projection onto the subspace spanned by δ_k and $H^\Lambda := H_\omega^\Lambda - \omega_k P_k$. This point of view helps for proving not only with a Wegner estimate (as can be seen in Appendix A.1 and [CGK09a]) but also a Minami estimate (see Theorem 3.14) since we can easily infer the monotonic growth (in the coupling constant ω_k) of the eigenvalues (as can be seen in the proof of Corollary 3.17).
However, if we try to adapt this approach to a random block operator $\mathbb{H}$ and consider it as a rank-2-perturbation (as can be seen in Lemma 4.29), we see immediately that the eigenvalues grow non-monotonically with respect to the coupling constant V_j (see Lemma 4.29).
Since one of our goals is to work on adapting the method of Combes, Germinet and Klein given in [CGK09a] for proving a Minami estimate to discrete random block operators (see Chapter 5), we can, at this point, learn from Lemma 4.29 that we cannot adapt Lemma 3.16 and Corollary 3.17 to random block operators directly. However, in Section 5.2 we shall provide a solution to this problem.
We can still use Lemma 4.29 though to achieve a better bounding constant for the Wegner estimate (cf. Remark 4.30).

Lemma 4.29. *Let $\mathbb{H}$ denote the random block operator defined in (4.13) and let $\mathbb{H}^\Lambda$ denote the random block operator restricted to $\Lambda \subset \mathbb{Z}^d$ under any given self-adjoint boundary condition. For $\epsilon > 0$ consider the switch function $\rho \in \mathcal{C}^1(\mathbb{R})$ such that ρ is non-decreasing with $0 \le \rho \le 1$ and*

$$\rho = \begin{cases} 0 & \textit{for } x < -\epsilon \\ 1 & \textit{for } x > \epsilon \end{cases}$$

and consider an $\eta > 0$ such that $\inf \operatorname{supp}\rho(\cdot - \eta) \ge \lambda_+$, i.e. $\inf \operatorname{supp}\rho(\cdot - \eta)$ is located in the positive part of the spectrum of $\mathbb{H}$. Furthermore, consider the operator $\mathbb{H}^\Lambda = \mathbb{H}^\Lambda(V)$ in the form of a rank-2-pertubration, i.e. for all $j \in \Lambda$ we have

$$\mathbb{H} = \mathbb{H}_0 + V_j \Pi_j := \begin{pmatrix} H^\Lambda(V) - V_j P_j & b \\ b & -H^\Lambda(V) + V_j P_j \end{pmatrix} + V_j \begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix},$$

where $\mathbb{H}_0$ is independent of V_j and P_j denotes the projection onto the subspace spanned by δ_j. Then we have for $T \in \mathbb{R}$

$$\left| \operatorname{tr}\rho(\mathbb{H}^\Lambda(V_j^\perp, T) - \eta) - \operatorname{tr}\rho(\mathbb{H}^\Lambda(V) - \eta) \right| \le 1.$$

Remark 4.30. Since we consider a rank-2-perturbation we would apriori expect a 2 in the estimate above. However, since we have the projection on the subspace spanned by $\delta_j \oplus 0$ with a positive sign and the projection on the subspace spanned by $0 \oplus \delta_j$ with a negative sign we get a monotonic behavior of the eigenvalues if we change only one of the corresponding coupling constants V_j. This effect, as shown below, yields an estimate with the constant 1, but the price we have to pay for this is the modulus.

Proof of Lemma 4.29. Let $E_n(V)$ for $1 \leq n \leq 2|\Lambda|$ denote the eigenvalues of $\mathbb{H}^\Lambda(V)$ ordered by magnitude and according to multiplicity, i.e.

$$E_1(V) \leq \cdots \leq E_{2|\Lambda|}(V). \tag{4.19}$$

Let $E_n(V_j^\perp, T)$ for $1 \leq n \leq 2|\Lambda|$ denote the eigenvalues of $\mathbb{H}^\Lambda(V_j^\perp, T)$ ordered accordingly. Let us first assume that $T \geq V_j$. Then we get by the min-max principle (cf. [Kir08, RS78]) for all $n \in \{1, \ldots, 2|\Lambda| - 1\}$

$$\begin{aligned} E_n(V_j^\perp, T) &= \sup_{\Psi_1,\ldots,\Psi_{n-1}} \inf_{\Phi \perp \langle \Psi_1,\ldots,\Psi_{n-1}\rangle} \langle\!\langle \Phi, \mathbb{H}_0 \Phi \rangle\!\rangle + T|\langle \delta_j, \phi_1\rangle|^2 - T|\langle \delta_j, \phi_2\rangle|^2 \\ &\leq \sup_{\Psi_1,\ldots,\Psi_{n-1}} \inf_{\Phi \perp \langle \Psi_1,\ldots,\Psi_{n-1}\rangle} \langle\!\langle \Phi, \mathbb{H}_0 \Phi \rangle\!\rangle + T|\langle \delta_j, \phi_1\rangle|^2 - V_j|\langle \delta_j, \phi_2\rangle|^2 \\ &\leq \sup_{\Psi_1,\ldots,\Psi_{n-1}} \inf_{\Phi \perp \langle \Psi_1,\ldots,\Psi_{n-1}, \delta_j \oplus 0\rangle} \langle\!\langle \Phi, \mathbb{H}_0 \Phi \rangle\!\rangle + V_j|\langle \delta_j, \phi_1\rangle|^2 - V_j|\langle \delta_j, \phi_2\rangle|^2 \\ &\leq \sup_{\Psi_1,\ldots,\Psi_n} \inf_{\Phi \perp \langle \Psi_1,\ldots,\Psi_n\rangle} \langle\!\langle \Phi, \mathbb{H}(V) \Phi \rangle\!\rangle \\ &= E_{n+1}(V), \end{aligned} \tag{4.20}$$

where $\langle \Psi_1, \ldots, \Psi_{n-1}\rangle$ denotes the linear span of the vectors $\Psi_1, \ldots, \Psi_{n-1} \in \mathcal{H}^2$. If we follow the same strategy, however starting with the other coupling constant V_j, we get for all $n \in \{2, \ldots, 2|\Lambda|\}$

$$\begin{aligned} E_{n-1}(V) &= \sup_{\Psi_1,\ldots,\Psi_{n-2}} \inf_{\Phi \perp \langle \Psi_1,\ldots,\Psi_{n-2}\rangle} \langle\!\langle \Phi, \mathbb{H}_0 \Phi \rangle\!\rangle + V_j|\langle \delta_j, \phi_1\rangle|^2 - V_j|\langle \delta_j, \phi_2\rangle|^2 \\ &\leq \sup_{\Psi_1,\ldots,\Psi_{n-2}} \inf_{\Phi \perp \langle \Psi_1,\ldots,\Psi_{n-2}\rangle} \langle\!\langle \Phi, \mathbb{H}_0 \Phi \rangle\!\rangle + T|\langle \delta_j, \phi_1\rangle|^2 - V_j|\langle \delta_j, \phi_2\rangle|^2 \\ &\leq \sup_{\Psi_1,\ldots,\Psi_{n-2}} \inf_{\Phi \langle \perp \Psi_1,\ldots,\Psi_{n-2}, 0 \oplus \delta_j\rangle} \langle\!\langle \Phi, \mathbb{H}_0 \Phi \rangle\!\rangle + T|\langle \delta_j, \phi_1\rangle|^2 - T|\langle \delta_j, \phi_2\rangle|^2 \\ &\leq \sup_{\Psi_1,\ldots,\Psi_{n-1}} \inf_{\Phi \langle \perp \Psi_1,\ldots,\Psi_{n-1}\rangle} \langle\!\langle \Phi, \mathbb{H}(V_j^\perp, T) \Phi \rangle\!\rangle \\ &= E_n(V_j^\perp, T). \end{aligned} \tag{4.21}$$

In case $V_j \geq T$ we can infer the same estimates by adapting the above procedure accordingly. Hence we have for all $n \in \{2, \ldots, 2|\Lambda| - 1\}$ and all $T \in \mathbb{R}$

$$E_{n-1}(V) \leq E_n(V_j^\perp, T) \leq E_{n+1}(V). \tag{4.22}$$

Furthermore, we can infer by (4.20)

$$\begin{aligned}
&\mathrm{tr}\rho(\mathbb{H}^\Lambda(V_j^\perp,T)-\eta)-\mathrm{tr}\rho(\mathbb{H}^\Lambda(V)-\eta)=\sum_{n=1}^{2|\Lambda|}\Big\{\rho(E_n(V_j^\perp,T)-\eta)-\rho(E_n(V)-\eta)\Big\}\\
&\leq\sum_{n=1}^{2|\Lambda|-1}\{\rho(E_{n+1}(V)-\eta)-\rho(E_n(V)-\eta)\}+\rho(E_{2|\Lambda|}(V_j^\perp,T)-\eta)-\rho(E_{2|\Lambda|}(V)-\eta)\\
&=\rho(E_{2|\Lambda|}(V)-\eta)-\rho(E_1(V)-\eta)+\rho(E_{2|\Lambda|}(V_j^\perp,T)-\eta)-\rho(E_{2|\Lambda|}(V)-\eta)\\
&\leq 1
\end{aligned}\tag{4.23}$$

and by (4.21)

$$\begin{aligned}
&\mathrm{tr}\rho(\mathbb{H}^\Lambda(V)-\eta)-\mathrm{tr}\rho(\mathbb{H}^\Lambda(V_j^\perp,T)-\eta)=\sum_{n=1}^{2|\Lambda|}\Big\{\rho(E_n(V)-\eta)-\rho(E_n(V_j^\perp,T)-\eta)\Big\}\\
&\leq\sum_{n=2}^{2|\Lambda|}\{\rho(E_n(V)-\eta)-\rho(E_{n-1}(V)-\eta)\}+\rho(E_1(V)-\eta)-\rho(E_1(V_j^\perp,T)-\eta)\\
&=\rho(E_{2|\Lambda|}(V)-\eta)-\rho(E_1(V)-\eta)+\rho(E_1(V)-\eta)-\rho(E_1(V_j^\perp,T)-\eta)\\
&\leq 1.
\end{aligned}\tag{4.24}$$

Hence, we can conclude

$$\Big|\mathrm{tr}\rho(\mathbb{H}^\Lambda(V_j^\perp,T)-\eta)-\mathrm{tr}\rho(\mathbb{H}^\Lambda(V)-\eta)\Big|\leq 1.\tag{4.25}$$

□

Remark 4.31. As noted in Remark 4.25 it suffices to prove the Wegner estimate assuming condition *(i)*. Furthermore, due to the symmetry of the spectrum of $\mathbb{H}$ (cf. Lemma 4.3) it is sufficient to restrict ourselves to the positive half of the spectrum, i.e. $E \geq 0$. In fact, due to Proposition 4.17 and due to condition *(i)* we have $E \geq \lambda$.

Equipped with these helpful tools we can turn to the proof of the Wegner estimate:

Proof of Theorem 4.24. Let $E_n(V)$ denote the n^{th} eigenvalue of $\mathbb{H}^\Lambda(V)$ ordered by magnitude and repeated according to multiplicity. Fix $\epsilon \in\ ]0,\min\{\lambda,1\}/3[$ and consider the respective switch function $\rho \in \mathcal{C}^1(\mathbb{R})$ defined in Lemma 4.29, i.e. ρ is non-decreasing, $0 \leq \rho \leq 1$ and

$$\rho=\begin{cases}0 & \text{for } x<-\epsilon\\ 1 & \text{for } x>\epsilon.\end{cases}\tag{4.26}$$

Then we have for $I :=]E-\epsilon, E+\epsilon]$

$$0\leq\chi_I(\eta)\leq\rho(\eta-E+2\epsilon)-\rho(\eta-E-2\epsilon)\tag{4.27}$$

for all $\eta \in \mathbb{R}$. Thus we get by the spectral theorem (cf. [RS80, Rud91])

$$\begin{aligned}\mathrm{tr}\big\{\chi_I(\mathbb{H}^\Lambda(V))\big\} &\leq \sum_{n=1}^{2|\Lambda|}\big\{\rho(E_n(V)-E+2\epsilon)-\rho(E_n(V)-E-2\epsilon)\big\}\\ &= -\sum_{n=1}^{2|\Lambda|}\int_{E-2\epsilon}^{E+2\epsilon}\frac{\partial}{\partial\eta}\rho(E_n(V)-\eta)d\eta\\ &= \sum_{n=1}^{2|\Lambda|}\int_{E-2\epsilon}^{E+2\epsilon}\rho'(E_n(V)-\eta)d\eta. \end{aligned} \tag{4.28}$$

By the chain rule we get

$$\sum_{j\in\Lambda}\frac{\partial}{\partial V_j}\rho(E_n(V)-\eta) = \rho'(E_n(V)-\eta)\sum_{j\in\Lambda}\frac{\partial E_n(V)}{\partial V_j}. \tag{4.29}$$

Since we assumed $H^\Lambda(V) \geq \lambda > 0$ by condition *(i)*, we conclude from Lemma 4.26

$$\begin{aligned}\rho'(E_n(V)-\eta) &\leq \frac{E_n(V)}{\lambda}\sum_{j\in\Lambda}\frac{\partial}{\partial V_j}\rho(E_n(V)-\eta)\\ &\leq \frac{E+1}{\lambda}\sum_{j\in\Lambda}\frac{\partial}{\partial V_j}\rho(E_n(V)-\eta)\end{aligned} \tag{4.30}$$

for all $n \in \mathbb{N}$ and $\eta \in [E-2\epsilon, E+2\epsilon]$. Let us remark here, that to gain the last inequality we used that $\epsilon < \min\{\lambda, 1\}/3$ which guarantees us that only the $E_n(V) \in]0, E+1[$ contribute. Furthermore, we applied here Lemma 4.26 which guarantees us that the j-sum is, for those $E_n(V)$, positive.
Now let us average over the random variable $\{V_j\}_{j\in\Lambda}$ to get

$$\begin{aligned}&\mathbb{E}\big\{\mathrm{tr}\chi_I(\mathbb{H}^\Lambda)\big\}\\ &\quad\leq \frac{E+1}{\lambda}\sum_{j\in\Lambda}\int_{E-2\epsilon}^{E+2\epsilon}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{\partial}{\partial V_j}\sum_{n=1}^{2|\Lambda|}\rho(E_n(V)-\eta)d\mu_V(V_j)\right)\\ &\qquad\qquad\times\left(\prod_{\substack{k\in\Lambda\\ k\neq j}}d\mu_V(V_k)\right)d\eta\end{aligned} \tag{4.31}$$

Since the function

$$V_j \longmapsto F(V_j) := \sum_{n=1}^{2|\Lambda|}\rho(E_n(V)-\eta) = \mathrm{tr}\left\{\rho(\mathbb{H}^\Lambda(V)-\eta)\right\} \tag{4.32}$$

is non-monotonic in its argument for given $\eta \in \mathbb{R}$ and $V_k \in \mathbb{R}$, for $k \neq j$, we have to use a different argument than for the proof of the Wegner estimate in [Kir08]. Since we have, by Lemma 4.29, that for all $x, y \in \mathbb{R}$

$$|F(x) - F(y)| \leq 1 \tag{4.33}$$

holds, we can apply Lemma 4.27 to (4.31) to get

$$\mathbb{E}\{\mathrm{tr}\chi_I(\mathbb{H}^\Lambda)\} \leq 4\epsilon|\Lambda|\frac{E+1}{\lambda}\|\phi_V\|_{BV}. \tag{4.34}$$

□

Corollary 4.32. *Under the conditions of Theorem 4.24 we have that the integrated density of states $\mathbb{N}$ of $\mathbb{H}$ is Lipschitz continuous and has a bounded density*

$$\mathbb{D} := \frac{d\mathbb{N}}{dE}.$$

In particular, we have for Lebesgue-almost all $E \in \mathbb{R}$

$$\mathbb{D}(E) \leq \frac{|E|+1}{\lambda}\|\phi_V\|_{BV},$$

in case condition (i) holds, and

$$\mathbb{D}(E) \leq \frac{|E|+1}{\beta}\|\phi_b\|_{BV},$$

in case condition (ii) holds.

Proof. The assertion follows immediately from Theorem 4.24 by Lemma 4.21 and dominated convergence. □

4.6 Anderson localization for random block operators

In this section we will state the result of Anderson localization for discrete random block operators which was obtained by Gebert in [Geb11] using multiscale analysis. This result guarantees us that we can find a regime of $\mathbb{P}$-almost sure dense pure point spectrum near the spectral gap of the spectrum of $\mathbb{H}$. It is therefore of vital importance for the main result of this thesis, Theorem 5.7, where we will be analyzing the correlation between the eigenvalues in the localized regime.

Theorem 4.33. *Let $\mathbb{H}$ be the random block operator defined in (4.13) such that $\mathbb{H} \geq \lambda$, $\lambda \geq 0$ and the common distribution laws μ_V resp. μ_b of V resp. b fulfill the assumptions of Theorem 4.24. Then we have $\mathbb{P}$-almost surely that there exists an interval $I = [-a, a]$, $a > 0$, with $\sigma(\mathbb{H}) \cap I \neq \emptyset$ and*

$$\sigma(\mathbb{H}) \cap I = \sigma_{pp}(\mathbb{H}) \cap I.$$

More strongly, the corresponding eigenfunctions decay exponentially.

Proof. The proof of the statement can be found in [Geb11]. □

5 Eigenvalue statistics for random block operators

In this chapter we will state and prove a structural theorem (cf. Theorem 5.7) yielding the independence of the eigenvalues in the localized regime for random block operators provided that we have a Wegner and a Minami estimate and that we have the exponential decay of the fractional moments of the Green's function (similar to the work of Klopp in [Klo] for the random Schrödinger case).
To achieve this we, will adapt the proof of Minami's theorem on the stochastic independence of the eigenvalues of the Anderson model in the localized regime to random block operators.
We will also present first steps on adapting the method of Combes, Germinet and Klein given in [CGK09a], for proving a Minami estimate, to the case of discrete random block operators and give suggestions for possible future studies on how one might prove the other assumptions of our structural theorem (cf. Theorem 5.7).

5.1 Local Poisson structure of the spectrum of random block operators

Definition 5.1. Let $\mathbb{H}$ be the random block operator defined in (4.13) then we denote for $z \notin \sigma(\mathbb{H})$ the resolvent of $\mathbb{H}$ by

$$\mathbb{G}(z) := (\mathbb{H} - z)^{-1}$$

and for $\Lambda \subset \mathbb{Z}^d$ we define

$$\mathbb{G}^{\Lambda}(z) := (\mathbb{H}^{\Lambda} - z)^{-1}$$

and according with Definition 4.9 we define the *kernel* of $\mathbb{G}(z)$ by

$$\begin{aligned}\mathbb{G}(z;n,m) :&= (\mathbb{H} - z)^{-1}(n,m) \\ &= \begin{pmatrix} \langle\!\langle \delta_n \oplus 0, \mathbb{G}(z)\delta_m \oplus 0 \rangle\!\rangle & \langle\!\langle \delta_n \oplus 0, \mathbb{G}(z) 0 \oplus \delta_m \rangle\!\rangle \\ \langle\!\langle 0 \oplus \delta_n, \mathbb{G}(z)\delta_m \oplus 0 \rangle\!\rangle & \langle\!\langle 0 \oplus \delta_n, \mathbb{G}(z) 0 \oplus \delta_m \rangle\!\rangle \end{pmatrix}\end{aligned}$$

for all $n, m \in \mathbb{Z}^d$.

Remark 5.2. 1. The above definition was chosen thusly to be in accordance with the notations chosen in [Geb11].

2. Analogously to Proposition 2.14 we can prove a geometric resolvent equation for random block operators, more precisely:

Proposition 5.3 (Geometric resolvent equation for random block operators)**.** *Let* $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$*,* $x \in \Lambda_1$*,* $\mathbb{H}$ *be the random block operator defined in (4.13),* H^{Λ_i} *for* $i = 1, 2$ *be its restriction under simple boundary conditions and let* $z \notin \big(\sigma(\mathbb{H}^{\Lambda_1}) \cup \sigma(\mathbb{H}^{\Lambda_2}) \cup \sigma(\mathbb{H}^{\Lambda_2 \setminus \Lambda_1})\big)$*. Then we have*

$$\mathbb{G}^{\Lambda_2}(z; x, x) = \mathbb{G}^{\Lambda_1}(z; x, x) + \sum_{\substack{(y,y') \in \partial\Lambda_1 \\ y \in \Lambda_1, y' \in \Lambda_2}} \mathbb{G}^{\Lambda_1}(z; x, y) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{G}^{\Lambda_2}(z; y', x).$$

Proof. The proof proceeds as in the standard random Schrödinger case, see [Kir08, Sec. 5.3]. For an explicit derivation of this result we refer to [Geb11]. □

Definition 5.4. For a matrix $A \in \mathbb{C}^{2\times 2}$ we shall denote the maximum norm of A by

$$\|A\|_\infty := \left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\|_\infty := \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$$

and its trace by

$$\mathrm{tr}_{2\times 2} A = a_{11} + a_{22}.$$

Remark 5.5. (1) We are now going to show that we can adapt Minami's proof for Theorem 3.11 to the case of random block operators. However, since not all the necessary ingredients have yet been proven, we shall assume them and show that with this structure the adaption is possible. To make comparison with the original result [Min96] easier we will follow the ideas, notations and steps expressed by Minami in [Min96].

(2) As for proving the local Poisson structure of the localized regime in the spectrum of the Anderson model (cf. Section 3.2), we must rescale the eigenvalues of $\mathbb{H}^\Lambda$ according to their average spacing to reveal the local spectral structure. Since our Wegner estimate (Theorem 4.24) tells us that the average level spacing of the eigenvalues behaves as $(2|\Lambda|)^{-1}$ as $|\Lambda|$ gets large we shall consider the set $\{\Xi_n(\Lambda, E)\}_{1 \le n \le 2|\Lambda|}$ with $\Xi_n(\Lambda, E) := 2|\Lambda|(E_j(\Lambda) - E)$ where $E_j(\Lambda)$ for $1 \le j \le 2|\Lambda|$ denote the eigenvalues of $\mathbb{H}^\Lambda$ and $E \in \mathbb{R}$ lying in the localized regime of the spectrum of $\mathbb{H}$ is chosen such that the density of states $\mathbb{D}(E)$ exists at E an obeys $\mathbb{D}(E) > 0$.

Definition 5.6. Let $\mathbb{H}^\Lambda$ denote the restriction of $\mathbb{H}$ to the finite-volume hypercube $\Lambda \subset \mathbb{Z}^d$ under simple boundary conditions and let us denote the eigenvalues of $\mathbb{H}^\Lambda$ by

$$E_1 \le \dots \le E_{2|\Lambda|} \tag{5.1}$$

ordered by magnitude and repeated according to multiplicity. Let $E \in \mathbb{R}$ be in the localized regime of $\sigma(\mathbb{H})$ such that the density of states $\mathbb{D}(E)$ exists at E and is positive. Define the family of point processes $\{\Xi(\Lambda, E)\}_\Lambda$ by

$$\Xi(\Lambda, E) := \sum_{j=1}^{2|\Lambda|} \delta_{2|\Lambda|(E_j(\Lambda) - E)}.$$

Theorem 5.7. *Consider the random block operator defined in (4.13)*

$$\mathbb{H}: \ \omega \longmapsto \mathbb{H}_\omega = \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix}$$

Assume that at least one of the following two conditions is satisfied:

(i) there exists a $\lambda > 0$ such that $H \geq \lambda 1$ holds $\mathbb{P}$-almost surely and μ_V is absolutely continuous with a piecewise continuous Lebesgue density ϕ_V of bounded variation and compact support,

(ii) there exists a $\beta > 0$ such that $b \geq \beta 1$ holds $\mathbb{P}$-almost surely and μ_b is absolutely continuous with a piecewise continuous Lebesgue density ϕ_b of bounded variation and compact support.

Let $\mathbb{H}^\Lambda$ denote the restriction of $\mathbb{H}$ to the finite-volume hypercube $\Lambda \subset \mathbb{Z}^d$ under simple boundary conditions and suppose that

(1) we have a Wegner estimate, and in particular that there exists a constant $D > 0$ such that we have for all $x \in \Lambda$

$$\mathbb{E}\left[\|\Im \mathbb{G}^\Lambda(z;x,x)\|_\infty\right] \leq D, \tag{5.2}$$

(2) for $E \in \mathbb{R}$ in the localized regime of $\sigma(\mathbb{H})$, the density of states $\mathbb{D}(E)$ exists at E and is positive,

(3) the fractional moments of the Green's function are bounded and decay exponentially fast, i.e. that we have for all $z \in \mathbb{C}$, all $x, y \in \mathbb{Z}^d$, all $\Lambda \subset \mathbb{Z}^d$ and $0 < s < 1$ that there exists a constant $C(s) > 0$ such that

$$\mathbb{E}\left[\|\mathbb{G}^\Lambda(z;x,y)\|_\infty^s\right] \leq C(s), \tag{5.3}$$

and that there exists an $s \in]0,1[$, a $C(s) > 0$, an $m(s) > 0$ and a $r(s) > 0$ such that for all $\Lambda \subset \mathbb{Z}^d$

$$\mathbb{E}\left[\|\mathbb{G}^\Lambda(z;x,y)\|_\infty^s\right] \leq C(s)\mathrm{e}^{-m(s)|x-y|_1} \tag{5.4}$$

with $x \in \Lambda$ and $y \in \partial\Lambda$ and $z \in \{z \in \mathbb{C} \ : \ \Im z > 0, \ |z - E| < r(s)\}$

(4) we have a Minami estimate, i.e. there exist constants $C > 0$ and $\rho > 0$ such that

$$\mathbb{E}\left[\mathrm{tr}\ \chi_I(\mathbb{H}_\omega^\Lambda)\left\{\mathrm{tr}\ \chi_I(\mathbb{H}_\omega^\Lambda) - 1\right\}\right] \leq C\left(|I|\ |\Lambda|\right)^{1+\rho} \tag{5.5}$$

holds for all bounded intervals I.

Then the point process $\Xi(\Lambda, E)$ converges weakly, as $|\Lambda| \to \infty$, to the Poisson point process Ξ with intensity measure $\mathbb{D}(E)dx$.

Remark 5.8. Similarly to the proof of the Wegner estimate (cf. Theorem 4.24) it is enough to prove Theorem assuming either *(i)* or *(ii)*. We shall choose to prove it in the case that *(i)* holds true. Furthermore, due to the symmetry of the spectrum of $\mathbb{H}$ (cf. Lemma 4.3) it is enough to consider only positive energies, i.e. $E \geq 0$.

Proof of Theorem 5.7.

Step 1.

Remark 5.9. A key aspect of Minami's proof is the that there exist many methods to properly deal with resolvents of operators. Therefore, we will prove in the first step that for weak convergence to hold (cf. Lemma 3.7) we don't need to consider the whole space $\mathcal{C}_c^+(\mathbb{R})$ but that it is enough to consider a certain kind of test function (cf. Definition 5.10) that will later on deliver us resolvents instead of arbitrary operator-valued functions.

Definition 5.10. Define $\mathcal{A}$ to be the class of functions $f : \mathbb{R} \to \mathbb{R}$ of the from

$$f(x) := \sum_{j=1}^{n} \frac{a_j \tau}{(x - \sigma_j)^2 + \tau^2}$$

with $n \geq 1$, $\tau > 0$ and $a_j > 0$,$\sigma_j \in \mathbb{R}$ for $1 \leq j \leq n$.

Lemma 5.11. *Let $|A|$ denote the Lebesgue measure of any real Borel set A. Let Ξ and Ξ_n, $n \in \mathbb{N}$, be point processes on $\mathbb{R}$ with intensity measures μ and μ_n, respectively. Assume that there exists a constant $c > 0$ such that for all real Borel sets A*

$$\mu_n(A) \leq c|A|, \quad \text{for } n \geq 1, \quad \text{and} \quad \mu(A) \leq c|A|$$

holds. Then the following two statements are equivalent:

i) Ξ_n *converges weakly to* Ξ *as* $n \to \infty$

ii) for all $f \in \mathcal{A}$

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left\{-\Xi_n(f)\right\}\right] = \mathbb{E}\left[\exp\left\{-\Xi(f)\right\}\right]$$

holds.

Proof. The proof of this statement uses standard density and approximation arguments and can be found in [Min96]. □

Step 2.

Remark 5.12. In this step we will check whether the point processes defined in Definition 5.6 as well as the Poisson point process on $\mathbb{R}$, Ξ, with intensity measure $\mu(dx) = \mathbb{D}(E)dx$ satisfy the conditions of Lemma 5.11.

Therefore, we define

$$f_\zeta(x) := \frac{\tau}{(x-\sigma)^2+\tau^2} \tag{5.6}$$

for any $\zeta := \sigma + i\tau \in \mathbb{C}_+$, i.e. $\tau > 0$. Then we get

$$\begin{aligned}\mathbb{E}\left[\Xi(\Lambda;E)(f_\zeta)\right] &= \mathbb{E}\left[\sum_{j=1}^{2|\Lambda|} \frac{\tau}{(2|\Lambda|(E_j(\Lambda)-E)-\sigma)^2+\tau^2}\right] \\ &= \frac{1}{2|\Lambda|}\mathbb{E}\left[\sum_{j=1}^{2|\Lambda|} \frac{(2|\Lambda|)^{-1}\tau}{(E_j(\Lambda)-E-(2|\Lambda|)^{-1}\sigma)^2+((2|\Lambda|)^{-1}\tau)^2}\right] \\ &= \frac{1}{2|\Lambda|}\mathbb{E}\left[\mathrm{tr}\Im(\mathbb{H}^\Lambda - E - (2|\Lambda|)^{-1}\zeta)^{-1}\right]. \end{aligned} \tag{5.7}$$

We know by the Wegner estimate (Theorem 4.24) that for all bounded intervals $I \subset \mathbb{R}$

$$\mathbb{E}\left\{\mathrm{tr}\chi_I(\mathbb{H}^\Lambda)\right\} \leq 2|I|\ |\Lambda|\frac{|E|+1}{\lambda}\|\phi_V\|_{BV} \tag{5.8}$$

holds. Hence, we define the real valued Borel measure

$$\nu(I) := \mathbb{E}\left\{\mathrm{tr}\chi_I(\mathbb{H}^\Lambda)\right\} \tag{5.9}$$

for all Borel sets $I \subset \mathbb{R}$. Define $z := E + (2|\Lambda|)^{-1}\zeta \in \mathbb{C}_+$, then we get by the Stieltjes transformation (cf. [PF92, Appendix A])

$$\begin{aligned}\mathbb{E}\left[\mathrm{tr}\Im(\mathbb{H}^\Lambda - z)^{-1}\right] &= \int_{\mathbb{R}} \frac{\Im z}{(\Re z - t)^2 + (\Im z)^2} d\nu(t) \\ &\leq 2|\Lambda|\frac{|E|+1}{\lambda}\|\phi_V\|_{BV}\int_{\mathbb{R}} \frac{\Im z}{(\Re z - t)^2 + (\Im z)^2} dt \\ &= 2|\Lambda|\frac{|E|+1}{\lambda}\|\phi_V\|_{BV}\pi. \end{aligned} \tag{5.10}$$

Thus we get

$$\begin{aligned}\mathbb{E}\left[\Xi(\Lambda;E)(f_\zeta)\right] &= \frac{1}{2|\Lambda|}\mathbb{E}\left[\mathrm{tr}\Im(\mathbb{H}^\Lambda - E - (2|\Lambda|)^{-1}\zeta)^{-1}\right] \\ &\leq \frac{|E|+1}{\lambda}\|\phi_V\|_{BV}\int_{\mathbb{R}} f_\zeta(x)dx. \end{aligned} \tag{5.11}$$

Hence, we can conclude from the Stieltjes–Perron inversion formula (see [PF92, Appendix A]) that

$$\mathbb{E}\left[\Xi(\Lambda;E)(dx)\right] \leq \frac{|E|+1}{\lambda}\|\phi_V\|_{BV}dx. \tag{5.12}$$

Since by Corollary 4.32 we have

$$\mathbb{D}(E) \leq \frac{|E|+1}{\lambda}\|\phi_V\|_{BV} \tag{5.13}$$

the conditions of Lemma 5.11 are satisfied and thus, in order to prove Theorem 5.7, it is enough to prove the following:

Let $n \in \mathbb{N}$ and $1 \leq j \leq n$, then for all $\tau > 0$, $a_j > 0$, $\sigma_j \in \mathbb{R}$ and $\zeta_j := \sigma_j + i\tau$

$$\lim_{|\Lambda|\to\infty} \mathbb{E}\left[\exp\left\{-\frac{1}{2|\Lambda|}\sum_{j=1}^{n} a_j \Im \mathrm{tr}\mathbb{G}^{\Lambda}(E + (2|\Lambda|)^{-1}\zeta_j)\right\}\right] = \mathcal{L}_P(\phi), \tag{5.14}$$

where we set

$$\phi(x) := \sum_{j=1}^{n} a_j \frac{\tau}{(x-\sigma_j)^2+\tau^2} \in \mathcal{A} \tag{5.15}$$

and $\mathcal{L}_P(\phi)$ denotes

$$\mathcal{L}_P(\phi) = \mathbb{E}\left[\exp\left\{-\Xi(\phi)\right\}\right] \tag{5.16}$$

where Ξ is the Poisson point process with intensity measure $\mathbb{D}(E)dx$.

Step 3.

Remark 5.13. The key idea in this step is to break down our big cube Λ into smaller cubes C_p and prove that we can approximate the point process

$$\Xi(\Lambda, E) := \sum_{j=1}^{2|\Lambda|} \delta_{2|\Lambda|(E_j(\Lambda)-E)} \tag{5.17}$$

asymptotically as $|\Lambda|$ gets large by the point process

$$\eta(\Lambda, E) := \sum_{p} \eta(C_p, E) := \sum_{p}\sum_{j=1}^{2|C_p|} \delta_{2|\Lambda|(E_j(C_p)-E)} \tag{5.18}$$

where $\{E_j(C_p)\}_{1\leq j\leq 2|C_p|}$ denote the eigenvalues of $\mathbb{H}^{C_p}$ (with simple boundary conditions).

With this aim in mind, let us define two diverging integer valued sequences $(N_L)_{L\in\mathbb{N}}$ and $(l_L)_{L\in\mathbb{N}}$ which behave for $L \to \infty$ as following

$$N_L = o(L) \quad \text{and} \quad l_L = o(L/N_L). \tag{5.19}$$

The exact choice of N_L and l_L will be apparent later. Now let us divide $]-L, L]^d \subset \mathbb{R}^d$ into N_L^d equal cubes D_p, for $p = 1, \ldots, N_L^d$, with the side length $(2L+1)/N_L$ and of the form $]a, b]^d$. Furthermore, define

$$C_p := D_P \cap \mathbb{Z}^d \tag{5.20}$$

and

$$\mathrm{int}(C_p) := \{x \in C_p \ : \ \mathrm{dist}(x, \partial C_p) > l_L\}. \tag{5.21}$$

As our next step we are going to plug this construction into the geometric resolvent equation (cf. Proposition 5.3) and receive the following perturbation formula for $z \in \mathbb{C}$ and $x \in \operatorname{int}(C_p)$

$$\mathbb{G}^{\Lambda}(z;x,x) = \mathbb{G}^{C_p}(z;x,x) + \sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathbb{G}^{C_p}(z;x,y)\begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}\mathbb{G}^{\Lambda}(z;y',x). \tag{5.22}$$

Recall the notations regarding the 2×2 matrix structure of $\mathbb{G}^{\Lambda}(z;x,x)$ introduced in Definition 5.4. Hence, we can infer

$$\left|\operatorname{tr}_{2\times 2}\Im\mathbb{G}^{\Lambda}(z;x,x)\right| \leq 2\|\Im\mathbb{G}^{\Lambda}(z;x,x)\|_{\infty}. \tag{5.23}$$

By the geometric resolvent equation (5.22) we then get

$$\begin{aligned}
&\left|\frac{1}{2|\Lambda|}\Im\operatorname{tr}\mathbb{G}^{\Lambda}(z) - \frac{1}{2|\Lambda|}\sum_p \Im\operatorname{tr}\mathbb{G}^{C_p}(z)\right| \\
&\qquad = \left|\frac{1}{2|\Lambda|}\sum_p\sum_{x\in C_p}\left\{\operatorname{tr}_{2\times 2}\Im\mathbb{G}^{\Lambda}(z;x,x) - \operatorname{tr}_{2\times 2}\Im\mathbb{G}^{C_p}(z;x,x)\right\}\right| \\
&\qquad \leq \frac{1}{|\Lambda|}\sum_p\sum_{x\in C_p\setminus\operatorname{int}(C_p)}\left\{\|\Im\mathbb{G}^{\Lambda}(z;x,x)\|_{\infty} + \|\Im\mathbb{G}^{C_p}(z;x,x)\|_{\infty}\right\} \\
&\qquad + \frac{1}{|\Lambda|}\sum_p\sum_{x\in\operatorname{int}(C_p)}\sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \|\mathbb{G}^{C_p}(z;x,y)\|_{\infty}\|\mathbb{G}^{\Lambda}(z;y',x)\|_{\infty} \\
&\qquad =: A_L + B_L
\end{aligned} \tag{5.24}$$

We want to show that $\mathbb{E}[A_L] \to 0$ and $\mathbb{E}[B_L] \to 0$ as $L \to \infty$. Since, by assumption 1,

$$\mathbb{E}[\|\Im\mathbb{G}^{\Lambda}(z;x,x)\|_{\infty}] \quad \text{and} \quad \mathbb{E}[\|\Im\mathbb{G}^{C_p}(z;x,x)\|_{\infty}] \tag{5.25}$$

and are both bounded by a constant independent of z, Λ, p and x we can conclude that for $L \to \infty$

$$\mathbb{E}[A_L] = \mathcal{O}\left((2L+1)^{-d}N_L^d((2L+1)/N_L)^{d-1}l_L\right) = \mathcal{O}(N_L L^{-1} l_l). \tag{5.26}$$

Before we can estimate B_L, let us consider $\mathbb{E}[B_L^{s/2}]$ for an $s \in]0,1[$. By the Hölder inequality we can then conclude

$$\begin{aligned}
\mathbb{E}[B_L^{s/2}] &\leq \frac{1}{|\Lambda|^{s/2}}\sum_p\sum_{x\in\operatorname{int}(C_p)}\sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathbb{E}\left[\|\mathbb{G}^{C_p}(z;x,y)\|_{\infty}^{s/2}\|\mathbb{G}^{\Lambda}(z;y',x)\|_{\infty}^{s/2}\right] \\
&\leq \frac{1}{|\Lambda|^{s/2}}\sum_p\sum_{x\in\operatorname{int}(C_p)}\sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \sqrt{\mathbb{E}\left[\|\mathbb{G}^{C_p}(z;x,y)\|_{\infty}^{s}\right]}\sqrt{\mathbb{E}\left[\|\mathbb{G}^{\Lambda}(z;y',x)\|_{\infty}^{s}\right]}.
\end{aligned} \tag{5.27}$$

The term $\mathbb{E}[\|\mathbb{G}^{\Lambda}(z;y',x)\|_{\infty}^{s}]$ is bounded by a constant $C^s < \infty$ independently of Λ, $z \in \mathbb{C}_+$, and $x, y' \in \Lambda$ by assumption (3). Since the second term $\mathbb{E}[\|\mathbb{G}^{C_p}(z;x,y)\|^s]$ is bounded exponentially by assumption (3) we can conclude

$$\begin{aligned}
\mathbb{E}[B_L^{s/2}] &\leq \frac{\sqrt{C^s}}{|\Lambda|^{s/2}} \sum_p \sum_{x \in \operatorname{int}(C_p)} \sum_{\substack{(y,y') \in \partial C_p \\ y \in C_p, y' \in \mathbb{Z}^d \setminus C_p}} \mathrm{e}^{-m(s)|x-y|_1} \\
&= \mathcal{O}\left((2L+1)^{-s/2} N_L^d \left(\frac{2L+1}{N_L}\right)^{d-1} \left(\frac{2L+1}{N_L} - l_L\right) \left(\frac{2L+1}{N_L}\right)^{d-1} \mathrm{e}^{-(m(s)/2) l_L} \right) \\
&= \mathcal{O}(L^{d(2-s/2)-1} N_L^{1-d} l_L \mathrm{e}^{-(m(s)/2) l_L}). \qquad (5.28)
\end{aligned}$$

If we now choose $N_L := (2L+1)^{\alpha}$ for $0 < \alpha < 1$ and $l_L := \beta \ln L$ for a β large enough we get that $\mathbb{E}[B_L^{s/2}] = o(1)$ for $L \to \infty$. Indeed, if we plug these definitions into (5.28) we get

$$\mathbb{E}[B_L^{s/2}] = \mathcal{O}(L^{d(2-s/2)-1+\alpha(1-d)-\beta(m(s)/2)} \ln L). \qquad (5.29)$$

Thus we have for

$$\beta > \frac{2}{m(s)} \left(d\left(2 - \frac{s}{2}\right) - 1 + \alpha(1-d) \right) \qquad (5.30)$$

that $\mathbb{E}[B_L^{s/2}] = o(1)$ for $L \to \infty$ and thus

$$\mathbb{E}[B_L] = o(1) \quad \text{and} \quad \mathbb{E}[A_L] = \mathcal{O}(L^{\alpha-1} \ln L) = o(1) \quad \text{for } L \to \infty. \qquad (5.31)$$

Therefore, with the definitions given in Remark 5.13, we can conclude that

$$\lim_{|\Lambda| \to \infty} \Big| \mathbb{E}\left[\exp\{-\Xi(\Lambda, E)(\phi)\}\right] - \mathbb{E}\left[\exp\{-\eta(\Lambda, E)(\phi)\}\right] \Big| = 0, \qquad (5.32)$$

where we have set $\phi \in \mathcal{A}$ as in (5.15). Therefore, it is enough to prove

$$\lim_{|\Lambda| \to \infty} \mathbb{E}\left[\exp\left\{ -\frac{1}{2|\Lambda|} \sum_p \sum_{j=1}^{n} a_j \Im \operatorname{tr} G^{C_p}(E + (2|\Lambda|)^{-1}\zeta_j) \right\} \right] = \mathcal{L}_P(\phi). \qquad (5.33)$$

To conclude the theorem from proving (5.33) we have to convince ourselves at this point that the conditions of Lemma 5.11 are still satisfied for the point process $\eta(\Lambda, E)$. By performing the exact same procedure as in *Step 2* for the point process $\eta(C_p, E)(dx)$ we can get

$$\mathbb{E}\left[\eta(C_p; E)(dx)\right] \leq \left(\frac{E+1}{\lambda} \|\phi_V\|_{BV} \right) \frac{|C_p|}{|\Lambda|} dx \leq \left(\frac{E+1}{\lambda} \|\phi_V\|_{BV} \right) N_L^{-d} dx, \qquad (5.34)$$

and thus

$$\mathbb{E}\left[\eta(\Lambda; E)(dx)\right] \leq \left(\frac{E+1}{\lambda} \|\phi_V\|_{BV} \right) dx. \qquad (5.35)$$

Hence, by Lemma 5.11, we can conclude the theorem from the following:

Proposition 5.14. *The point process $\eta(\Lambda, E)$, defined in Remark 5.13, converges weakly, for $|\Lambda| \to \infty$, to the Poisson point process Ξ with intensity measure $\mathbb{D}(E)dx$.*

Step 4.

Remark 5.15. In the next two steps we are going to prove Proposition 5.14 by using the results from Lemma 5.18 and the assumption (4).

Definition 5.16. (Uniformly asymptotically negligible array) Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a family of point processes such that each ξ_n is formed by a superposition of other point processes θ_i^n for $1 \leq i \leq m_n$ with $m_n \in \mathbb{N}$, i.e.

$$\xi_n = \sum_{i=1}^{m_n} \theta_i^n.$$

If for all $n \in \mathbb{N}$ the families $\{\theta_i^n :\ 1 \leq i \leq m_n\}$ are mutually independent and if for all bounded Borel sets A

$$\lim_{n\to\infty} \sup_{1\leq i\leq m_n} \mathbb{P}\left[\theta_i^n(A) > 0\right] = 0$$

holds, we shall call them *uniformly asymptotically negligible array.*

Remark 5.17. More details about uniformly asymptotically negligible arrays can be found in [DVJ08, Sec. 11.2], indeed the statement of the following lemma is a special case of Theorem 11.2.V in [DVJ08, Sec. 11.2].

Lemma 5.18. *The superposition of the point processes $\eta(C_p, E)$ converges weakly to the Poisson point process Ξ with intensity measure $\mathbb{D}(E)dx$ if for all bounded Borel sets A*

$$\sum_p \mathbb{P}\left[\eta(C_p, E)(A) \geq 1\right] \longrightarrow \mathbb{D}(E)|A|, \tag{5.36}$$

$$\sum_p \mathbb{P}\left[\eta(C_p, E)(A) \geq 2\right] \longrightarrow 0 \tag{5.37}$$

holds for $|\Lambda| \to \infty$.

Proof. Since the point processes $\eta(C_p, E)$ are independent for all p and, since $\eta(C_p, E)(A)$ is integer valued for all bounded real Borel sets A, (5.34) and the Markov inequality imply that we have for all bounded Borel sets A

$$\lim_{L\to\infty} \sup_{1\leq p\leq N_L^d} \mathbb{P}\left[\eta(C_p, E)(A) > 0\right] = 0.$$

Hence, the $\eta(C_p, E)$ indeed form a uniformly asymptotically negligible array and thus the statement follows from Theorem 11.2.V in [DVJ08]. □

However, to conclude to theorem, we still have to verify the conditions (5.36) and (5.37). By assumption (4) we get for all finite intervals A

$$\begin{aligned}\sum_{j\geq 2}\mathbb{P}\left[\eta(C_p,E)(A)\geq j\right] &= \sum_{j\geq 2}(j-1)\mathbb{P}\left[\eta(C_p,E)(A)=j\right]\\ &\leq \sum_{j\geq 2}j(j-1)\mathbb{P}\left[\eta(C_p,E)(A)=j\right]\\ &= \mathbb{E}\left[\eta(C_p,E)(A)\left\{\eta(C_p,E)(A)-1\right\}\right]\\ &= \mathbb{E}\left[\mathrm{tr}\;\chi_{\tilde{A}}(\mathbb{H}^{C_p})\left\{\mathrm{tr}\;\chi_{\tilde{A}}(\mathbb{H}^{C_p})-1\right\}\right]\\ &\leq C\frac{|A|^2}{|\Lambda|^2}|C_p|^2 = o(N_L^{-d}),\end{aligned} \tag{5.38}$$

where $\tilde{A} := |\Lambda|^{-1} - E$, which implies condition (5.37). Furthermore, this yields

$$\begin{aligned}\mathbb{P}\left[\eta(C_p,E)(A)\geq 1\right] &= \mathbb{E}\left[\eta(C_p,E)(A)\right] - \sum_{j\geq 2}\mathbb{P}\left[\eta(C_p,E)(A)\geq j\right]\\ &= \mathbb{E}\left[\eta(C_p,E)(A)\right] + o(N_L^{-d}).\end{aligned} \tag{5.39}$$

Therefore, it is enough to prove

$$N_L^d\mathbb{E}\left[\eta(C_p,E)(A)\right] \longrightarrow \mathbb{D}(E)|A| \quad \text{for } L\to\infty \tag{5.40}$$

and for this purpose it is enough, by the previous construction, to prove for all $\zeta = \sigma + i\tau \in \mathbb{C}_+$

$$N_L^d\mathbb{E}\left[\eta(C_p,E)(f_\zeta)\right] \longrightarrow \mathbb{D}(E)\pi \quad \text{for } L\to\infty, \tag{5.41}$$

which we will prove in the following step.

Step 5. As in *Step 3*, we have for $\lambda := E + (2|\Lambda|)^{-1}\zeta$

$$\begin{aligned}\mathbb{E}\left[\eta(C_p,E)(f_\zeta)\right] &= \frac{1}{2|\Lambda|}\mathbb{E}\left[\Im\;\mathrm{tr}\mathbb{G}^{C_p}(\lambda)\right]\\ = \frac{1}{2|\Lambda|}\mathbb{E}&\left[\sum_{x\in\mathrm{int}(C_p)}\mathrm{tr}_{2\times 2}\Im\mathbb{G}^{C_p}(\lambda;x,x) + \sum_{x\in C_p\setminus\mathrm{int}(C_p)}\mathrm{tr}_{2\times 2}\Im\mathbb{G}^{C_p}(\lambda;x,x)\right].\end{aligned} \tag{5.42}$$

First we are going to perform an analogous argument as for the estimate of $\mathbb{E}[A_L]$ in *Step 3*. Let us therefore choose N_L and l_L as in *Step 3*. We can then conclude that the part of (5.42) where we sum over all $x \in C_p\setminus\mathrm{int}(C_p)$ is of order

$$\mathcal{O}\left((2L+1)^{-d}\left(\frac{2L+1}{N_L}\right)^{d-1} l_L\right) = \mathcal{O}(L^{\alpha(1-d)-1}\ln L) = o(N_L^{-d}). \tag{5.43}$$

On the other hand we have, by the geometric resolvent equation, for $x \in \mathrm{int}(C_P)$ and $z \in \mathbb{C}_+$

$$\mathbb{G}^{C_p}(z;x,x) = \mathbb{G}(z;x,x) - \sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathbb{G}^{C_p}(z;x,y)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathbb{G}(z;y',x) \tag{5.44}$$

and in particular

$$\begin{aligned}\mathrm{tr}_{2\times 2}\mathbb{G}^{C_p}(z;x,x) =& \mathrm{tr}_{2\times 2}\mathbb{G}(z;x,x)\\ &- \sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathrm{tr}_{2\times 2}\left[\mathbb{G}^{C_p}(z;x,y)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathbb{G}(z;y',x)\right].\end{aligned} \tag{5.45}$$

By the definition of λ we see that $\Im\lambda = (2|\Lambda|)^{-1}\tau$. Hence we can deduce, by using the standard bound on resolvents, that

$$\|\mathbb{G}^{C_p}(z;x,x)\|_\infty \leq \frac{4}{\Im z} = \frac{4|\Lambda|}{\tau} \quad \text{and} \quad \|\mathbb{G}(z;x,x)\|_\infty \leq \frac{4}{\Im z} = \frac{4|\Lambda|}{\tau}. \tag{5.46}$$

Let us now recall that we have exponential decay of the fractional moments of the Green's function by assumption (3) and that they are bounded by a constant independent of the points in $\mathbb{Z}^d$ as well as of $z \in \mathbb{C}_+$. Thus we can conclude for $s \in]0,1[$

$$\begin{aligned}&\Big|\mathbb{E}[\mathrm{tr}_{2\times 2}\Im\mathbb{G}^{C_p}(\lambda;x,x)] - \mathbb{E}[\mathrm{tr}_{2\times 2}\Im\mathbb{G}(\lambda;x,x)]\Big|\\
&\quad\leq \sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathbb{E}\left\{\left|\mathrm{tr}_{2\times 2}\left[\mathbb{G}^{C_p}(z;x,y)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathbb{G}(z;y',x)\right]\right|\right\}\\
&\quad\leq 2\sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathbb{E}\Big\{\|\mathbb{G}^{C_p}(z;x,y)\|_\infty\|\mathbb{G}(z;y',x)\|_\infty\Big\}\\
&\quad\leq 2\left(\frac{|\Lambda|}{\tau}\right)^{2-s}\sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \mathbb{E}\Big\{\|\mathbb{G}^{C_p}(z;x,y)\|_\infty^{s/2}\|\mathbb{G}(z;y',x)\|_\infty^{s/2}\Big\}\\
&\quad\leq 2\left(\frac{|\Lambda|}{\tau}\right)^{2-s}\sum_{\substack{(y,y')\in\partial C_p \\ y\in C_p, y'\in\mathbb{Z}^d\setminus C_p}} \sqrt{\mathbb{E}\Big\{\|\mathbb{G}^{C_p}(z;x,y)\|_\infty^{s}\Big\}}\sqrt{\mathbb{E}\Big\{\|\mathbb{G}(z;y',x)\|_\infty^{s}\Big\}}\\
&\quad= \mathcal{O}\left((2L+1)^{d(2-s)}\left(\frac{2L+1}{N_L}\right)^{d-1}\mathrm{e}^{-(m(s)/2)l_L}\right)\\
&\quad= \mathcal{O}\Big(L^{d(3-s)-1}N_L^{1-d}\mathrm{e}^{-(m(s)/2)l_L}\Big).\end{aligned} \tag{5.47}$$

If we now choose $N_L = (2L+1)^\alpha$ with $0 < \alpha < 1$ as before and $l_L = \gamma \ln L$ with γ large enough, say

$$\gamma > \frac{2}{m(s)}\Big\{(3-s)d - 1 + \alpha(d-1)\Big\}, \tag{5.48}$$

we have

$$\mathbb{E}[\mathrm{tr}_{2\times 2}\Im \mathbb{G}^{C_p}(\lambda; x, x)] = \mathbb{E}[\mathrm{tr}_{2\times 2}\Im \mathbb{G}(\lambda; x, x)] + o(1), \quad \text{as } L \to \infty. \tag{5.49}$$

In particular the term $o(1)$ in the expression above is uniform in $x \in \mathbb{Z}^d$. By the Stieltjes transformation applied to the integrated density of states $\mathbb{N}$ (cf. [PF92, Appendix A]), we have for all $x \in \mathbb{Z}^d$

$$\mathbb{E}[\mathrm{tr}_{2\times 2}\Im \mathbb{G}(\lambda; x, x)] = 2\Im \int_{\mathbb{R}} \frac{1}{t-\lambda} d\mathbb{N}(t). \tag{5.50}$$

However, we assumed that $\mathbb{D}(E)$ exists at E which implies, in particular, that $\mathbb{N}(t)$ is differentiable at $t = E$ with Lebesgue derivative $\mathbb{D}(E)$. Since for $L \to \infty$ we have that $\lambda \to E$ from above which implies by the Sokhotski-Plemelj formula [PF92, Appendix A, A.6] that

$$\Im \int_{\mathbb{R}} \frac{1}{t-\lambda} d\mathbb{N}(t) \longrightarrow \pi \left.\frac{d\mathbb{N}(t)}{dt}\right|_{t=E} = \pi \mathbb{D}(E) \tag{5.51}$$

as $L \to \infty$. Thus we can infer

$$\begin{aligned}\mathbb{E}\left[\eta(C_p, E)(f_\zeta)\right] &= \frac{|\mathrm{int} C_p|}{2|\Lambda|} 2\Im \int_{\mathbb{R}} \frac{1}{t-\lambda} d\mathbb{N}(t) + o(N_L^{-d}) \\ &= \pi N_L^{-d} \Im \int_{\mathbb{R}} \frac{1}{t-\lambda} d\mathbb{N}(t) + o(N_L^{-d})\end{aligned} \tag{5.52}$$

Hence

$$N_L^d \mathbb{E}\left[\eta(C_p, E)(f_\zeta)\right] \longrightarrow \pi \mathbb{D}(E), \quad \text{as } L \to \infty, \tag{5.53}$$

which finishes the proof. □

5.2 Minami estimate for random block operators

The original method of Minami (cf. [Min96]) for proving a Minami estimate for the Anderson model involves a rank-2-perturbation. However, if we try this approach for random block operators of the form (4.13) we face serious difficulties due to the $-H$ in the block structure. It is at this point unclear whether this approach would yield anything at all.

Therefore, it seems reasonable to try to adapt the method of [CGK09a] (cf. Section 3.3) to random block operators. This approach consists of two main parts: a Wegner estimate and an estimate that allows us to factorize the expectation over the random variables (cf. Corollary 3.17) by making one of the factors in (4) independent of one specific random variable. In this section we aim to gain a similar factorization formula for random block operator.

Remark 5.19. The first ingredient we need is that the eigenvalues of $\mathbb{H}$ are $\mathbb{P}$-almost surely of multiplicity one. Unfortunately we do not have a Minami estimate for random block operators, and therefore cannot use the strategy proposed in [KM06] which would allow us to conclude simplicity of the eigenvalues from a Minami estimate. However, in [Geb11, Appendix A] we can find a different approach which does not rely on a Minami estimate. We will state it and present the proof for the readers' convenience in the following while sticking closely to the ideas and notations found in [Geb11, Appendix A].

Theorem 5.20. *Let $\mathbb{H}^\Lambda$ be the restriction of a random block operator as defined in (4.13) to $\Lambda \subset \mathbb{Z}^d$ under any kind of boundary conditions as defined in Section 4.4, then we have*

$$\mathbb{P}(\mathbb{H}^\Lambda \textit{ has a degenerate eigenvalue}) = 0.$$

Proof. The argument can be found in [Geb11, Appendix A] and is stated, for the readers' convenience, in Appendix A.2. □

With this we can conclude a more exact version of Corollary 4.5

Corollary 5.21. *The spectrum of the operator $\mathbb{H}^2$ has multiplicity 2 with a possible exception at 0.*

Definition 5.22. Let $\mathbb{H}$ be the random block operator defined in (4.13) and let $\mathbb{H}^\Lambda$ denote its restriction to the finite-volume hypercube $\Lambda \subset \mathbb{Z}^d$ under either Dirichlet, Neumann or simple boundary conditions. Then we denote by

$$\mathbb{H}^{\Lambda, D_j}$$

for all $j \in \mathbb{Z}^d$ the operator $\mathbb{H}^\Lambda$ with additional Dirichlet boundary conditions in the points $\delta_j \oplus 0$ and $0 \oplus \delta_j$, i.e. for all $\Psi \in \mathrm{dom}(\mathbb{H}^{\Lambda, D_j})$ we have $\Psi \perp 0 \oplus \delta_j$ and $\Psi \perp \delta_j \oplus 0$.

Remark 5.23. As the spectrum of $\mathbb{H}$ is symmetric around 0 (cf. Lemma 4.3) it suffices for a Minami estimate to consider only positive energies, i.e. $E \geq 0$.

Proposition 5.24. *Let $\mathbb{H}$ be the random self-adjoint block operator on the Hilbert space $\mathcal{H}^2$ defined in (4.13) and let $\mathbb{H}^\Lambda$ denote its restriction to the finite-volume hypercube $\Lambda \subset \mathbb{Z}^d$ under either Dirichlet, Neumann or simple boundary conditions. Then we have for all $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$ and all $j \in \Lambda$*

$$\mathrm{tr}\chi_{[a,b[}(\mathbb{H}^\Lambda) \leq 1 + \mathrm{tr}\chi_{[a,b[}(\mathbb{H}^{\Lambda, D_j}).$$

Proof. Let us denote the eigenvalues of $\mathbb{H}^\Lambda$ by

$$E_1 < \cdots < E_{2|\Lambda|} \tag{5.54}$$

ordered by magnitude (the multiplicity of the eigenvalues is, however, 1 due to the above) and let

$$\lambda_1 \leq \cdots \leq \lambda_{2|\Lambda|} \tag{5.55}$$

denote the eigenvalues of $(\mathbb{H}^\Lambda)^2$ ordered by magnitude including multiplicity which is exactly 2 as follows from Corollary 5.21. In particular we have

$$\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 < \cdots < \lambda_{2|\Lambda|-1} = \lambda_{2|\Lambda|}. \tag{5.56}$$

Fix $j \in \Lambda$ and let analogously

$$E_1^j < \cdots < E_{2(|\Lambda|-1)}^j \tag{5.57}$$

denote the eigenvalues of $\mathbb{H}^{\Lambda,D_j}$ and

$$\lambda_1^j \leq \cdots \leq \lambda_{2(|\Lambda|-1)}^j \tag{5.58}$$

the eigenvalues of $(\mathbb{H}^{\Lambda,D_j})^2$ ordered by magnitude including multiplicity which is again exactly 2 as follows from Corollary 5.21. In particular we have

$$\lambda_1^j = \lambda_2^j < \lambda_3^j = \lambda_4^j < \cdots < \lambda_{2|\Lambda|-3}^j = \lambda_{2|\Lambda|-2}^j. \tag{5.59}$$

Since we have for all $j \in \Lambda$

$$(\mathbb{H}^{\Lambda,D_j})^2 = \big((\mathbb{H}^\Lambda)^2\big)^{D_j}, \tag{5.60}$$

if follows again by the symmetry of the spectrum and by Corollary 5.21 that we have the following identifications

$$E_k = -\sqrt{\lambda_{2k}} \quad \text{and } E_{|\Lambda|+k} = \sqrt{\lambda_{2k-1}} \quad \text{for } 1 \leq k \leq |\Lambda| \tag{5.61}$$

and

$$E_k^j = \sqrt{\lambda_{2k}^j} \quad \text{and } E_{|\Lambda|+k}^j = \sqrt{\lambda_{2k-1}^j} \quad \text{for } 1 \leq k \leq (|\Lambda|-1). \tag{5.62}$$

Since $\mathrm{dom}(\mathbb{H}^{\Lambda,D_j}) \subset \mathrm{dom}(\mathbb{H}^\Lambda)$ we get by the min-max principle (cf. [Kir08, RS78])

$$\begin{aligned}
\lambda_{2k-1} &= \sup_{\Psi_1,\ldots,\Psi_{2k-1}} \inf_{\substack{\Phi\perp\Psi_1,\ldots,\Psi_{2k-1}\\ \Phi\in\mathrm{dom}(\mathbb{H}^\Lambda)}} \langle\!\langle \Phi, \mathbb{H}^\Lambda\Phi\rangle\!\rangle \\
&\leq \sup_{\Psi_1,\ldots,\Psi_{2k-1}} \inf_{\substack{\Phi\perp\Psi_1,\ldots,\Psi_{2k-1}\\ \Phi\in\mathrm{dom}(\mathbb{H}^{\Lambda,D_j})}} \langle\!\langle \Phi, \mathbb{H}^\Lambda\Phi\rangle\!\rangle \\
&= \sup_{\Psi_1,\ldots,\Psi_{2k-1}} \inf_{\substack{\Phi\perp\Psi_1,\ldots,\Psi_{2k-1}\\ \Phi\in\mathrm{dom}(\mathbb{H}^{\Lambda,D_j})}} \langle\!\langle \Phi, \mathbb{H}^{\Lambda,D_j}\Phi\rangle\!\rangle \\
&= \lambda_{2k-1}^j = \lambda_{2k}^j
\end{aligned} \tag{5.63}$$

and

$$\begin{aligned}
\lambda_{2k-1}^j &= \sup_{\Psi_1,\ldots,\Psi_{2k-1}} \inf_{\substack{\Phi\perp\Psi_1,\ldots,\Psi_{2k-1}\\ \Phi\in\mathrm{dom}(\mathbb{H}^{\Lambda,D_j})}} \langle\!\langle \Phi, \mathbb{H}^{\Lambda,D_j}\Phi\rangle\!\rangle \\
&= \sup_{\Psi_1,\ldots,\Psi_{2k-1}} \inf_{\substack{\Phi\perp\Psi_1,\ldots,\Psi_{2k-1},0\oplus\delta_j,\delta_j\oplus 0\\ \Phi\in\mathrm{dom}(\mathbb{H}^\Lambda)}} \langle\!\langle \Phi, \mathbb{H}^\Lambda\Phi\rangle\!\rangle \\
&\leq \sup_{\Psi_1,\ldots,\Psi_{2k+1}} \inf_{\substack{\Phi\perp\Psi_1,\ldots,\Psi_{2k+1}\\ \Phi\in\mathrm{dom}(\mathbb{H}^\Lambda)}} \langle\!\langle \Phi, \mathbb{H}^\Lambda\Phi\rangle\!\rangle \\
&= \lambda_{2k+1} = \lambda_{2k+2},
\end{aligned} \tag{5.64}$$

for all $1 \le k \le (|\Lambda| - 1)$. Together we have for $1 \le k \le (|\Lambda| - 1)$

$$\lambda_{2k-1} = \lambda_{2k} \le \lambda^j_{2k-1} = \lambda^j_{2k} \le \lambda_{2k+1} = \lambda_{2k+2} \tag{5.65}$$

which translates with (5.61) and (5.62) to

$$E_{|\Lambda|+k} \le E^j_{(|\Lambda|-1)+k} \le E_{|\Lambda|+(k+1)} \tag{5.66}$$

for $1 \le k \le (|\Lambda| - 1)$ and in particular to

$$E_{2|\Lambda|-1} \le E^j_{2(|\Lambda|-1)} \le E_{2|\Lambda|}. \tag{5.67}$$

Hence, we can conclude for all $c > 0$

$$0 \le \mathrm{tr}\chi_{[c,\infty[}(\mathbb{H}^\Lambda) - \mathrm{tr}\chi_{[c,\infty[}(\mathbb{H}^{\Lambda,D_j}) \le 1. \tag{5.68}$$

Thus we get for all $a, b \in \mathbb{R}_{\ge 0}$ with $a < b$

$$\begin{aligned}\mathrm{tr}\chi_{[a,b[}(\mathbb{H}^\Lambda) &= \mathrm{tr}\chi_{[a,\infty[}(\mathbb{H}^\Lambda) - \mathrm{tr}\chi_{[b,\infty[}(\mathbb{H}^\Lambda)\\ &\le \mathrm{tr}\chi_{[a,\infty[}(\mathbb{H}^\Lambda) - \mathrm{tr}\chi_{[b,\infty[}(\mathbb{H}^{\Lambda,D_j})\\ &\le \mathrm{tr}\chi_{[a,\infty[}(\mathbb{H}^\Lambda) - \mathrm{tr}\chi_{[a,\infty[}(\mathbb{H}^{\Lambda,D_j}) + \mathrm{tr}\chi_{[a,b[}(\mathbb{H}^{\Lambda,D_j})\\ &\le 1 + \mathrm{tr}\chi_{[a,b[}(\mathbb{H}^{\Lambda,D_j}).\end{aligned} \tag{5.69}$$

□

Remark 5.25. Unlike in the case of the Anderson model (cf. Section 3.3), it is at this point not quite clear how Proposition 5.24 and the Wegner estimate 4.24 can be used to get the desired Minami estimate. Indeed, they seem to point towards the desired result, but only in a very specific case, since we have for all $i \in \Lambda$:

$$\begin{aligned}&\mathbb{E}\left[\mathrm{tr}\ \chi_I(\mathbb{H}^\Lambda_\omega)\left\{\mathrm{tr}\ \chi_I(\mathbb{H}^\Lambda_\omega) - 1\right\}\right] \le \mathbb{E}\left[\mathrm{tr}\ \chi_I(\mathbb{H}^\Lambda_\omega)\left\{\mathrm{tr}\ \chi_I(\mathbb{H}^{\Lambda,D_i}_\omega)\right\}\right]\\ &\le \frac{E+1}{\lambda}\sum_{j\in\Lambda}\int_{E-2\epsilon}^{E+2\epsilon}\mathbb{E}_{V_j^\perp}\left(\int_{\mathbb{R}}\frac{\partial}{\partial V_j}\sum_{n=1}^{2|\Lambda|}\rho(E_n(V)-\eta)\left\{\mathrm{tr}\chi_I(\mathbb{H}^{\Lambda,D_i}_\omega)\right\}d\mu_V(V_j)\right)d\eta\\ &\le \begin{cases}\left(2\epsilon\frac{E+1}{\lambda}\|\phi_V\|_{BV}\right)^2 2|\Lambda|, & \text{for } j = i\\ ? \quad \text{for } j \ne i.\end{cases}\end{aligned} \tag{5.70}$$

Let us note at this point more explicitly that it is the fact that

$$V_j \longmapsto \frac{\partial}{\partial V_j}\sum_{n=1}^{2|\Lambda|}\rho(E_n(V)-\eta) \tag{5.71}$$

can be negative which does not let us conclude the Minami estimate easily. Let us further note that this problem arises due to the non-monotonic growth of the eigenvalues of $\mathbb{H}^\Lambda$ with respect to a single V_j, $j \in \Lambda$.

5.3 Exponential decay of the fractional moments of the Green's function

In this section we give a short outlook on possible strategies to prove the condition of Theorem 5.7 on the exponential decay of the fractional moments of the Green's function, i.e. we need to prove that there exists an $s \in]0,1[$, a $C(s) > 0$, an $m(s) > 0$ and a $r(s) > 0$ such that for all $\Lambda \subset \mathbb{Z}^d$ and all $x \in \Lambda$, $y \in \partial\Lambda$

$$\mathbb{E}\left[\|\mathbb{G}^{\Lambda}(z;x,y)\|_{\infty}^{s}\right] \leq C(s)\mathrm{e}^{-m(s)|x-y|_1} \tag{5.72}$$

with $z \in \{z \in \mathbb{C} \ : \ \Im z > 0, \ |z-E| < r(s)\}$, holds.

Remark 5.26. Actually, we don't need to prove the above statement in its full glory for the purpose of proving Theorem 5.7. In particular, we only need to prove the above exponential decay for $z := E + (2|\Lambda|)^{-1}\zeta$, where $E \in \sigma(\mathbb{H})$ lies in the localized regime such that the density of state $\mathbb{D}(E)$ exists at E and is positive, and $\zeta := \sigma + i\tau \in \mathbb{C}_+$ for large enough Λ such that we are close enough to the spectrum, i.e. $z \in \{z \in \mathbb{C} \ : \ \Im z > 0, \ |z-E| < r(s)\}$ as above (cf. *Step 3* in the proof of Theorem 5.7).

The standard procedures to prove the exponential decay of the fractional moments (cf. [AM93, ASFH01]), in the case of random Schrödinger operators, are based on rank-2-perturbations. If we adapt this method directly to random block operators, and use a rank-4-perturbation of the form

$$\mathbb{H} = \mathbb{H}_0 + V_j \begin{pmatrix} P_j & 0 \\ 0 & -P_j \end{pmatrix} + V_l \begin{pmatrix} P_l & 0 \\ 0 & -P_l \end{pmatrix} =: \mathbb{H}_0 + V_j\Pi_j + V_l\Pi_l \tag{5.73}$$

where $\mathbb{H}_0 := \mathbb{H} - V_j\Pi_j - V_l\Pi_l$, and then follow the argument given in [AM93], we immediately run into difficulties. At the moment it is unclear how to prove this statement via this method.

However, as noted in Section 2.4 Aizenman, Schenker, Friedrich and Hundertmark showed in [ASFH01] that in the case of random Schrödinger operators there exists a direct link between the exponential decay of the fractional moments and the *results* of multiscale analysis and the latter is something we do have for random block operators thanks to Gebert (cf. [Geb11]). As can be found in [Geb11], we have for an interval $I = [E_1, E_2]$ close to the gap of the spectrum that there exists a $p > 2d$, an α with $1 < \alpha < \frac{2p}{p+2d}$ and a $\gamma > 0$ such that for any two disjoint cubes $\Lambda_1 := \Lambda_L(n)$ and $\Lambda_2 := \Lambda_L(m)$

$$\mathbb{P}\Big[\exists E \in I : \ \forall n \in \Lambda_{L^{1/2}}, m \in \partial^-\Lambda_L : \\ \|\mathbb{G}^{\Lambda_1}(E;n,m)\|_\infty > \mathrm{e}^{-\gamma L} \text{ and } \|\mathbb{G}^{\Lambda_2}(E;n,m)\|_\infty > \mathrm{e}^{-\gamma L}\Big] \leq \frac{1}{L^{2p}}, \tag{5.74}$$

where $\Lambda_L(n)$ denotes the hypercube Λ_L centered at the point $n \in \mathbb{Z}^d$. Basically we can apply the same strategy as for the prove of the above statement (as can be found in [Geb11]; see also [Kir08, Sec. 9-11]) to gain the weaker form of the multiscale analysis result (cf. Section 2.30). Hence, we could prove that there exists an $\alpha > 1$, $p > 2d$ and a $\gamma > 0$ such that for all $E \in I$

$$\mathbb{P}\left[\forall n \in \Lambda_{L^{1/2}}, m \in \partial^- \Lambda_L : \ \|\mathbb{G}^{\Lambda_L}(E; n, m)\|_\infty > \mathrm{e}^{-\gamma L}\right] \leq \frac{1}{L^p} \tag{5.75}$$

holds. The advantage of this result is that it only considers one cube but it is generally weaker than (5.74) since it is not uniform in the energies E, however, it should be enough for our purposes. It should even be possible to gain subexponential decay of the probability in (5.75) as L gets large by adapting the idea of *Bootstrap multiscale analysis*, as was shown for the case of random Schrödinger operators by Germinet and Klein in [GK01], to the case of random block operators. Building on this, we can follow the idea proposed in [ASFH01, Sec. 4.4] and divide our space Ω into the "good set"

$$\Omega_G := \{\omega \ : \ \|\mathbb{G}^{\Lambda_L}(E; 0, m)\|_\infty \leq \mathrm{e}^{-\gamma |m|_1}\} \tag{5.76}$$

and the "bad set"

$$\Omega_B := \{\omega \ : \ \|\mathbb{G}^{\Lambda_L}(E; 0, m)\|_\infty > \mathrm{e}^{-\gamma |m|_1}\}. \tag{5.77}$$

for $m \in \partial^\pm$ Hence, we get for $s \in]0, 1[$ and all $y \in \partial^\pm$

$$\begin{aligned} \mathbb{E}\left[\|\mathbb{G}^\Lambda(z; 0, y)\|_\infty^s\right] &= \mathbb{E}\left[\|\mathbb{G}^\Lambda(z; 0, y)\|_\infty^s \mathbb{1}_{\Omega_G}\right] + \mathbb{E}\left[\|\mathbb{G}^\Lambda(z; 0, y)\|_\infty^s \mathbb{1}_{\Omega_B}\right] \\ &\leq C\mathrm{e}^{-\gamma L} + \left(\mathbb{E}\left[\|\mathbb{G}^\Lambda(z; 0, y)\|_\infty^t\right]\right)^{\frac{s}{t}} \left(\mathbb{E}\left[\mathbb{1}_{\Omega_B}\right]\right)^{1-\frac{s}{t}}, \end{aligned} \tag{5.78}$$

with $t \geq s$. Since we have $\Im z = (2|\Lambda|)^{-1}\tau$ we get for $\delta > 0$ and $0 \leq \kappa \leq 1$

$$\mathbb{E}\left[\|\mathbb{G}^\Lambda(z; 0, y)\|_\infty^s\right] \leq C\mathrm{e}^{-\gamma L} + \left(\frac{4|\Lambda|}{\tau}\right)^s \left(\mathrm{e}^{-\delta L^\kappa}\right)^{1-\frac{s}{t}}. \tag{5.79}$$

However, we see in [ASFH01] that to gain exponential decay of the fractional moments of the Green's function we also need to prove that the fractional moments are bounded by a constant independent of Λ, $x, y \in \Lambda$ and of z. What we gain easily is that we have for all $x, y \in \Lambda$ and all $z := E + (2|\Lambda|)^{-1}\zeta$

$$\|\mathbb{G}^\Lambda(z; x, y)\|_\infty \leq \frac{4}{\Im z} = \frac{4|\Lambda|}{\tau}. \tag{5.80}$$

If we can indeed show subexponential decay of the probability in (5.75) it ought to be enough to suppress the $|\Lambda|$ factor arising from the rather rough inequality (5.80). However, we still need to work on how exactly to conclude the desired exponential decay of the fractional moments of the Green's function for all $x, y \in \Lambda$ from the above.

A Appendix

For the readers' convenience we shall state in the following the proof of the Wegner estimate for the Anderson Model (cf. Theorem 2.25 and [CGK09a]) and the proof of Theorem 5.20 (cf. [Geb11, Appendix A]).

A.1 Proof of the Wegner estimate for the Anderson model

In the following we will present the proof of the Wegner estimate for the Anderson Model (cf. Theorem 2.25) for the case that μ has bounded Lebesgue density. The proof, including the general case of an arbitrary probability distribution μ, can be found in [CGK09a].

Proof of Theorem 2.25. Let g denote the bounded Lebesgue density of μ and consider the self-adjoint operator H_ω^Λ which can be written in the form

$$H_\omega^\Lambda = H_0^\Lambda + \sum_{j\in\Lambda} \omega_j P_j, \tag{A.1}$$

where P_j denotes the orthogonal projection onto the subspace spanned by δ_j. For $k \in \Lambda$ we can interprate H_ω^Λ to be a rank-1-perturbation of the following form

$$H_\omega^\Lambda = H^\Lambda + \omega_k P_k, \tag{A.2}$$

with $H^\Lambda := H_\omega^\Lambda - \omega_k P_k$ which is therefore independent of ω_k. Given $z := \lambda + i\epsilon \in \mathbb{C}$ with $\Im z = \epsilon > 0$, the resolvent equation yields

$$(H_\omega^\Lambda - z)^{-1} = (H^\Lambda - z)^{-1} - w_k (H_\omega^\Lambda - z)^{-1} P_k (H^\Lambda - z)^{-1}. \tag{A.3}$$

We can infer from the by (A.3) induced quadratic form evaluated at δ_k that

$$\langle \delta_k, (H_\omega^\Lambda - z)^{-1} \delta_k \rangle = \left(\langle \delta_k, (H^\Lambda - z)^{-1} \delta_k \rangle^{-1} + \omega_k \right)^{-1}. \tag{A.4}$$

Define

$$\langle \delta_k, (H^\Lambda - z)^{-1} \delta_k \rangle^{-1} =: a \in \mathbb{C} \tag{A.5}$$

which is number independent of ω_k since H_ω^Λ is independent of ω_k. It then follows from (A.4) that

$$\int_{\mathbb{R}} \Im \langle \delta_k, (H_\omega^\Lambda - z)^{-1} \delta_k \rangle d\omega_k = \int_{\mathbb{R}} \frac{\Im \overline{a}}{(\Re a + \omega_k)^2 + (\Im \overline{a})^2} d\omega_k = \pi \tag{A.6}$$

holds. Now we will make use of Stone's formula [RS80, Theorem VII.13] which yields for any self-adjoint operator A on a Hilbert space and $c < d \in \mathbb{R}$ the following identity

$$\lim_{\varrho \to 0} \frac{1}{2\pi i} \int_c^d \left[(A - \kappa - i\varrho)^{-1} - (A - \kappa + i\varrho)^{-1} \right] d\kappa = \frac{1}{2} \left[\chi_{[c,d]}(A) + \chi_{]c,d[}(A) \right] \quad \text{(A.7)}$$

where the limit holds in the strong sense. Let the interval I be of the form $[a, b]$, $]a, b]$, $[a, b[$ or $]a, b[$ then together with (A.6) Stone's formula yields

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} \langle \delta_k, \left[\chi_{[a,b]}(H_\omega^\Lambda) + \chi_{]a,b[}(H_\omega^\Lambda) \right] \delta_k \rangle d\omega_k &= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_a^b \int_{\mathbb{R}} \Im \langle \delta_k, (H_\omega^\Lambda - \lambda - i\epsilon)^{-1} \delta_k \rangle d\omega_k d\lambda \\ &\leq (b - a). \end{aligned} \quad \text{(A.8)}$$

In particular this yields that we have for any $c \in \mathbb{R}$

$$\int_{\mathbb{R}} \langle \delta_k, \chi_{\{c\}}(H_\omega^\Lambda) \delta_k \rangle d\omega = 0. \quad \text{(A.9)}$$

Hence, we get for any bounded interval I

$$\int_{\mathbb{R}} \langle \delta_k, \chi_I(H_\omega^\Lambda) \delta_k \rangle d\mu(\omega) = \int_{\mathbb{R}} \langle \delta_k, \chi_I(H_\omega^\Lambda) \delta_k \rangle dg(\omega) d\omega \leq \|g\|_\infty |I|. \quad \text{(A.10)}$$

Now we are nearly done, however, before we can conclude the statement we will introduce the following notation. Let $\mathbb{E}_{\omega_j}$ be the expectation with respect to the random variable ω_j and accordingly we will write $\mathbb{E} = \mathbb{E}_\omega$ for the expectation with respect to all random variables. Furthermore, let $\omega_j^\perp := (\omega_n)_{n \in \mathbb{Z}^d \setminus \{j\}}$ and $\mathbb{E}_{\omega_j^\perp}$ denote the corresponding expectation. Thus we can conclude

$$\mathbb{E} \left[\operatorname{tr} \chi_I(H_\omega^\Lambda) \right] = \sum_{k \in \Lambda} \mathbb{E}_{\omega_k^\perp} \left\{ \mathbb{E}_{\omega_k} \left[\langle \delta_k, \chi_I(H_\omega^\Lambda) \delta_k \rangle \right] \right\} \leq Q_\mu(|I|) |\Lambda|. \quad \text{(A.11)}$$

□

A.2 Proof of the simplicity of the eigenvalues of $\mathbb{H}$

In the following we will state the proof of Theorem 5.20 which can be found in [Geb11, Appendix A]. Thus, let $\mathbb{H}$ be the random block operator defined in (4.13). Since we assumed that the common distribution of V, μ_V, and respectively of b, μ_b, are absolutely continuous with respect to the Lebesgue measure we can use the following result to prove Theorem 5.20.

Lemma A.1. *Let λ be the Lebesgue measure on $\mathbb{R}^d$ and let $A \subset \mathbb{R}^d$ be a Borel set with $\lambda(A) > 0$. Then we have*

$$\exists a \in A : \ \exists (c_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \ c_n \to 1, \text{ for } n \to \infty, \ \text{such that: } \{ac_n\}_n \subset A.$$

Proof. Let us define for $a \in \mathbb{R}^d$ the line starting in 0 and ending in a by $[0 : a] := \{ta \;:\; t \in [0, \infty[\}$. We will prove the statement by contradiction. Assume that $\forall a \in A: \forall c_n \to 1: ac_n \notin A$ for n large. Then we have

$$\forall a \in A \; \exists \epsilon(a): \; B_{\epsilon(a)}(a) \cap [0 : a] \cap A = \{a\}. \tag{A.12}$$

Hence, there are no accumulation points in $A \cap [0 : a]$ and thus we have that $A \cap [0 : a]$ is discrete. Therefore, it is of measure 0 with respect to the Lebesgue measure on $\mathbb{R}$. By transforming into spherical coordinates and using Fubini we can conclude that $\lambda(A) = 0$ contradicting the assumption that $\lambda(A) > 0$. □

Remark A.2. With the help of this lemma we can turn to the proof of Theorem 5.20 as can be found in [Geb11].

Proof of Theorem 5.20. Define $I := \{1, \ldots, 2|\Lambda|\} \subset \mathbb{N}$ and let the set $\{E_n(V, b)\}_{n \in I}$ denote the eigenvalues of $\mathbb{H}^\Lambda$ ordered by magnitude and repeated according to multiplicity which depend on the random potential V and the off-diagonal random entries b. Since we have that

$$\begin{aligned} \mathbb{P}\big(\mathbb{H}^\Lambda \text{ has a degenerate eigenvalue }\big) &= \mathbb{P}\big(\exists\, l \neq k \in I: \; E_l(V, b) = E_k(V, b)\big) \\ &\leq \binom{2|\Lambda|}{2} \max_{k \neq l} \mathbb{P}\big(E_l(V, b) = E_k(V, b)\big), \end{aligned} \tag{A.13}$$

it is sufficient to proof that $\mathbb{P}\big(E_l(V, b) = E_k(V, b)\big) = 0$ for all $k \neq l$. We shall prove this by contradiction. Hence, fix $k \neq l$ and assume

$$\mathbb{P}\big(E_l(V, b) = E_k(V, b)\big) > 0. \tag{A.14}$$

Since we assumed that for all $n \in \Lambda$ V_n and b_n are absolutely continuous with respect to the Lebesgue measure we have that the set $A := \{(V, b) \in \mathbb{R}^{2|\Lambda|} \;:\; E_l(V, b) = E_k(V, b)\} \subset \mathbb{R}^{2|\Lambda|}$ has positive Lebesgue measure $\lambda(A) > 0$. Lemma A.1 then equips us with a point $a := (\tilde{V}, \tilde{b}) \in A$ and real valued sequence $(c_n)_n$ with $c_n \to 1$ such that the set $\{ac_n\}_n \subset A$ and has the accumulation point a.
Now consider the operator

$$T(\beta) := \begin{pmatrix} H_0^\Lambda & 0 \\ 0 & -H_0^\Lambda \end{pmatrix} + \beta \begin{pmatrix} \tilde{V} & \tilde{b} \\ \tilde{b} & -\tilde{V} \end{pmatrix}, \tag{A.15}$$

where $\beta \in \mathbb{C}$. Let us denote the eigenvalues of $T(\beta)$ by $\vartheta_n(\beta)$. Then we have by the above that for all n $\vartheta_k(c_n) = \vartheta_l(c_n)$ and in particular $\vartheta_k(1) = \vartheta_l(1)$. However, we know by analytic perturbation theory (cf. [Kat76]), that the $\vartheta_n(\beta)$ are analytic in some complex neighborhood U of 1. Since we have that $\vartheta_k(\beta) = \vartheta_l(\beta)$ on a set with an accumulation point we can conclude by the identity theorem of complex analysis that $\vartheta_k(\beta) = \vartheta_l(\beta)$ holds for all $\beta \in U$. Since we get for all real β a neighborhood in which the eigenvalues are analytic we can conclude that $\vartheta_k(\beta) = \vartheta_l(\beta)$ holds for all $\beta \in \mathbb{R}$ and in particular for $\beta = 0$. Recall that we assumed H_ω to be the Anderson model and hence $H_0^\Lambda = \Delta^\Lambda$. Hence, we know that all eigenvalues of H_0^Λ are distinct and that $H_0^\Lambda > 0$ which is a contradiction. □

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Masterarbeit selbstständig verfasst und keine weitern als die genannten Hilfsmittel und Quellen verwendet habe.

München, den 1. August 2011

Martin Vogel