Tutorial: closable operators, closure, closed operators

Let T be a linear operator on a Hilbert space $\mathcal{H}$, defined on some subspace $\mathcal{D}(T) \subset \mathcal{H}$, the domain of T.

When, motivated by several important examples (e.g., the Hellinger-Toeplitz theorem, the position operator on $\mathbb{R}$ [Exercise 47], the momentum operator [Problem 49], the kinetic energy operator [Exercise 49], the Schrödinger Hamiltonians [Problem 48 (ii)], the fact that canonical commutation relations cannot hold for bounded operators [Problem 50]), we want to relax boundedness and thus we assume that $\|Tz_n\| \xrightarrow{n\to\infty} \infty$ along a sequence of vectors $z_n \in \mathcal{H}$, $\|z_n\| = 1$, then two phenomena obviously occur:

1. *we loose continuity* (recall: linear bounded $\Leftrightarrow$ linear continuous);
2. *we loose bounded linear extension*: since $\sup_{\|x\|=1} \|Tx\| = \infty$ it is not possible any longer to use the bounded linear extension theorem to extend T by continuity from $\mathcal{D}(T)$ to a linear operator defined on the closure $\overline{\mathcal{D}(T)}$. In particular, if T is densely defined we do not have a bounded linear extension theorem any longer to extend T to the whole $\mathcal{H}$. (WARNING: this does not exclude that there exist operators on $\mathcal{H}$ that are *unbounded* and *everywhere defined* (indeed there are, can you figure out an example?), it only means that we loose a useful tool, the bounded linear extension by continuity.)

Both phenomena lead in a natural way to the notion of closable/closed operators.

Consider first the loss of continuity. It may happen that $\mathcal{D}(T) \ni x_n \to x \in \mathcal{H}$ but Tx_n has no limit, or that $\mathcal{D}(T) \ni x_n \to x$, $\mathcal{D}(T) \ni \widetilde{x}_n \to x$, but Tx_n and $T\widetilde{x}_n$ have different limits. Moreover, if $x \in \mathcal{D}(T)$ it could be that $Tx_n \to y \neq Tx$. Any of these possibilities prevents T to be extended "by continuity" to all the limit points of $\mathcal{D}(T)$, i.e., the whole $\mathcal{H}$.

Sometimes, a "less problematic" situation occurs: not along all sequences $\mathcal{D}(T) \ni x_n \to x \in \mathcal{H}$ has Tx_n a limit, nevertheless for all sequences in $\mathcal{D}(T)$ converging to x along which T has a limit, this limit is *unique*. This "not so bad" circumstance makes T closable.

Definition 1. T is CLOSABLE if, given an arbitrary $x \in \mathcal{H}$ limit point of $\mathcal{D}$, for all the approximating sequences $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ of $x \in \mathcal{H}$ such that Tx_n has a limit, such a limit is the same.

It T is closable, there is a natural candidate for its closure.

Definition 2. If T is closable, the CLOSURE of T is the operator $\overline{T}$ whose domain and action are

- $\mathcal{D}(\overline{T}) := \{x \in \mathcal{H} \,|\, \exists\, y \in \mathcal{H}$ such that, for any sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{D}(T)$ with $x_n \to x$, $Tx_n \to y\}$
- $\overline{T}x := y$ for any $x \in \mathcal{D}(\overline{T})$.

In fact, one easily checks that the Definition 2 is well-posed because y is uniquely identified by x and $\overline{T}$ defines a linear operator. Also, it is clear that $T \subset \overline{T}$ for every closable T, that is, $\mathcal{D}(T) \subset \mathcal{D}(\overline{T})$ and $\overline{T}x = Tx$ for all $x \in \mathcal{D}(T)$ (just consider the sequence with $x_n = x$ $\forall n$).

Definition 3. T is CLOSED if $T = \overline{T}$. More precisely, T is closed when the following holds true: if $x \in \mathcal{H}$ is a limit point of $\mathcal{D}(T)$ such that $\mathcal{D}(T) \ni x_n \to x$ and $Tx_n \to y$ for some $y \in \mathcal{H}$, then $x \in \mathcal{D}(T)$ and $Tx = y$.

Consider the following three facts for a linear operator T on $\mathcal{H}$:

(i) $\mathcal{D}(T) \ni x_n \to x \in \mathcal{H}$, (ii) $Tx_n \to y \in \mathcal{H}$, (iii) $Tx = y\,.$

Then T is closed if (i)+(ii)⇒(iii), whereas T is (everywhere defined and) bounded if (i)⇒(ii)+(iii). Thus, the notion of closed operator does not show up within $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$.

The above definitions can be rewritten in the language of the GRAPH of T, i.e., the set

$$\Gamma(T) = \{(x, Tx) \in \mathcal{H} \oplus \mathcal{H} \,|\, x \in \mathcal{D}(T)\}\,.$$

Recall that $\mathcal{H} \oplus \mathcal{H}$ is naturally equipped with the direct sum topology, which makes it a Hilbert space, induced by the scalar product

$$\langle (z, w), (\widetilde{z}, \widetilde{w}) \rangle_{\mathcal{H}\oplus\mathcal{H}} = \langle z, \widetilde{z} \rangle + \langle w, \widetilde{w} \rangle.$$

In particular, $\overline{\Gamma(T)}$ denotes the closure of the graph of T in $\mathcal{H} \oplus \mathcal{H}$. In this language, it is immediate to recognise the following definitions to be equivalent to the previous ones.

Definition 1'. T is CLOSABLE if $\overline{\Gamma(T)} = \Gamma(S)$ for some linear operator S.

Definition 2'. If T is closable, its CLOSURE is that linear operator $\overline{T}$ identified by $\overline{\Gamma(T)} = \Gamma(\overline{T})$.

Definition 3'. T is CLOSED if $\overline{\Gamma(T)} = \Gamma(T)$.

Note that S in Definition 1' is uniquely identified by T (if T is closable) because $\Gamma(S) = \Gamma(\widetilde{S}) \Rightarrow S = \widetilde{S}$. Thus, if T is closed according to Definition 3', it is indeed true that $T = \overline{T}$.

In the language of the graph it is also clear that if $\Gamma(T) \subset \Gamma(R)$ for some linear operator R then $T \subset R$, i.e., R is an extension of T. Thus, once again we see that $T \subset \overline{T}$ for every closable operator T.

Notice also the following (easy to prove). $\Gamma(T) \subset \Gamma(R)$ for some linear operator R means that T admits linear extensions. If T is closable, in particular, $\overline{T}$ is an extension of T. It is a distinguished extension, because it is closed (indeed $\overline{\Gamma(\overline{T})} = \Gamma(\overline{T})$) and because if R is any *closed* extension of T, i.e.,

$$T \subset R = \overline{R}\,, \quad \text{or equivalently} \quad \Gamma(T) \subset \Gamma(R) = \overline{\Gamma(R)},$$

then necessarily $R = \overline{T}$. In other words, the closure $\overline{T}$ of a closable operator T is the smallest closed extension of T.

(Once again, not to overlook it: it could be that $\overline{\Gamma(T)}$ is not the graph of a linear operator, in which case T is not closable.)

In many contexts closable operators form the most reasonable class of unbounded operators to study. Somehow non-closable operators are "too pathological". For instance, we saw that a non-closable operator has empty resolvent set (Problem 50).

Remarkably, "closablity" of T is rather encoded in T^*. More precisely, let us quote the following results from class.

Theorem. Let T be a densely defined operator on $\mathcal{H}$. Then:

- T^* is closed. (Irrespectively of whether T is or not.)
- T is closable ⇔ T^* is densely defined, in which case $\overline{T} = T^{**}$.
- If T is closable, then T and $\overline{T}$ have the same adjoint: $T^* = \overline{T}^*$.

Comments.

(1) T^* is always closed. One may say that the construction of the adjoint produces an operator that is "more stable" than T. Consider for example the operator (Problem 49)

$$T_k = -\mathrm{i}\frac{d}{dt} \qquad \text{on the domain} \qquad \mathcal{D}(T_k) = \{\, f \in C^k([0,1]) \,|\, f(0) = f(1) = 0 \,\}$$

for $k \in \mathbb{N}$. Clearly $T_1 \supset T_2 \supset T_3 \supset \cdots$, moreover (as a consequence of the discussion in Problem 49) each T_k is closable but none of them is closed, and $\overline{T_1} = \overline{T_2} = \overline{T_3} = \cdots \equiv \overline{T} = -\mathrm{i}\frac{d}{dt}$ on the domain $\mathcal{D}(\overline{T}) = \{f \in AC[0,1] \,|\, f(0) = f(1) = 0\}$, whereas $T_1^* = T_2^* = T_3^* = \cdots \equiv T^* = -\mathrm{i}\frac{d}{dt}$ on the domain $\mathcal{D}(T^*) = \{f \in L^2[0,1] \,|\, f, f' \in AC[0,1]\}$. Irrespectively of which T_k one starts from, it is T^* the "important" operator, which in turn determines (via $T^{**} = \overline{T}$) the closure $\overline{T}$.

(2) Mind the pitfall: T^* is always closed, this does not imply that T is. For the densely defined operator T of Problem 47, for example, T^* is not densely defined (and is zero in $\mathcal{D}(T^*)$). T^* is closed but since its domain is not dense then T cannot be closable.

(3) The double adjoint T^{**} is a restriction of the adjoint, $T^{**} \subset T^*$, that for closable operator gives precisely the closure $\overline{T}$ of T. Thus, T^{**} is the smallest closed extension of T, if T is closable. Note that in the solution to Problem 49 (iii) the closure of the (closable) operator A_0 is computed in two alternative, equivalent ways: using the definition of closure, or (after computing A_0^* and therefore A_0^{**}) using the formula $\overline{A_0} = A_0^{**}$.

(4) A symmetric operator is always closable. Indeed, T symmetric means that T is densely defined and $T \subset T^*$, so T^* is densely defined too and by the theorem above T is closable. Its closure is $\overline{T} = T^{**}$. A self-adjoint operator is always closed, because it coincides with its adjoint. Recap:

- T symmetric: $T \subset \overline{T} = T^{**} \subset T^*$,
- T symmetric and closed: $T = \overline{T} = T^{**} \subset T^*$,
- T essentially self-adjoint: $T \subset \overline{T} = T^{**} = T^*$,
- T self-adjoint: $T = \overline{T} = T^{**} = T^*$.

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