Mathematical Quantum Mechanics – Mid-term exam, 13.12.2012
Mathematische Quantenmechanik – Zwischenklausur, 13.12.2012

Name:/*Name*: ______________________________

Matriculation number:/*Matrikelnr.:* ____________ **Semester:**/*Fachsemester:* ____________

Degree course:/*Studiengang:* ❏ Diplom ❏ Bachelor, PO ________ ❏ Lehramt Gymnasium (modularisiert)
❏ TMP ❏ Master, PO ________ ❏ Lehramt Gymnasium (nicht modul.)
❏ ______________________________

Major:/*Hauptfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ ________

Minor:/*Nebenfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ ________

Credits needed for:/*Anrechnung der Credit Points für das:* ❏ Hauptfach ❏ Nebenfach

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❏ Yes ❏ No

problem	**1**	**2**	**3**	**4**	**5**	**6**	$\sum$
total marks	**20**	**20**	**20**	**20**	**20**	**20**	120
scored marks							

INSTRUCTIONS:

- This booklet is made of fourteen pages, including the cover, numbered from 1 to 14. The test consists of six problems. Each problem is worth 20 marks. 100 marks are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-size "cheat sheet" (Spickzettel).
- Raise up your hand to request extra sheets or scratch paper. You are not allowed to use your own paper.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 100 minutes.

GOOD LUCK!

Name

PROBLEM 1. (20 marks)

Compute

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x\,.$$

SOLUTION:

SOLUTION TO PROBLEM 1 (CONTINUATION):

Name

PROBLEM 2. (20 marks)

Prove that for any $u \in \mathcal{D}'(\mathbb{R})$ satisfying

$$x\,u' + u = 0$$

in the distribution sense, there exist constant $c_1, c_2 \in \mathbb{C}$ such that

$$u \;=\; c_1\,\delta_0 + c_2\,\mathrm{PV}\Big(\frac{1}{x}\Big)\,.$$

(Notation: $\mathcal{D}'(\mathbb{R})$ is the space of distributions on $\mathbb{R}$, δ_0 is the delta distribution at zero, $\mathrm{PV}(\frac{1}{x})$ is the principal value distribution.)

Hints: (1) You may use the fact that the solution to $T' = 0$ in $\mathcal{D}'(\Omega)$ is the constant distribution. Use this to find the action of u on test functions of the form $x\phi$, $\phi \in \mathcal{D}(\mathbb{R})$.

(2) You may also use the following straightforward consequence of Taylor's formula, precisely as we did in homework Exercise 9(ii): if $\phi \in C_0^\infty(\mathbb{R})$ and $\chi_\phi \in C_0^\infty(\mathbb{R}, [0,1])$ is such that $\chi_\phi \equiv 1$ on the support of ϕ, then $\exists \psi \in C^\infty(\mathbb{R})$ such that

$$\phi(x) \;=\; \phi(x)\chi_\phi(x) \;=\; \big(\phi(0) + x\psi(x)\big)\chi_\phi(x) \qquad \forall x \in \mathbb{R}.$$

SOLUTION:

SOLUTION TO PROBLEM 2 (CONTINUATION):

Name

PROBLEM 3. (20 marks)

Let $d \in \mathbb{N}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $H^1(\mathbb{R}^d)$ and let $f \in H^1(\mathbb{R}^d)$. Assume that

$$f_n \rightharpoonup f \qquad \text{as } n \to \infty \qquad \text{weakly in } H^1(\mathbb{R}^d)\,.$$

Prove that

$$f_n \rightharpoonup f \qquad \text{as } n \to \infty \qquad \text{weakly in } L^2(\mathbb{R}^d)\,.$$

SOLUTION:

SOLUTION TO PROBLEM 3 (CONTINUATION):

Name

PROBLEM 4. (20 marks)

Consider the complex Hilbert space $\mathcal{H} = L^2[-1,1]$ and the bounded linear operators

$$\begin{aligned} A : \mathcal{H} \to \mathcal{H}, &\qquad (Af)(x) = xf(x), \\ B : \mathcal{H} \to \mathcal{H}, &\qquad (Bf)(x) = x^2 f(x), \end{aligned}$$

for all $x \in [-1,1]$ and $f \in L^2[-1,1]$. Correspondingly, let $\mathcal{A}_A$ and $\mathcal{A}_B$ be the C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ generated respectively by A and by B. (That is, $\mathcal{A}_A$ consists of the closure in the operator norm of all polynomials of A. A^* is not mentioned simply because $A^* = A$. The same for $\mathcal{A}_B$.)

(i) Consider the functions $\mathbf{1}$ and θ in $\mathcal{H}$ defined by

$$\mathbf{1}(x) := 1 \quad \forall x \in [-1,1]\,, \qquad\qquad \theta(x) := \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [-1,0) \end{cases}.$$

Prove that $\mathbf{1}$ is cyclic for $\mathcal{A}_A$ while θ is not.

(ii) Prove that $\mathcal{A}_B$ has *no* cyclic vectors.
(*Hint:* if f was cyclic, define $g(x) := \overline{f(-x)}\,\mathrm{sgn}(x)$.)

SOLUTION:

SOLUTION TO PROBLEM 4 (CONTINUATION):

Name

PROBLEM 5. (20 marks)

Consider the three-dimensional Hamiltonian $H = -\Delta + V$ where $V \in L^\infty(\mathbb{R}^3)$. Assume that for some $E \in \mathbb{R}$ and some $\psi \in L^2(\mathbb{R}^3)$ one has

$$-\Delta\psi + V\psi = E\psi$$

as an identity of L^2-functions. Prove that $\psi \in L^\infty(\mathbb{R}^3)$.

(*Hint:* for such a ψ prove and use the inequality $\|\psi\|_\infty \leqslant a\left(\|\Delta\psi\|_2 + \|\psi\|_2\right)$ for some $a > 0$.)

SOLUTION:

SOLUTION TO PROBLEM 5 (CONTINUATION):

Name

PROBLEM 6. (20 marks)

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a measurable function. For any $\lambda > 0$ let $E_0(\lambda)$ denote the ground state energy of $-\Delta + \lambda V$. Assume that

(a) $V \in L^{3/2}(\mathbb{R}^3)$,

(b) $\lim_{|x|\to\infty} V(x) = 0$,

(c) $E_0(\lambda_0) < 0$ for some $\lambda_0 > 0$.

(d) $V(x) < 0$ for almost every $x \in \mathbb{R}^3$.

Prove that under the assumptions (a), (b), (c), and (d), the function $[\lambda_0, +\infty) \ni \lambda \mapsto E_0(\lambda)$ is strictly decreasing.

SOLUTION:

SOLUTION TO PROBLEM 6 (CONTINUATION):