LMU | LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

Winter term 2012-2013 / *Wintersemester 2012-2013*

Mathematical Quantum Mechanics – Final exam, 9.2.2013

Mathematische Quantenmechanik – Endklausur, 9.2.2013

Name:/*Name*: ______________________________

Matriculation number:/*Matrikelnr.:* ____________________ **Semester:**/*Fachsemester:* ______________

Degree course:/*Studiengang:* ❏ Diplom ❏ Bachelor, PO ________ ❏ Lehramt Gymnasium (modularisiert)
❏ TMP ❏ Master, PO ________ ❏ Lehramt Gymnasium (nicht modul.)
❏ ______________________________

Major:/*Hauptfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ __________

Minor:/*Nebenfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ __________

Credits needed for:/*Anrechnung der Credit Points für das:* ❏ Hauptfach ❏ Nebenfach

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❏ Yes ❏ No

problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	$\sum$
total marks	**16**	**16**	**16**	**16**	**16**	**16**	**16**	**16**	**16**	144
scored marks										

homework bonus		**mid-term performance**		**final test performance**		**total performance**		**FINAL MARK**	

INSTRUCTIONS:

- This booklet is made of twenty-two pages, including the cover, numbered from 1 to 22. The test consists of nine problems. Each problem is worth 16 marks. 100 marks are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-size "cheat sheet" (Spickzettel).
- Raise up your hand to request extra sheets or scratch paper. You are not allowed to use your own paper.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in German or in English. Put your name on every sheet you hand in.
- You have 140 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
nach § Abs. Nr. Buchstabe LPO I

UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der ____________
Herr / Frau ____________ *aus* ____________
geboren am ______ *in* ______ *hat im* **WiSe** *–Halbjahr* **2012-2013**
meine Übungen **zur Mathematisches Quantenmekanik** ____________

mit ____________ *besucht.*
Er / Sie hat ____________
schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden. ____________

MÜNCHEN, *den* **9 Februar 2013**

Name

PROBLEM 1. (16 marks)

Let $\mathbb{S}^1$ denote the unit circle in the complex plane, consider it homeomorphic to $[0,2\pi)$ with periodicity. Let $h \in L^2(\mathbb{S}^1)$ be such that $h(x) = \overline{h(-x)}$ for a.e. $x \in \mathbb{S}^1$. Consider the map

$$T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)\,, \qquad Tf := h * f$$

(the convolution with the function h).

(i) Prove that T is an everywhere defined bounded linear operator.

(ii) Prove that T is a compact and self-adjoint operator.

(iii) Find an explicit orthonormal system $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{S}^1)$ and a collection $\{\lambda_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$ such that

$$T = \sum_{n\in\mathbb{Z}} \lambda_n |e_n\rangle\langle e_n|$$

where the series is meant to converge in operator norm.

(*Hint:* you may find it useful to use the fact that the Fourier transform turns convolution into multiplication.)

SOLUTION:

SOLUTION TO PROBLEM 1 (CONTINUATION):

Name

PROBLEM 2. (16 marks)

Consider the measure μ on $\mathbb{R}$ defined by

$$\mu := \mu_{\text{Lebesgue}} + \delta_4 + \delta_{-4}$$

(where δ_a denotes the Dirac measure centred at a) and the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x) := \begin{cases} x^2\,, & x \geqslant 0 \\ 3\,, & x < 0 \end{cases}.$$

Correspondingly, consider the operator $M_\phi : L^2(\mathbb{R}, \mathrm{d}\mu) \to L^2(\mathbb{R}, \mathrm{d}\mu)$ defined by

$$M_\phi f := \phi f\,, \qquad \mathcal{D}(M_\phi) := \{f \in L^2(\mathbb{R}, \mathrm{d}\mu) \,|\, \phi f \in L^2(\mathbb{R}, \mathrm{d}\mu)\}$$

(i.e., the self-adjoint operator of multiplication by ϕ).

(i) Find the spectrum of M_ϕ.

(ii) Find the eigenvalues of M_ϕ.

SOLUTION:

SOLUTION TO PROBLEM 2 (CONTINUATION):

Name

PROBLEM 3. (16 marks)

Let $\mathcal{H}$ be a Hilbert space and let A be a (possibly unbounded) self-adjoint operator acting on $\mathcal{H}$. Define

$$U := (A - \mathrm{i})(A + \mathrm{i})^{-1}\,.$$

(i) Prove that U is a unitary operator on $\mathcal{H}$.

(ii) Prove that $\ker(U - \mathbb{1}) = \{\mathbf{0}\}$.

SOLUTION:

SOLUTION TO PROBLEM 3 (CONTINUATION):

Name

PROBLEM 4. (16 marks)

Let $\mathcal{H}$ be a Hilbert space and let A be a (possibly unbounded) self-adjoint operator acting on $\mathcal{H}$. Denote by $\{E^{(A)}_\Omega\}_\Omega$ the projection-valued measure associated with A and by $\sigma(A)$ the spectrum of A. Prove the following:

$$\sigma(A) = \left\{ \lambda \in \mathbb{R} \;\middle|\; E^{(A)}_{(\lambda-\varepsilon,\lambda+\varepsilon)} \neq \mathbb{O} \;\; \forall \varepsilon > 0 \right\}.$$

SOLUTION:

SOLUTION TO PROBLEM 4 (CONTINUATION):

Name

PROBLEM 5. (16 marks)

For every $x \in [0,1]$ and $n \in \mathbb{N}$ define $f_n(x) := n^{-1/4}\, x^{\frac{3n+1}{2n}}$. Consider the sequence $\{f_n\}_{n=1}^{\infty}$.

(i) Prove that $f_n \rightharpoonup 0$ weakly in $H^1(0,1)$ as $n \to \infty$.

(ii) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge weakly to 0 in $H^2(0,1)$? Justify your answer.

SOLUTION:

SOLUTION TO PROBLEM 5 (CONTINUATION):

Name

PROBLEM 6. (16 marks)

Let A and B be two bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ such that

$$\mathbb{O} \leqslant A \leqslant B\,.$$

Prove that

$$\mathbb{O} \leqslant A^{1/3} \leqslant B^{1/3}\,.$$

(*Hint:* consider $\displaystyle\int_0^{+\infty} \frac{1}{\lambda^{2/3}} \frac{1}{x+\lambda}\,\mathrm{d}\lambda$ for $x > 0$.)

SOLUTION:

SOLUTION TO PROBLEM 6 (CONTINUATION):

Name

PROBLEM 7. (16 marks)

Let $\mathcal{H}$ be a separable Hilbert space and let T be a bounded linear operator acting on $\mathcal{H}$.

(i) Assume that $T = T_1 T_2$ for some Hilbert-Schmidt operators T_1 and T_2 acting on $\mathcal{H}$. Prove that T is of trace class.

(ii) Assume that T is of trace class. Prove that there exist two Hilbert-Schmidt operators T_1 and T_2 acting on $\mathcal{H}$ such that $T = T_1 T_2$.

SOLUTION:

SOLUTION TO PROBLEM 7 (CONTINUATION):

Name

PROBLEM 8. (16 marks)

Consider the Hamiltonian $H = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ acting on $L^2(\mathbb{R})$, where

- V is real-valued,
- $V \in L^\infty(\mathbb{R})$,
- and $\lim\limits_{x\to\pm\infty} V(x) = 0$.

For any given $E > 0$ construct a sequence $\{\psi_n\}_{n=1}^\infty$ in $C_0^\infty(\mathbb{R})$ such that

- $\|\psi_n\|_2 = 1 \;\; \forall n \in \mathbb{N}$
- and $\lim\limits_{n\to\infty} \|(H-E)\psi_n\|_2 = 0\,.$

SOLUTION:

SOLUTION TO PROBLEM 8 (CONTINUATION):

Name

PROBLEM 9. (16 marks)

Consider the Hamiltonian $H = -\Delta - V$ acting on $L^2(\mathbb{R}^3)$, where

$$V(\mathbf{x}) \;=\; Z\,|\mathbf{x}|^{-1}\,e^{-|\mathbf{x}|}\,, \qquad \mathbf{x} \neq 0\,,$$

and Z is a positive parameter. (You may think of H as the Hamiltonian of an hydrogenic atom where the Coulomb interaction is exponentially suppressed at large distances.)

(i) Prove that there exists a universal constant $Z_0 > 0$ such that if $Z < Z_0$ then the ground state energy E_0 of H is non negative.

(ii) Prove that there exists a universal constant $C_0 > 0$ such that the ground state energy $E_0^f(N)$ of a system of N non-interacting spinless fermions, each subject to the same potential V, is bounded below by $E_0^f(N) \geqslant -C_0\,Z^{5/2}$ uniformly in N.

SOLUTION:

SOLUTION TO PROBLEM 9 (CONTINUATION):