Mathematical Quantum Mechanics

TMP Programme Munich – winter term 2012/2013

PROBLEMS IN CLASS – Supplementary problems on self-adjoint operators and spectral theorem.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Problem 14. (Momentum operator on $[0, 2\pi]$.)

Consider the operators A_0 and A on the Hilbert space $L^2[0, 2\pi]$ given by

$$A_0 f = -\mathrm{i} f' , \qquad \mathcal{D}(A_0) = \{ f \in C^1([0, 2\pi]) \,|\, f(0) = f(2\pi) = 0 \} ,$$
$$A f = -\mathrm{i} f' , \qquad \mathcal{D}(A) = \{ f \in C^1([0, 2\pi]) \,|\, f(0) = f(2\pi) \} .$$

(i) Show that both A_0 and A are symmetric and that $A_0 \subset A$.

(ii) Find A_0^*.

(iii) Find A^* and show that A is essentially self-adjoint.

(iv) Show that A_0 has no eigenvalues.

(v) Show that A admits an orthonormal basis of eigenvectors.

Problem 15. (Properties of the adjoint of a densely defined operator.)

Let A and B be two *densely defined* operators on a Hilbert space $\mathcal{H}$. Show the following.

(i) $(\alpha A)^* = \overline{\alpha} A^* \ \forall \alpha \in \mathbb{C}$.

(ii) If $\mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(B^*)$ are dense in $\mathcal{H}$, then $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$, $\mathcal{D}(A^* + B^*) = \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$, and $(A + B)^* \supset A^* + B^*$.

(iii) If $\mathcal{D}(AB)$ is dense, then $(AB)^* \supset B^* A^*$.

(iv) If $A \subset B$ then $A^* \supset B^*$.

(v) If A is self-adjoint, A has no symmetric extension.

(vi) $\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$. (Compare with the bounded case: Problem 13 (i).)

Problem 16. Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. Show the following.

(i) If A is injective and $\operatorname{Ran} A$ is dense in $\mathcal{H}$ then $(A^{-1})^* = (A^*)^{-1}$.

(ii) If A is self-adjoint and injective, then A^{-1} is self-adjoint too.

Problem 17. Let $\mathcal{H}$ be a Hilbert space and let A, B be bounded self-adjoint operators on $\mathcal{H}$.

(i) Assume that $A \leqslant B$. Show that $C^*AC \leqslant C^*BC$ for all $C \in \mathcal{B}(\mathcal{H})$.

(ii) Assume that $\mathbb{O} \leqslant A \leqslant B$. Show that $\|A\| \leqslant \|B\|$.

(iii) Assume that $A \geqslant \mathbb{O}$. Show that A is invertible if and only if $A \geqslant c\mathbb{1}$ for some $c > 0$.

(iv) Assume that $\mathbb{O} \leqslant A \leqslant B$. Show that for every $\lambda > 0$ $A + \lambda\mathbb{1}$ and $B + \lambda\mathbb{1}$ are positive and invertible and $(B + \lambda\mathbb{1})^{-1} \leqslant (A + \lambda\mathbb{1})^{-1}$

(v) Assume that $\mathbb{O} \leqslant A \leqslant B$ and that A is invertible. Show that B is invertible too and $B^{-1} \leqslant A^{-1}$. (*Hint:* (iii) and (iv) above.)

Problem 18. (Absolute value, positive, negative part of a self-adjoint operator: with and without the functional calculus.)

Let $\mathcal{H}$ be a Hilbert space and let $A = A^* \in \mathcal{B}(\mathcal{H})$.

(i) Explain why the operator $|A| = \sqrt{A^*A}$ constructed with Hilbert space techniques (see, e.g., Reed Simon, Theorem VI.9) and the operator $|A|$ constructed by means of the continuous functional calculus are actually the same.

(ii) Show that the limit, in operator norm, of a sequence of positive operators on $\mathcal{H}$ is positive.

(iii) Show that $A_n := 2(\frac{4}{n}\mathbb{1} + (|A|-A)^2)^{-1}(|A|-A)^2|A|$ is bounded and positive in $\mathcal{B}(\mathcal{H})$ for every $n \in \mathbb{N}$. (*Hint:* the operators A, $|A|$, $(\frac{4}{n}\mathbb{1} + (|A|-A)^2)^{-1}$ commute and the latter is positive. No functional calculus argument is needed here, although it would help.)

(iv) Show that $A_n \xrightarrow[n\to\infty]{\|\,\|} |A|-A$ and deduce by (ii) that $A \leqslant |A|$.

(v) Re-prove that $A \leqslant |A|$ using the continuous functional calculus.

(vi) Show that there is a unique pair of positive operators A_+, A_- in $\mathcal{B}(\mathcal{H})$ such that $A_+A_- = \mathbb{O}$ and $A = A_+ - A_-$. (*Hint:* both to prove that $A_+ \geqslant 0$, $A_- \geqslant 0$, and to prove uniqueness, you need $A \leqslant |A|$.)

Problem 19. (Operator monotone functions.)

A continuous, real-valued function f on an interval I is said OPERATOR MONOTONE (on the interval I) if $A \leqslant B \Rightarrow f(A) \leqslant f(B)$ for every bounded, self-adjoint operators A, B on a Hilbert space $\mathcal{H}$ such that $\sigma(A) \subset I$, $\sigma(B) \subset I$.

(i) Show that the function $f_\alpha(t) = \dfrac{t}{(1+\alpha\, t)}$ is operator monotone on $\mathbb{R}^+$ if $\alpha \geqslant 0$.

(ii) Show that the function $f_\alpha(t)$ considered in (i) is operator monotone on $[0, 1]$ if $\alpha \in (-1, 0]$.

(iii) Let A, B be bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $\mathbb{O} \leqslant A \leqslant B$. Show that $\mathbb{O} \leqslant \sqrt{A} \leqslant \sqrt{B}$, in other words, $x \mapsto \sqrt{x}$ is operator monotone on $\mathbb{R}^+$.

(*Hint:* Problem 17 (iv) and the identity $\sqrt{x} = \dfrac{1}{\pi}\displaystyle\int_0^{+\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda}}\Big(1 - \frac{\lambda}{\lambda + x}\Big)$, valid $\forall x \geqslant 0$.)

(iv) Same assumption as in (iii). Show that $\mathbb{O} \leqslant A^\alpha \leqslant B^\alpha \ \forall \alpha \in [0,1]$. (*Hint:* same strategy as in (iii), use now $x^\alpha = \dfrac{\sin \alpha\pi}{\pi} \displaystyle\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1-\alpha}} \frac{x}{\lambda + x}$ valid $\forall x \geqslant 0$, $\forall \alpha \in (0,1)$.)

(v) Produce a counterexample to the conclusion in (iv) when $\alpha > 1$.

Problem 20. (Spectral resolution of the position operator.)

Consider the position operator on $L^2[0,1]$, i.e., the map $A : L^2[0,1] \to L^2[0,1]$, $(A\psi)(x) = x\psi(x)$ a.e. in $[0,1]$. (Recall from homework that $A = A^*$, $\|A\| = 1$, $\sigma(A) = [0,1]$.)

(i) Give the explicit action of the operator $f(A)$ on $L^2[0,1]$ where $f : [0,1] \to \mathbb{C}$ is a given bounded, Borel-measurable function. Use the measurable functional calculus to answer this question (see (iii) below, instead).

(ii) Exhibit the projection-valued measure $\{E_\Omega\}_\Omega$ associated with A, that is, give the explicit action of E_Ω on $L^2[0,1]$ for each Borel set $\Omega \subset \sigma(A)$.

(iii) Conversely, given the projection-valued measure $\{E_\Omega\}_\Omega$ associated with A determined in (ii), construct $f(A)$ (i.e., give its explicit action) using the spectral resolution for A.

Problem 21. (The restriction to a spectral subspace.)

Let $\mathcal{H}$ be a Hilbert space, A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$ and $\{E_\Omega\}_\Omega$ be the projection-valued measure associated with A.

(i) Show that the subspace $\operatorname{ran} E_\Omega$ is invariant under A for any Borel set $\Omega \subset \sigma(A)$.

(ii) Show that if Ω is a closed Borel set in $\sigma(A)$ then $\sigma\big(A\big|_{\operatorname{ran} E_\Omega}\big) \subset \Omega$.

(*Hint:* spectral theorem, multiplication operator form.)

Problem 22. (More applications of spectral theorem: unitary group; norm of the resolvent.)

Let $\mathcal{H}$ be a Hilbert space and A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$.

(i) Show that the operator $U(t) = e^{\mathrm{i}tA}$ constructed with the functional calculus for A is a unitary operator for all $t \in \mathbb{R}$ and that

$$U(t)^* = U(-t)\,, \qquad U(t)U(s) = U(t+s) \qquad \forall t,s \in \mathbb{R}\,.$$

(ii) Prove that the operator-valued function $t \mapsto U(t)$ defined in (i) is differentiable with respect to the operator norm topology and $U'(t) = \mathrm{i}AU(t) = \mathrm{i}U(t)A \ \forall t \in \mathbb{R}$.

(iii) Let $\lambda \notin \sigma(A)$. Show that $\|(\lambda - A)^{-1}\| = \dfrac{1}{d(\lambda, \sigma(A))}$.

Problem 23. Let A be the integral operator on $L^2[0,1]$ defined by $(Af)(x) = \int_0^1 \min(x,y)f(y)\mathrm{d}y$ for a.e. $x \in [0,1]$.

(i) Prove that A is bounded and self-adjoint.

(ii) Reduce A to the form of a multiplication by a function, that is, produce a measure space $(\mathcal{M},\mu)$, an isomorphism $U : L^2[0,1] \to L^2(\mathcal{M},\mathrm{d}\mu)$, and a bounded measurable function $F:\mathcal{M} \to \mathbb{R}$ such that UAU^* acts on $L^2(\mathcal{M},\mathrm{d}\mu)$ as the operator of multiplication by F.

Problem 24. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of densely defined self-adjoint operators on a Hilbert space $\mathcal{H}$ and let A be another self-adjoint operator on $\mathcal{H}$. Assume that

$$\lim_{n\to\infty} \big\| e^{itA_n}\varphi - e^{itA}\varphi \big\| \; = \; 0 \qquad \forall\varphi \in \mathcal{H}\,, \qquad \forall t \in \mathbb{R}\,.$$

Show that

$$\lim_{n\to\infty} \big\| R_z(A_n)\varphi - R_z(A)\varphi \big\| \; = \; 0 \qquad \forall\varphi \in \mathcal{H}$$

where $R_z(A_n) = (z\mathbb{1} - A_n)^{-1}$, $R_z(A) = (z\mathbb{1} - A)^{-1}$ for an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$.
(*Hint:* represent the resolvent $R_z(A)$ with an integral involving e^{itA}, then use the spectral theorem.)

Problem 25.

(i) Let $\{E_\Omega\}_{\Omega\in\Sigma_B(\mathbb{R})}$ be a projection-valued measure on a Hilbert space ($\Sigma_B(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$). Show that the three properties that define $\{E_\Omega\}_{\Omega\in\Sigma_B(\mathbb{R})}$, i.e.,

1. each E_Ω is an orthogonal projection on $\mathcal{H}$,
2. $E_\emptyset = \mathbb{O}$, $E_\mathbb{R} = \mathbb{1}$,
3. $E_{\bigcup_{n=1}^\infty \Omega_n} = \sum_{n=1}^\infty E_{\Omega_n}$ strongly, for pairwise disjoint $A_1, A_2, \dots \in \Sigma_B(\mathbb{R})$,

imply

4. $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_2}E_{\Omega_1} = E_{\Omega_1\cap\Omega_2}$ $\forall \Omega_1,\Omega_2 \in \Sigma_B(\mathbb{R})$.

Problem 26. Consider the following statement for an operator $A = A^* \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space: $\mathbb{O} \leqslant A \leqslant \mathbb{1}$ if and only if $A^2 \leqslant A$.

(i) Prove the statement without the spectral theorem (only with Hilbert space techniques).

(ii) Prove the statement using the spectral theorem.

Problem 27. (Riesz projection.)

Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ and Λ be a non-empty compact subset of $\sigma(A)$. Consider in the complex plane a closed, piecewise smooth, positively oriented curve Γ such that the intersection between $\sigma(A)$ and the region enclosed by Γ is Λ, and also $\Gamma \cap \sigma(A) = \emptyset$. (Hence Λ is separated by a gap from the rest of the spectrum of A.) Show that

$$\chi_{\Lambda}(A) \;=\; \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{1}{z-A}\, \mathrm{d}z$$

where χ_{Λ} is the characteristic function of Λ.

Problem 28. (Stone's theorem.)

Preliminary remark: if A is a densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and $\forall t \in \mathbb{R}$ one defines $U(t) := e^{\mathrm{i}tA}$ with the functional calculus, then $\{U(t)\}_{t\in\mathbb{R}}$ is a unitary group. The proof is the same for bounded or unbounded A, see Problem 19 (i). If A is bounded, such a group is differentiable in the *norm* operator topology, with $U'(t) = \mathrm{i}AU(t)$ (Problem 19 (ii)), whereas if A is unbounded the same proof gives that $\{U(t)\}_{t\in\mathbb{R}}$ is a *strongly* continuous group and $U'(t) = \mathrm{i}AU(t)$ holds in the *strong* operator topology.

Assume now that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group on a Hilbert space $\mathcal{H}$, i.e., each $U(t)$ is unitary, $U(t+s) = U(t)U(s)$ $\forall t,s \in \mathbb{R}$, and $\forall \psi \in \mathcal{H}$ $U(t)\psi \to U(t_0)\psi$ if $t \to t_0$.

(i) Let $\mathcal{D} \subset \mathcal{H}$ be the subspace of all finite linear combinations of vectors $\varphi_f \in \mathcal{H}$ of the form $\varphi_f = \int_{-\infty}^{+\infty} f(t)U(t)\varphi\, \mathrm{d}t$ for some $\varphi \in \mathcal{H}$ and some $f \in C_0^\infty(\mathbb{R})$, where the integral can be taken to be a Riemann integral since $U(t)$ is strongly continuous. Show that $\mathcal{D}$ is dense in $\mathcal{H}$.

(ii) For $\varphi_f \in \mathcal{D}$ define $A\varphi_f := -\mathrm{i}\,\varphi_{-f'}$. Show that A is symmetric.
(*Hint:* compute $\lim_{s\to 0} \frac{U(s)-\mathbb{1}}{s}$ on each φ_f.)

(iii) Show that both A and $U(t)$ leave $\mathcal{D}$ invariant and commute on $\mathcal{D}$.

(iv) Show that A is essentially self-adjoint.
(*Hint:* if u is a solution to $A^*u = \pm \mathrm{i}u$, consider the function $t \mapsto \langle U(t)\varphi, u\rangle$ $\forall \varphi \in \mathcal{D}$.)

(v) Show that $U(t) = e^{\mathrm{i}t\overline{A}}$.
(*Hint:* set $w(t) := U(t)\varphi - V(t)\varphi$, $\varphi \in \mathcal{D}$, where $V(t) := e^{\mathrm{i}t\overline{A}}$. Compute $\frac{d}{dt}\|w(t)\|^2$.)

Problem 29. Let A be a densely defined (possibly unbounded), self-adjoint operator in a Hilbert space $\mathcal{H}$. Denote by $\{E_\Omega\}_\Omega$ the projection-valued measure associated with A.

Let $\psi_1, \dots, \psi_N$ be N linearly independent vectors in the domain of A and let $\mu \in \mathbb{R}$ be such that

$$\langle \psi, A\psi\rangle \;<\; \mu \|\psi\|^2$$

for any non-zero element $\psi \in \mathrm{span}\{\psi_1, \dots, \psi_N\}$.

Show that $\dim \mathrm{R}(E_{(-\infty,\mu]}) \geqslant N$. ($R(T)$ denotes the range of an operator T.)