PROBLEMS IN CLASS – Tutorials of 18 and 19 December 2012
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Problem 11. Recall that an orthogonal projection acting on a Hilbert space $\mathcal{H}$ is an operator $P \in \mathcal{B}(\mathcal{H})$ such that $P = P^* = P^2$. Recall also that in the Hilbert space case the symbol $\oplus$ denotes the orthogonal sum (see the Projection Theorem).

(i) Show that the kernel and the range of an orthogonal projection P are two closed subspaces that decompose $\mathcal{H}$ in the orthogonal decomposition $\mathcal{H} = \mathrm{Ker}P \oplus \mathrm{Ran}P$.

(ii) Conversely, show that if K, R are two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = K \oplus R$ then there exists $P \in \mathcal{B}(\mathcal{H})$ such that P is the orthogonal projection *onto* R.

Assume in the following that P is an orthogonal projection on $\mathcal{H}$ other than the identity.

(iii) Find the point spectrum $\sigma_{\mathrm{p}}(P)$.

(iv) Find spectrum $\sigma(P)$.

(v) For every $\lambda \notin \sigma(P)$ give the explicit action of the resolvent operator $(\lambda \mathbb{1} - P)^{-1}$.

Problem 12. Consider the measurable functions f_0 and g_0 such that $f_0(x) = e^{-x^2}$, $g_0(x) = \dfrac{e^{-|x|}}{|x|^{1/4}}$ and the linear map $f \mapsto Tf$ such that $(Tf)(x) = \Big(\int_{\mathbb{R}} g_0(y)\, f(y)\, \mathrm{d}y \Big) f_0(x)$ for a.e. $x \in \mathbb{R}$.

(i) Show that T is a bounded linear operator on $L^2(\mathbb{R})$.

(ii) Compute $\|T\|$.

(iii) Find the adjoint operator T^* of T.

Problem 13. Let $\mathcal{H}$ be a Hilbert space.

(i) Show that $\mathrm{Ker}\, T^* = (\mathrm{Ran}\, T)^\perp$ for every $T \in \mathcal{B}(\mathcal{H})$.

(ii) Show that $\mathcal{H} = \overline{\mathrm{Ran}\, T} \oplus \mathrm{Ker}\, T^*$ for every $T \in \mathcal{B}(\mathcal{H})$.

(iii) Show that if $N \in \mathcal{B}(\mathcal{H})$ is normal then $\mathrm{Ker}\, N = \mathrm{Ker}\, N^* = (\mathrm{Ran}\, N)^\perp = (\mathrm{Ran}\, N^*)^\perp$.

(A normal operator N is an operator such that $NN^* = N^*N$, i.e., N commutes with its adjoint. Self-adjoint operators, as well as unitary operators, are normal.)