Mathematical Quantum Mechanics

TMP Programme Munich – winter term 2012/2013

HOMEWORK ASSIGNMENT 13
Hand-in deadline: Monday 11 February 2013 by 6 p.m. in the "MQM" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: www.math.lmu.de/~michel/WS12_MQM.html

Exercise 49. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $[0,1]$. Define

$$V(x) := \begin{cases} 0\,, & x \in \mathbb{Q}\cap[0,1] \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{|x-q_n|}}\,, & x\in[0,1]\setminus\mathbb{Q} \end{cases}.$$

(i) Show that $V \in L^1[0,1]$.

(ii) Consider the following self-adjoint operators acting on the Hilbert space $L^2[0,1]$: the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the domain $\mathcal{D}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}) = \{f \in H^2(\mathbb{R})\,|\, f(0) = f(1) = 0\}$, and the operator of multiplication by V on its natural domain $\mathcal{D}(V)$ (see Exercise 39). Show that $\mathcal{D}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V) \equiv \mathcal{D}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2})\cap\mathcal{D}(V) = \{\mathbf{0}\}$ (where $\mathbf{0}$ is the zero function).

(iii) Produce a *dense* subspace of $L^2[0,1]$ where the quadratic form of the energy of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V$ is well defined and finite.

Exercise 50. Let $\mathcal{H}$ be a complex Hilbert space. Let A and B be two bounded and self-adjoint operators acting on $\mathcal{H}$. Denote by $\mathbb{1}$ and $\mathbb{O}$ respectively the identity and the zero operator on $\mathcal{H}$.

(i) Assume that $A \geqslant \mathbb{1}$. Prove that A is invertible and $\mathbb{O} \leqslant A^{-1} \leqslant \mathbb{1}$.

(ii) Assume that $\mathbb{O} \leqslant A \leqslant B$. Prove that both $A+\lambda\mathbb{1}$ and $B+\lambda\mathbb{1}$ are invertible for any $\lambda > 0$ and that $(B+\lambda\mathbb{1})^{-1} \leqslant (A+\lambda\mathbb{1})^{-1}$. (*Hint:* use (i).)

(iii) Assume that $\mathbb{O} \leqslant A \leqslant B$. Prove that $\sqrt{A} \leqslant \sqrt{B}$. (*Hint:* prove and use the identity

$$\sqrt{x} \;=\; \frac{1}{\pi}\int_0^{\infty} \frac{1}{\sqrt{\lambda}}\Big(1-\frac{\lambda}{x+\lambda}\Big)\,\mathrm{d}\lambda \qquad \forall x \geqslant 0\,,$$

then use (ii).)

(iv) Assume that $\mathbb{O} \leqslant A \leqslant B$. Show that this implies neither $A^2 \leqslant B^2$ nor $AB \geqslant \mathbb{O}$ (i.e., disprove both inequalities with counterexamples, you can think of 2×2 matrices), but prove that $\sqrt{A}B\sqrt{A} \geqslant \mathbb{O}$ and $\sqrt{B}A\sqrt{B} \geqslant \mathbb{O}$ are correct.

Exercise 51.

(i) Let $f \in H^1(0,1)$. Prove:

$$\sup_{x\in[0,1]} |f(x)|^2 \leqslant \varepsilon \int_0^1 |f'(t)|^2 \mathrm{d}t + (1+\frac{1}{\varepsilon}) \int_0^1 |f(t)|^2 \mathrm{d}t \qquad \forall \varepsilon > 0\,.$$

(ii) Let $V \in L^2_{\mathrm{loc}}(\mathbb{R})$ be such that $\sup_{n\in\mathbb{Z}} \int_n^{n+1} |V(x)|^2 \mathrm{d}x < \infty$. Prove:

$$\forall \varepsilon > 0 \;\; \exists\, b > 0 \quad \text{such that} \quad \|Vf\|_2 \leqslant \varepsilon \|f''\|_2 + b\|f\|_2 \qquad \forall f \in H^2(\mathbb{R}).$$

Exercise 52. Let A be a densely defined (possibly unbounded), self-adjoint operator in a Hilbert space $\mathcal{H}$. Denote by $\{E_\Omega\}_\Omega$ the projection-valued measure associated with A. Let $\psi_1, \dots, \psi_N$ be N linearly independent vectors in the domain of A and let $\mu \in \mathbb{R}$ be such that

$$\langle \psi, A\psi \rangle \; < \; \mu \|\psi\|^2$$

for any non-zero element $\psi \in \mathrm{span}\{\psi_1, \dots, \psi_N\}$. Prove that $\dim \mathrm{Ran}(E_{(-\infty,\mu]}) \geqslant N$.