Mathematical Quantum Mechanics

TMP Programme Munich – winter term 2012/2013

HOMEWORK ASSIGNMENT 11-12
Hand-in deadline: Tuesday 22 January 2013 by 6 p.m. in the "MQM" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 41. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Prove that

$$T \text{ is compact} \qquad \Leftrightarrow \qquad \begin{cases} \text{for every sequence } \{x_n\}_{n=1}^{\infty} \text{ in } \mathcal{H}, \\ x_n \xrightarrow[n\to\infty]{\text{weakly}} x \;\;\Rightarrow\;\; Tx_n \xrightarrow[n\to\infty]{\|\,\|} Tx\,. \end{cases}$$

(*Hint:* Uniform Boundedness for $\Rightarrow$. Banach-Alaoglu for $\Leftarrow$.)

Exercise 42. Let H and H_0 be two self-adjoint operators on a given Hilbert space $\mathcal{H}$. Consider the corresponding wave operators Ω_+ and Ω_- defined by

$$\mathcal{D}(\Omega_\pm) := \big\{\psi\in\mathcal{H}\,|\,\exists \lim_{t\to\pm\infty} e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}\psi\big\}\,,\qquad \Omega_\pm\psi := \lim_{t\to\pm\infty} e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}\psi\,.$$

(i) Prove that both $\mathcal{D}(\Omega_\pm)$ and $\mathrm{Ran}(\Omega_\pm)$ are closed subspaces of $\mathcal{H}$ and that $\Omega_\pm : \mathcal{D}(\Omega_\pm)\to\mathrm{Ran}(\Omega_\pm)$ is unitary.

(ii) Let $t\in\mathbb{R}$. Prove that $e^{-\mathrm{i}tH_0}$ leaves invariant either subspaces of the decomposition $\mathcal{H} = \mathcal{D}(\Omega_\pm)\oplus\mathcal{D}(\Omega_\pm)^\perp$ and that $e^{-\mathrm{i}tH}$ leaves invariant either subspaces of the decomposition $\mathcal{H} = \mathrm{Ran}(\Omega_\pm)\oplus\mathrm{Ran}(\Omega_\pm)^\perp$.

(iii) Prove that $\Omega_\pm H_0\psi = H\Omega_\pm\psi \;\forall\psi\in\mathcal{D}(\Omega_\pm)\cap\mathcal{D}(H_0)$.
(*Hint:* prove first $\Omega_\pm e^{-\mathrm{i}tH_0}\phi = e^{-\mathrm{i}tH}\Omega_\pm\phi \;\forall\phi\in\mathcal{D}(\Omega_\pm)$. The conclusion then follows from Stone's theorem – this was only stated in class and in the lecture notes, so here you may just proceed under the additional assumption that $\Omega_\pm\phi\in\mathcal{D}(H)$; this extra fact is not needed, it is a consequence of Stone's theorem.)

Exercise 43.

(i) Let H and H_0 be two self-adjoint operators on a given Hilbert space $\mathcal{H}$. Denote with Ω_+ and Ω_- the corresponding wave operators (see Exercise 42). Assume that $\mathcal{D}(H)\subset\mathcal{D}(H_0)$. Prove that

$$t_0\geqslant 0\,,\quad \psi\in\mathcal{D}(H_0)\,,\;\text{ and } \int_{t_0}^{+\infty}\big\|(H-H_0)\,e^{\mp\mathrm{i}tH_0}\psi\big\|\,\mathrm{d}t < +\infty \qquad\Rightarrow\qquad \psi\in\mathcal{D}(\Omega_\pm)\,.$$

(ii) Let $V\in L^2(\mathbb{R}^3)$, real-valued. Consider the self-adjoint operators $H_0=-\Delta$ and $H=-\Delta+V$ on the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^3)$, with domain $\mathcal{D}(H_0)=\mathcal{D}(H)=H^2(\mathbb{R}^3)$. Let Ω_+ and Ω_- be the corresponding wave operators (see Exercise 42). Prove that $\mathcal{D}(\Omega_\pm)=\mathcal{H}$.
(*Hint:* part (i) and the L^∞/L^1-dispersive estimate, with a density argument on ψ.)

Exercise 44. Let $\mathcal{H} = L^2(\mathbb{R})$. For each $t \in \mathbb{R}$ define $U_t : \mathcal{H} \to \mathcal{H}$ by

$$(U_t f)(x) := e^{-t/2} f(e^{-t} x) \quad \text{for a.e. } x\,, \qquad f \in \mathcal{H}.$$

(i) Prove that $\{U_t \,|\, t \in \mathbb{R}\}$ is a strongly continuous, one-parameter unitary group on $\mathcal{H}$.

(ii) Define $D := \text{strong-}\lim_{t\to 0} \dfrac{U_t - \mathbb{1}}{\mathrm{i}\, t}$ on the subspace $C_0^\infty(\mathbb{R})$. Show that D define a symmetric operator and find its explicit action.