HOMEWORK ASSIGNMENT 10
Hand-in deadline: Tuesday 15 January 2013 by 6 p.m. in the "MQM" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 37. Consider the map $V : L^2[0,1] \to L^2[0,1]$, $(Vf)(x) := \int_0^x f(y)\mathrm{d}y$ for a.a. $x \in [0,1]$.

(i) Show that V is bounded.

(ii) Show that V is compact.

(iii) Show that V has *no* eigenvalues.

(iv) Show that $\sigma(V) = \{0\}$.

Exercise 38. Let $d \in \mathbb{N}$. Let A be a compact operator on $L^2(\mathbb{R}^d)$ such that its singular value decomposition[(1)] $A = \sum_j \mu_j |v_j\rangle\langle u_j|$ satisfies $\sum_j \mu_j^2 < \infty$. Show that there exists $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$(Af)(\cdot) = \int_{\mathbb{R}^d} k(\cdot, y) f(y)\,\mathrm{d}y \quad \forall f \in L^2(\mathbb{R}^d) \qquad \text{and} \qquad \|k\|_2^2 = \sum_j \mu_j^2\,.$$

[(1)] See the Section "Crash course on operators on a Hilbert space" of the handout "Many-body quantum dynamics": $\{u_j\}_j$ and $\{v_j\}_j$ are two orthonormal systems of $L^2(\mathbb{R}^d)$, $\{\mu_j\}_j$ are non-negative numbers.

Exercise 39. Let (X, μ) be a measure space. (It is understood that the measure μ is σ-finite.) Let $\phi : X \to \mathbb{C}$ be a (possibly unbounded) measurable function. Consider the linear map M_ϕ on $L^2(X, \mathrm{d}\mu)$ whose domain and action are defined by

$$\begin{aligned} \mathcal{D}(M_\phi) &:= \{f \in L^2(X, \mathrm{d}\mu) \,|\, \phi f \in L^2(X, \mathrm{d}\mu)\} \\ (M_\phi f)(x) &:= \phi(x) f(x) \quad \mu\text{-a.e.} \end{aligned}$$

(i) Show that $\mathcal{D}(M_\phi)$ is dense in $L^2(X, \mathrm{d}\mu)$.

(ii) Show that $M_\phi^* = M_{\overline{\phi}}$ (in particular, M_ϕ is self-adjoint $\Leftrightarrow$ ϕ is real-valued).
(Warning: you need to check the domain, see Def. 7.2 in the handout "Many-body".)

(iii) Define $\operatorname{ess\,ran}\phi := \big\{\lambda \in \mathbb{C} \,|\, \forall \varepsilon > 0 \;\; \mu(\{x \in X \,|\, |\lambda - \phi(x)| < \varepsilon\}) > 0\big\}$, the "essential range" of ϕ. Show that $\sigma(M_\phi) = \operatorname{ess\,ran}\phi$.

(iv) Show that λ is an eigenvalue of M_ϕ $\Leftrightarrow$ $\mu(\{\phi^{-1}(\lambda)\}) > 0$.

Exercise 40. Consider the position and the momentum operators on $\mathbb{R}$, i.e.:

- Q is the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}x)$ with domain $\mathcal{D}(Q)$ given in Ex. 39,
- $P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ on $L^2(\mathbb{R}, \mathrm{d}x)$ with domain $\mathcal{D}(P) = H^1(\mathbb{R}, \mathrm{d}x)$.

(i) Show that Q is self-adjoint, has no eigenvalue, and $\sigma(Q) = \mathbb{R}$.

(ii) Show that P is self-adjoint, has no eigenvalue, and $\sigma(P) = \mathbb{R}$.

(*Hint:* you may either find first the adjoint operator, or apply the results from Ex. 39 above.)