HOMEWORK ASSIGNMENT 09

Hand-in deadline: Tuesday 8 January 2013 by 6 p.m. in the "MQM" drop box.

Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 33. Let $d \in \mathbb{N}$, $d \geqslant 3$. Consider the Hamiltonian $H = -\Delta - V$ in d dimensions, where the potential V is such that $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and $\lim_{|x|\to\infty} V(x) = 0$. Assume that for some $E < 0$ and some $\psi \in H^2(\mathbb{R}^d)$ one has $H\psi = E\psi$ as an identity in $L^2(\mathbb{R}^d)$.

(i) Let $g \in C^2(\mathbb{R}^d)$ be real-valued, bounded, and with bounded first and second order partial derivatives. Prove that

$$\big\langle g\psi, (H-E)\, g\psi \big\rangle \;=\; \langle \psi\,, |\nabla g|^2 \psi\rangle\,.$$

(ii) Pick $\xi \in C^\infty(\mathbb{R},[0,1])$ such that $\xi(t) = 0$ for $t \leqslant \frac{1}{2}$ and $\xi(t) = 1$ for $t \geqslant 1$. Set

$$\chi_{R}(x) := \xi\Big(\frac{|x|}{R}\Big) \qquad x \in \mathbb{R}^d,\ R \geqslant 1,$$

$$f_\varepsilon(x) := \frac{\beta|x|}{1+\varepsilon|x|} \qquad x \in \mathbb{R}^d,\ \varepsilon > 0,\ \beta \in (0, \sqrt{-E}\,)\,.$$

Deduce from (i) that for every sufficiently large $R \geqslant 1$ there are constants $\delta_R, C_{\beta,R} > 0$ such that

$$\big\|\, \chi_R\, e^{f_\varepsilon}\, \psi \,\big\|_2^2 \;\leqslant\; \frac{C_{\beta,R}}{-E-\beta^2-\delta_R}\,\|\psi\|_2^2 \qquad \forall \varepsilon > 0\,.$$

(Notice that $\delta_R \xrightarrow{R\to\infty} 0$ and $C_{\beta,R}$ is bounded in β as $\beta \to 0$.)

(iii) Deduce from (ii) that for every sufficiently large $R \geqslant 1$

$$\big\|\, \mathbb{1}_{\{|x|\geqslant R\}}\, e^{\beta|x|}\, \psi \,\big\|_2^2 \;\leqslant\; \frac{C_{\beta,R}}{-E-\beta^2-\delta_R}\,\|\psi\|_2^2$$

(i.e., the bound state ψ is *exponentially localised*).

Exercise 34. Consider the three-dimensional system consisting of two fixed nuclei each with charge Z, placed at a distance R apart, and 2 electrons subject to their mutual repulsion and to the attraction of the nuclei. The spin of the particles, the nucleus-nucleus repulsion, and the fermionic symmetry shall be neglected in this problem. Let $E_{\mathrm{GS}}(R)$ be the ground state energy of such a system. Prove that

$$\lim_{R\to\infty} E_{\mathrm{GS}}(R) \;=\; -\frac{Z^2}{2}\,.$$

(*Hint:* a good trial function for the upper bound, an IMS-type localisation in both variables for the lower bound, i.e., write $1 = \chi_0^2(x_j) + \chi_R^2(x_j) + \overline{\chi}^2(x_j)$, $j = 1, 2$, for suitable χ_0, χ_R, $\overline{\chi}$.)

Exercise 35.

(i) Let $d \in \mathbb{R}$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a measurable function. Set

$$k_1(x) := \int_{\mathbb{R}^3} |k(x,y)|\mathrm{d}y \quad \text{for a.e.} x \in \mathbb{R}^d\,, \qquad k_2(y) := \int_{\mathbb{R}^3} |k(x,y)|\mathrm{d}x \quad \text{for a.e.} y \in \mathbb{R}^d\,.$$

Assume that $k_1, k_2 \in L^\infty(\mathbb{R}^d)$. Prove that $(Af)(x) := \int_{\mathbb{R}^d} k(x,y)f(y)\mathrm{d}y$, $f \in L^2(\mathbb{R}^d)$, defines a linear bounded operator $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with $\|A\| \leqslant \|k_1\|_\infty^{1/2}\|k_2\|_\infty^{1/2}$.

(ii) Let $k(x,y) := \pi^{-4}(x^2+y^2+1)^{-2}$, $x, y \in \mathbb{R}^3$. Define A as in part (i). Can the operator A have an eigenvalue 1? Justify your answer.

Exercise 36. Let $d \in \mathbb{R}$. Let $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Set

$$(Af)(x) := \int_{\mathbb{R}^d} k(x,y)f(y)\mathrm{d}y \qquad \text{for a.e. } x \in \mathbb{R}^d\,, \quad f \in L^2(\mathbb{R}^d)\,. \tag{$*$}$$

(i) Show that $(*)$ defines a bounded operator $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with $\|A\| \leqslant \|k\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$.

(ii) Produce a sequence $\{A_n\}_{n=1}^\infty$ of finite rank operators in $\mathcal{B}(L^2(\mathbb{R}^d))$ such that $A_n \xrightarrow{n\to\infty} A$ in the operator norm. (Recall: "finite rank" means that the image is finite-dimensional.)

(iii) Let $\{f_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. Show that

$$\sum_{n=1}^\infty \|Af_n\|_{L^2(\mathbb{R}^d)}^2 \;=\; \|k\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}^2$$

irrespectively of the choice of the orthonormal basis $\{f_n\}_{n=1}^\infty$.