**HOMEWORK ASSIGNMENT 06**

**Hand-in deadline:** Tuesday 4 December 2012 by 6 p.m. in the "MQM" drop box.

**Rules:** Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

**Info:** `www.math.lmu.de/~michel/WS12_MQM.html`

**Exercise 21.** Let $d \in \mathbb{N}$.

(i) Construct two functions $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^d)$ with the following properties $\forall x \in \mathbb{R}^d$: $0 \leqslant \chi_j(x) \leqslant 1$, $j = 1, 2$, $\chi_1(x) = 1$ if $|x| \leqslant 1$, $\chi_1(x) = 0$ if $|x| \geqslant 2$, and $\chi_1^2(x) + \chi_2^2(x) = 1$.

(ii) Let $\{\chi_j\}_{j=1}^M$ ($M \in \mathbb{N}$) be a family of bounded functions in $C^\infty(\mathbb{R}^d)$ such that $\sum_{j=1}^M \chi_j^2(x) = 1$ $\forall x \in \mathbb{R}^d$. Prove the following identity of operators on $\mathcal{S}(\mathbb{R}^d)$:

$$-\Delta \;=\; \sum_{j=1}^{M} \big(\chi_j(-\Delta)\chi_j - |\nabla \chi_j|^2\big)\,.$$

**Exercise 22.** Consider the Hamiltonian $H = -\Delta + V$ in $d$ dimensions and its ground state energy

$$E_0 \;=\; \inf_{\substack{\|\psi\|_2=1\\ \psi\in\mathcal{M}}} \left[\int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)|\psi(x)|^2\,\mathrm{d}x\right]$$

with $\mathcal{M} := H^1(\mathbb{R}^d) \cap \{\psi \,|\, \int V_- |\psi|^2 \mathrm{d}x < \infty\}$. The potential $V$ is assumed not to vanish almost everywhere.

(i) Let $d \geqslant 3$. Assume that $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and $|\{x \in \mathbb{R}^d \text{ s.t. } |V(x)| \geqslant \varepsilon\}| < \infty$ $\forall \varepsilon > 0$ (no assumption on the sign of $V$). $|\Omega|$ denotes the Lebesgue measure of the set $\Omega$. Prove that $E_0 \leqslant 0$.

(ii) Assume that $V \in L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ for some $\varepsilon > 0$ and that $V(x) \leqslant 0$. Prove that $E_0 < 0$. (*Hint:* a convenient trial function that involves logarithm.)

**Exercise 23.** Consider the following two Hamiltonians in three dimensions and the corresponding ground state energies ($\mathbf{R}$ is a fixed parameter in $\mathbb{R}^3$):

$$\begin{cases} H &= -\Delta - \dfrac{1}{|\mathbf{x}|} \\ \\ E_{\mathrm{GS}} &= \inf\limits_{\substack{\psi\in H^1(\mathbb{R}^3) \\ \|\psi\|_2=1}} \langle \psi, H\psi\rangle \end{cases} , \qquad \begin{cases} H^{(\mathbf{R})} &= -\Delta - \dfrac{1}{|\mathbf{x}|} - \dfrac{1}{|\mathbf{x}-\mathbf{R}|} \\ \\ E^{(\mathbf{R})}_{\mathrm{GS}} &= \inf\limits_{\substack{\psi\in H^1(\mathbb{R}^3) \\ \|\psi\|_2=1}} \langle \psi, H^{(\mathbf{R})}\psi\rangle \end{cases} .$$

(i) Prove that

$$E^{(\mathbf{R})}_{\mathrm{GS}} \leqslant E_{\mathrm{GS}} - \frac{1}{2}\,e^{-|\mathbf{R}|} \qquad \forall\,\mathbf{R}\in\mathbb{R}^3$$

(*Hint:* ground state wave-function of the Hydrogen atom as a trial function.)

(ii) Prove that there exists constants $c, r > 0$ such that

$$E^{(\mathbf{R})}_{\mathrm{GS}} \geqslant E_{\mathrm{GS}} - \frac{c}{|\mathbf{R}|}\,, \qquad |\mathbf{R}| \geqslant r.$$

(*Hint:* Exercise 21(ii).)

**Exercise 24.** Consider the Schrödinger Hamiltonian $H = -\Delta + V$ in $d$ dimensions. Assume that $V(\lambda x) = \frac{1}{\lambda}V(x)$ $\forall\lambda > 0$ and $\forall x \in \mathbb{R}^d$ (this is the case, for instance, for $V(x) = \frac{c}{|x|}$). Let $\psi \in L^2(\mathbb{R}^d)$, $\|\psi\|_2 = 1$, such that $\Delta\psi \in L^2(\mathbb{R}^d)$, $V\psi \in L^2(\mathbb{R}^d)$, and $H\psi = E\psi$ for some $E \in \mathbb{R}$, the equality being in the sense of $L^2$ functions. Prove that

$$E = -\langle\psi, (-\Delta)\psi\rangle = \frac{1}{2}\langle\psi, V\psi\rangle$$

and that therefore $E \leqslant 0$.

(*Hint:* introduce $U_\lambda : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $(U_\lambda\phi)(\cdot) := \lambda^{-d/2}\phi(\cdot/\lambda)$ and check that both $U_\lambda H\psi$ and $HU_\lambda\psi$ belong to $L^2(\mathbb{R}^d)$. Use this to compute the expectation of $(U_\lambda H - HU_\lambda)$ in the state $\psi$ when $\lambda \to 1$.)