HOMEWORK ASSIGNMENT 05

Hand-in deadline: Tuesday 27 November 2012 by 6 p.m. in the "MQM" drop box.

Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 17. Let $d \in \mathbb{N}$. Consider the free Schrödinger evolution operator

$$U_t \;:=\; e^{\mathrm{i}t\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad t \in \mathbb{R},$$

defined in Exercise 3.

(i) Prove that $(U_t)_{t\in\mathbb{R}}$ is a strongly continuous, not norm continuous, unitary group on $L^2(\mathbb{R}^d)$.

(ii) Prove that $U_t \xrightarrow{t\to\infty} \mathbb{O}$ weakly in the sense of operators.

Exercise 18. Let $\mathcal{A}$ be a C^*-algebra with identity.

(i) Let $A \in \mathcal{A}$ and let $\sigma(A)$ denote the spectrum of A. Show that $\sigma(A)$ is a compact subset of $\mathbb{C}$ contained in the circle of radius $\|A\|$ centred at the origin.
(*Hint:* expand the function $\mathbb{C} \ni \lambda \mapsto (\lambda - a)^{-1}$ around a fixed λ_0 and around infinity.)

(ii) Let A be a normal element of $\mathcal{A}$ (i.e., $AA^* = A^*A$). Prove that $\|A^n\| = \|A\|^n \ \forall n \in \mathbb{N}$.

(iii) Assume that $\mathcal{A}$ is non-commutative. Prove that the only scalar commutator in $\mathcal{A}$, i.e., the only possible identity $QP - PQ = \alpha\mathbb{1}$ for some $Q, P \in \mathcal{A}$, is the case $\alpha = 0$.

Exercise 19. Let $\mathcal{A}$ be a C^*-algebra with identity.

(i) Let $\pi : \mathcal{A} \to \mathcal{M}_2(\mathbb{C})$ (where $\mathcal{M}_2(\mathbb{C})$ is the 2×2 matrices with complex entries) be a representation of $\mathcal{A}$ on $\mathbb{C}^2$ such that

$$\big\{\, M \in \mathcal{M}_2(\mathbb{C}) \,|\, M\pi(A) = \pi(A)M \ \ \forall A \in \mathcal{A} \,\big\} \;=\; \Big\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,|\, \lambda \in \mathbb{C} \Big\}\,.$$

Let $\mathbf{x} \in \mathbb{C}^2$, $\mathbf{x} \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Prove that $\mathrm{Span}\big\{\pi(A)\mathbf{x} \,|\, A \in \mathcal{A}\big\}$ is dense in $\mathbb{C}^2$.

(ii) Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. ($\mathcal{B}(\mathcal{H})$ is the C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ equipped with the usual $*$-algebraic and normed space structure.) Denote scalar product and norm in $\mathcal{H}$ by $\langle\, , \,\rangle$ and $\|\ \|$ respectively. Given $\psi \in \mathcal{H}$ consider the map $\omega_\psi : \mathcal{A} \to \mathbb{C}$, $\omega_\psi(A) := \langle \psi, \pi(A)\psi\rangle$. Prove that ω_ψ is a bounded linear map with norm $\|\psi\|^2$.

Exercise 20.

Let $\mathcal{H} = \mathbb{C}^2$ be the Hilbert space of polarisation states in the x, y plane of a photon flying along the z axis and denote by $|x\rangle := \binom{1}{0}$, resp. $|y\rangle := \binom{0}{1}$, the state of polarisation in the positive x direction, resp. y-direction. Let $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\sigma_0 := \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (the Pauli matrices on $\mathbb{C}^2$).

(i) Let $P_\theta : \mathcal{H} \to \mathcal{H}$, $\theta \in [0, 2\pi)$, be the orthogonal projection onto the state of polarisation in the direction $(\cos\theta, \sin\theta)$ and let $\sigma(\theta) := 2P_\theta - \mathbb{1}$. Prove that

$$\sigma(\theta) = \sigma_3 \cos 2\theta + \sigma_1 \sin 2\theta \,.$$

and that $\sigma(\theta)$ has only eigenvalues ± 1.

(ii) Consider an EPR pair of transverse photons flying apart in opposite directions along the z axis in the polarisation state $\Psi_{\mathrm{EPR}} := \frac{1}{\sqrt{2}}(|x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle) \in \mathcal{H} \otimes \mathcal{H}$. Prove that

$$\langle \alpha\beta \rangle \;:=\; \langle\, \Psi_{\mathrm{EPR}} \,|\, \sigma(\alpha) \otimes \sigma(\beta) \,|\, \Psi_{\mathrm{EPR}} \,\rangle \;=\; \cos 2(\alpha - \beta)\,.$$

(iii) Recall that Bell's inequality relative to the simultaneous measurement of polarisation in the state Ψ_{EPR} considered in (ii) reads

$$|\, \langle \alpha\beta \rangle - \langle \alpha\gamma \rangle \,| \;\leqslant\; 1 - \langle \beta\gamma \rangle\,.$$

Find angles α, β, and γ such that Bell's inequality is violated.