HOMEWORK ASSIGNMENT 04
Hand-in deadline: Tuesday 20 November 2012 by 6 p.m. in the "MQM" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 13.

Let $p \in [1, +\infty)$. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^p(\mathbb{R})$ and let $f \in L^p(\mathbb{R})$.

(i) Assume that $\displaystyle\int_{\mathbb{R}} f_n(x)g(x)\,\mathrm{d}x \xrightarrow{n\to\infty} \int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x \quad \forall g \in C_0^\infty(\mathbb{R})$. Is it true that $f_n \rightharpoonup f$ as $n \to \infty$ weakly in $L^p(\mathbb{R})$? Give a proof or a counterexample (applicable to a generic p).

(iii) Fix now $p = 2$. Assume that, as $n \to \infty$, $f_n \rightharpoonup f$ weakly in $L^2(\mathbb{R})$ and $\|f_n\|_2 \to \|f\|_2$. Prove that $f_n \to f$ in the L^2-norm sense.

Exercise 14.

(i) Let $(f_n)_{n=1}^{\infty}$ be a sequence in $H^1(\mathbb{R})$ and let $f, g \in L^2(\mathbb{R})$ be such that $f_n \rightharpoonup f$ and $f_n' \rightharpoonup g$ weakly in $L^2(\mathbb{R})$ as $n \to \infty$. (Here f_n' is the weak derivative of f_n.) Prove that $f \in H^1(\mathbb{R})$ and that $f' = g$.

(ii) Set $f_n(x) := n^{-\frac{1}{4}} x^{\frac{1}{2}+\frac{1}{2n}}$, $n \in \mathbb{N}$, $x \in [0, 1]$. In which of the following senses does the sequence $(f_n)_{n=1}^{\infty}$ converge and, if it does, what is the limit?

- In norm in $L^2[0, 1]$.
- In norm in $H^1(0, 1)$.
- Weakly in $L^2[0, 1]$.
- Weakly in $H^1(0, 1)$.

Exercise 15.

Consider the following functionals acting on the given Banach space:

$$\Phi_1[f] := \int_0^{2\pi} |f(x)|^4 \mathrm{d}x - \mathrm{i}\int_0^{2\pi} |f(x)|^3 \mathrm{d}x \qquad \text{on the space } L^6[0, 2\pi],$$

$$\Phi_2[f] := \int_0^{2\pi} f(x)\,\mathrm{d}x \qquad \text{on the space } L^p[0, 2\pi],\ p \in [1, +\infty),$$

$$\Phi_3[f] := \left(\int_{-\infty}^{+\infty} |f(x)|^2\,\mathrm{d}x\right)^{1/2} \qquad \text{on the space } L^2(\mathbb{R}),$$

$$\Phi_4[f] := \begin{cases} 1 & \text{if } \|f\|_p = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{on the space } L^p(\mathbb{R}),\ p \in [1, +\infty).$$

(i) Decide in each case if the functional is norm-continuous in the given space.

(ii) Decide in each case if the functional is weakly continuous in the given space.

Exercise 16.

Let $d \in \mathbb{N}$.

(i) Take $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Consider the free Schrödinger evolution $e^{\mathrm{i}t\Delta}\psi$ of ψ (discussed in Exercise 3.(i)). Prove that

$$\|e^{\mathrm{i}t\Delta}\psi\|_2 \;=\; \|\psi\|_2 \qquad \text{and} \qquad \|e^{\mathrm{i}t\Delta}\psi\|_\infty \leqslant \frac{1}{(4\pi t)^{d/2}}\,\|\psi\|_1 \qquad \forall t > 0.$$

(ii) Prove that for every $t > 0$ the operator $e^{\mathrm{i}t\Delta}$ extends uniquely to a bounded linear operator $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, $p \in [1, 2]$, $p^{-1} + q^{-1} = 1$, with

$$\|e^{\mathrm{i}t\Delta}\psi\|_q \;\leqslant\; \frac{1}{(4\pi t)^{d(\frac{1}{2}-\frac{1}{q})}}\,\|\psi\|_p \qquad \forall \psi \in L^p(\mathbb{R}^d)\,.$$

(*Hint:* use the Riesz-Thorin interpolation theorem stated in class.)