HOMEWORK ASSIGNMENT 03
Hand-in deadline: Tuesday 13 November 2012 by 6 p.m. in the "MQM" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 9. Consider the distribution δ in $\mathcal{D}'(\mathbb{R})$ centred at zero. The distribution $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\delta$ is also denoted by $\delta^{(k)}$. For $k = 1, 2, 3$ the notation δ', δ'', δ''' (respectively) is also used. $\delta^{(0)} = \delta$.

(i) Let $f \in C^\infty(\mathbb{R})$. Prove the following identity in $\mathcal{D}'(\mathbb{R})$:

$$f\delta''' \;=\; -f'''(0)\delta + 3f''(0)\delta' - 3f'(0)\delta'' + f(0)\delta'''.$$

(ii) Show that the general solution to the distributional equation

$$x^2 T \;=\; 0 \qquad\qquad (T \in \mathcal{D}'(\mathbb{R}))$$

is $T = c_0\delta + c_1\delta'$, $c_0, c_1 \in \mathbb{C}$.

(iii) Determine all solutions $T \in \mathcal{D}'(\mathbb{R})$ to the distributional differential equation

$$T' \;=\; \delta_1 - \delta_{-1} + x^{2012}\,\mathrm{sgn}(x)\mathbb{1}_{\{|x|\geqslant 1\}}$$

where δ_a is the delta distribution at the point a and $\mathrm{sgn}(x)$ is the sign of x.

Exercise 10. Let $d \in \mathbb{N}$, $p, q \in \mathbb{R}^d$, and $\theta > 0$. Consider the coherent state (see Ex. 1.(ii))

$$\psi_{q,p,\theta}(x) \;:=\; \frac{1}{(\theta\sqrt{\pi})^{d/2}}\, e^{ipx}\, e^{-\frac{|x-q|^2}{2\theta^2}}\,, \qquad x \in \mathbb{R}^d\,.$$

(i) Let $t \in \mathbb{R}$. Compute $e^{\mathrm{i}t\Delta}\psi_{q,p,\theta}$ explicitly using the kernel of the free Schrödinger evolution determined in Ex. 3.(i).

(ii) Prove that, apart from an irrelevant (possibly x-dependent) phase factor, $e^{\mathrm{i}t\Delta}\psi_{q,p,\theta}$ is still a coherent state of the form $\psi_{q(t),p(t),\theta(t)}(x)$ where $q(t) = q + 2pt$, $p(t) = p$, and $\theta(t) = \sqrt{\theta^2 + 4t^2/\theta^2}\,$.

Exercise 11. Let $m > 0$. Consider the measurable functions G and G_m on $\mathbb{R}^3$ defined by

$$G(x) \;:=\; \frac{1}{4\pi|x|}\,, \qquad G_m(x) \;:=\; \left(\frac{1}{m^2+(2\pi\,\cdot)^2}\right)^{\vee}(x)$$

($f^{\vee}$ denotes the inverse Fourier transform of f).

(i) Prove that $\dfrac{G_m(x)}{G(x)} \xrightarrow{|x|\to 0} 1$.

(ii) Prove that $-\dfrac{\ln G_m(x)}{m|x|} \xrightarrow{|x|\to\infty} 1$.

(iii) Prove that $(-\Delta+m^2)G_m = \delta$ as an identity of distributions in $\mathcal{D}'(\mathbb{R}^3)$.

Exercise 12.

(i) Prove that for every $\varphi \in C_0^\infty(\mathbb{R})$ the limit $\lim_{\varepsilon\downarrow 0}\displaystyle\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{1}{x}\,\varphi(x)\,\mathrm{d}x$ exists and is finite.

(ii) Show that the linear map $\mathrm{PV}(\frac{1}{x}) : C_0^\infty(\mathbb{R}) \to \mathbb{C}$ defined by

$$\mathrm{PV}\Big(\frac{1}{x}\Big)(\varphi) \;:=\; \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{1}{x}\,\varphi(x)\,\mathrm{d}x$$

is continuous in the topology of the space of test functions $\mathcal{D}(\mathbb{R})$, and therefore is a distribution.

(iii) Prove the following identities in $\mathcal{D}'(\mathbb{R})$:

$$\frac{1}{x\pm\mathrm{i}0} \;:=\; \lim_{\varepsilon\downarrow 0}\frac{1}{x\pm\mathrm{i}\varepsilon} \;=\; \mathrm{PV}\Big(\frac{1}{x}\Big) \mp \mathrm{i}\pi\delta_0$$

where δ_0 is the delta distribution centred at the origin.