Mathematical Quantum Mechanics

TMP Programme Munich – winter term 2012/2013

HOMEWORK ASSIGNMENT 02
Hand-in deadline: Tuesday 6 November 2012 by 6 p.m. in the "MQM" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Exercise 5.

(i) Prove for all $\psi \in \mathcal{S}(\mathbb{R}^3)$ that

$$\begin{aligned} \pi\int_{\mathbb{R}^3}\frac{1}{|x|}\,\psi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^3}\frac{1}{|k|^2}\,\widehat{\psi}(k)\,\mathrm{d}k\,,\\ \int_{\mathbb{R}^3}\frac{1}{|x|^2}\,\psi(x)\,\mathrm{d}x &= \pi\int_{\mathbb{R}^3}\frac{1}{|k|}\,\widehat{\psi}(k)\,\mathrm{d}k\,. \end{aligned}$$

(*Hint:* it is enough you to prove *only one* of the two identities above, because...)

(ii) Consider the three-dimensional hydrogenic Hamiltonian $H^{(Z)} = -\Delta - \dfrac{Z}{|x|}$ $(Z > 0)$. Recall from class that its ground state energy is

$$\mathcal{E}_{\mathrm{GS}}(Z) \;:=\; \inf_{\substack{\psi\in\mathcal{M}\\ \|\psi\|_2=1}} \langle\psi, H^{(Z)}\psi\rangle \;=\; -\frac{Z^2}{4}$$

where $\mathcal{M} = \{\psi\,|\,\psi, \nabla\psi, |\cdot|^{-1/2}\psi \in L^2(\mathbb{R}^3)\}$. Estimate $\mathcal{E}_{\mathrm{GS}}(Z)$ from above by means of the trial functions

$$\psi_{q,p,\theta}(x) \;:=\; \frac{1}{(\theta\sqrt{\pi})^{3/2}}\,e^{ipx}\,e^{-\frac{|x-q|^2}{2\theta^2}}\,,\qquad x\in\mathbb{R}^3\,,$$

$q,p \in \mathbb{R}^3$, $\theta > 0$ (i.e., the "coherent states" determined in Exercise 1.(ii)) and prove that

$$\inf_{q,p\in\mathbb{R}^3,\theta>0}\langle\psi_{q,p,\theta}, H^{(Z)}\psi_{q,p,\theta}\rangle \;=\; -\frac{2Z^2}{3\pi}\,.$$

Exercise 6. Decide which of the following sequences in $L^1_{\mathrm{loc}}(\mathbb{R})$ converge in $\mathcal{D}'(\mathbb{R})$ and compute the limit when it exists.

(i) $\{f_n\}_{n=1}^\infty$ with $f_n(x) := \dfrac{n}{\pi(1+n^2x^2)}$

(ii) $\{g_n\}_{n=1}^\infty$ with $g_n(x) := \dfrac{\sin nx}{\pi x}$

(iii) $\{h_n\}_{n=1}^\infty$ with $h_n(x) := n^2 x\cos(nx)$

(iv) $\{k_n\}_{n=1}^\infty$ with $k_n(x) := \Big(\dfrac{n^2}{1+n^2(nx-1)^2}\Big)^2$.

Exercise 7. Let $d \in \mathbb{N}$.

(i) Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that

$$|\psi(x)| \leqslant \frac{C}{(1+|x|)^\alpha} \qquad \forall x \in \mathbb{R}^d$$

for some constants $C > 0$, $\alpha > d$. Prove that $\widehat{\psi} \in C^k(\mathbb{R}^d)$ for every integer $k < \alpha - d$.

(ii) Let $\psi \in C_0^\infty(\mathbb{R})$, $\psi \not\equiv 0$. Prove that $\widehat{\psi} \notin C_0^\infty(\mathbb{R})$.

(*Hint:* go for a contradiction and extend $\widehat{\psi}(k)$ to the whole $\mathbb{C}$-plane.)

Exercise 8. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$, $d \in \mathbb{N}$, $d \geqslant 3$, and for every $x \equiv (x_1, \dots, x_d) \in \mathbb{R}^d$ define

$$h_\varepsilon(x) := \Big(\frac{x_1}{\varepsilon + |x|^2} |\psi(x)|^2, \dots, \frac{x_d}{\varepsilon + |x|^2} |\psi(x)|^2 \Big), \qquad \varepsilon > 0\,.$$

Apply the divergence theorem from Calculus to the vector field h_ε (in particular: show that such a theorem is applicable) in a convenient way so to prove Hardy's inequality

$$\int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x \leqslant \Big(\frac{2}{d-2}\Big)^2 \int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,\mathrm{d}x\,.$$

Optional (not needed for the mark): modify the definition of h_ε so to prove, along the same line,

$$\int_{\mathbb{R}^d} \frac{|\psi(x)|^p}{|x|^p}\,\mathrm{d}x \leqslant \Big(\frac{p}{d-p}\Big)^p \int_{\mathbb{R}^d} |\nabla\psi(x)|^p\,\mathrm{d}x\,, \qquad 1 < p < d\,.$$