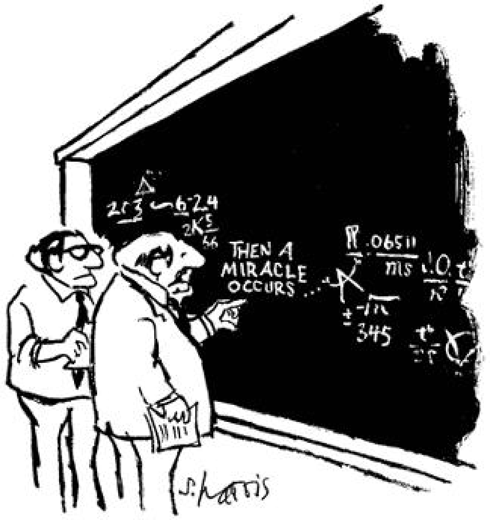
HOMEWORK ASSIGNMENT 01

Hand-in deadline: Tuesday 30 October 2012 by 6 p.m. in the "MQM" drop box.

Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/WS12_MQM.htm`



"I think you should be more explicit here in step two."

Exercise 1. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ ($d \in \mathbb{N}$) with $\|\psi\|_2 = 1$. Let $q, p \in \mathbb{R}^d$.

(i) Show that

$$\Big(\int_{\mathbb{R}^d} (x-q)^2\,|\psi(x)|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}^d} (k-p)^2\,|\widehat{\psi}(k)|^2\,\mathrm{d}k\Big) \;\geqslant\; \frac{d^2}{16\,\pi^2}\,.$$

Here $\widehat{\psi}$ denotes the Fourier transform of ψ.[1]

(ii) Find all ψ's in $\mathcal{S}(\mathbb{R}^d)$ such that

$$\|\psi\|_2 = 1\,, \qquad \langle \psi, x\psi\rangle = q\,, \qquad \langle \psi, -\mathrm{i}\nabla_x \psi\rangle = 2\pi p\,,$$

and such that the inequality in (i) becomes an equality. Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^d)$.

Exercise 2. Let A be a symmetric, positive definite $d \times d$ real matrix and let $b \in \mathbb{R}^d$ ($d \in \mathbb{N}$). Define $f : \mathbb{R}^d \to \mathbb{R}$ by

$$f(x) \;:=\; \exp(-x\cdot Ax + b\cdot x)\,, \qquad x \in \mathbb{R}^d\,,$$

Show that the Fourier transform $\widehat{f}$ of f is given by

$$\widehat{f}(k) \;=\; \frac{\pi^{d/2}}{\sqrt{\det A}}\exp\Big(-\frac{1}{4}(2\pi k + \mathrm{i}b)\cdot A^{-1}(2\pi k + \mathrm{i}b)\Big)\,, \qquad k \in \mathbb{R}^d\,.$$

Here $a \cdot c$ means $\sum_{j=1}^d a_j c_j$ for $a = (a_1, \dots, a_d), c = (c_1, \dots, c_d) \in \mathbb{C}^d$ and Ax is the usual row times column product.

[1] Recall that the convention for this course is $\widehat{\psi}(k) = \int e^{-2\pi \mathrm{i} k\cdot x} f(x)\,\mathrm{d}x$.

Exercise 3. For every $t \in \mathbb{R}$ define the bounded linear operator $e^{\mathrm{i}t\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$e^{\mathrm{i}t\Delta} \;:=\; \mathcal{F}^{-1} \circ \mathcal{M}_{\exp(-\mathrm{i}t|2\pi k|^2)} \circ \mathcal{F}$$

where $d \in \mathbb{N}$, $\mathcal{F}$ is the Fourier transform operator on $L^2(\mathbb{R}^d)$, and $\mathcal{M}_f$ is the multiplication operator by the function f.

(i) Prove that $\forall \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ one has

$$(e^{\mathrm{i}t\Delta}\psi)(x) \;=\; e^{-\mathrm{i}\frac{\pi}{4}d} \int_{\mathbb{R}^d} \frac{e^{\mathrm{i}\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}}\, \psi(y)\, \mathrm{d}y$$

for (Lebesgue-)almost every $x \in \mathbb{R}^d$.

(ii) Using (i) prove that

$$\Bigg\| \, e^{\mathrm{i}t\Delta}\psi \;-\; e^{-\mathrm{i}\frac{\pi}{4}d} \int_{\mathbb{R}^d} \frac{e^{\mathrm{i}\frac{|x-y|^2}{4t}}\; e^{-\frac{|y|^2}{R^2}}}{(4\pi t)^{d/2}}\, \psi(y)\, \mathrm{d}y \,\Bigg\|_{L^2} \xrightarrow{R\to\infty} 0 \qquad \forall \psi \in L^2(\mathbb{R}^d)\,.$$

(iii) Prove that, $\forall \psi \in \mathcal{S}(\mathbb{R}^d)$, $u(t,x) := (e^{\mathrm{i}t\Delta}\psi)(x)$ is a solution to

$$\mathrm{i}\partial_t u(t,x) \;=\; -\Delta_x u(t,x)$$

in the class $C^\infty((\mathbb{R}\backslash\{0\})_t \times \mathbb{R}^d_x)$.

Exercise 4. Let $d \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R}^d)$.

(i) Show that

$$u(x) \;:=\; \frac{1}{(4\pi)^{d/2}} \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}y\; \frac{e^{-t-\frac{|x-y|^2}{4t}}}{t^{d/2}} f(y)$$

is finite for every $x \in \mathbb{R}^d$.

Optional (not needed for the mark): make sure that you can prove that in fact $u \in \mathcal{S}(\mathbb{R}^d)$.

(ii) Choose now $d = 1$ and consider the function u defined in (i). Show that u is the *unique* solution to

$$-u''(x) + u(x) = f(x)$$

in the class $\mathcal{S}(\mathbb{R})$.

(*Notice:* question (ii) clearly includes the request to prove that such u belongs to $\mathcal{S}(\mathbb{R})$.)