EXERCISE SHEET 9, issued on Tuesday 11 January 2011
Due: Tuesday 18 January 2011 by 8,15 a.m. in the designated “MQM” box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.

Exercise 33 [15 points]. Consider the Schrödinger Hamiltonian $H = -\Delta + V$ in d dimensions. Assume that $U_{-t} V U_t = e^{-t} V$ for all $t \in \mathbb{R}$, where for any real t the map $U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by $(U_t\psi)(x) := e^{-td/2}\psi(e^{-t}x)$ $\forall x \in \mathbb{R}^d$, $\forall \psi \in L^2(\mathbb{R}^d)$.

(i) Prove that $t \mapsto U_t$ is a strongly continuous unitary group on $L^2(\mathbb{R}^d)$, i.e., U_t is unitary $\forall t \in \mathbb{R}$, $U_0 = \mathbb{1}$, $U_{t+s} = U_t U_s$ $\forall t, s \in \mathbb{R}$, $\|U_t\psi - \psi\|_2 \xrightarrow{t \to 0} 0$ $\forall \psi \in L^2(\mathbb{R}^d)$.

(ii) Let $\psi \in L^2(\mathbb{R}^d)$, $\|\psi\|_2 = 1$, such that $\Delta\psi \in L^2(\mathbb{R}^d)$, $V\psi \in L^2(\mathbb{R}^d)$, and $H\psi = E\psi$ (in L^2) for some $E \in \mathbb{R}$. Prove that

$$E \;=\; -\langle \psi, (-\Delta)\psi \rangle \;=\; \frac{1}{2}\langle \psi, V\psi \rangle$$

and that therefore $E \leqslant 0$. (*Hint:* check that both $U_t H\psi$ and $H U_t \psi$ belong to $L^2(\mathbb{R}^d)$, then use this fact and (i) to compute the expectation of $(U_t H - H U_t)$ in the state ψ when $t \to 0$.)

Exercise 34 [11 points]. Consider two families $\{\phi_j\}_{j=1}^N$ and $\{\psi_\ell\}_{\ell=1}^N$ of functions in $L^2(\mathbb{R}^d)$ (d and N being positive integers).

(i) Prove that $\langle \phi_1 \wedge \cdots \wedge \phi_N, \psi_1 \wedge \cdots \wedge \psi_N \rangle_{L^2(\mathbb{R}^{Nd})} = \det \begin{pmatrix} \langle \phi_1, \psi_1 \rangle & \cdots & \langle \phi_1, \psi_N \rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_N, \psi_1 \rangle & \cdots & \langle \phi_N, \psi_N \rangle \end{pmatrix}$.

(ii) Let A be a $N \times N$ matrix with complex entries. Define the functions

$$\xi_i \;:=\; \sum_{j=1}^N A_{ij}\psi_j\,, \qquad i = 1, 2, \dots, N\,.$$

Prove that

$$\xi_1 \wedge \cdots \wedge \xi_N \;=\; (\det A)\, \psi_1 \wedge \cdots \wedge \psi_N\,.$$

Exercise 35 [14 points]. Consider a family $\{\psi_\ell\}_{\ell=1}^N$ of functions in $L^2(\mathbb{R}^d)$ (d and N being positive integers) such that $\langle \psi_\ell, \psi_{\ell'} \rangle = \delta_{\ell,\ell'}$. Let $\Psi := \psi_1 \wedge \cdots \wedge \psi_N \in L^2(\mathbb{R}^{Nd})$. For any $k \in \{1, \dots, N\}$ let $\gamma_\Psi^{(k)}$ be the k-particle reduced density matrix associated with Ψ and let $\rho_\Psi^{(k)}$ be the corresponding k-particle density. Denoting by $\gamma_\Psi^{(k)}(x_1, \dots, x_k; x'_1, \dots, x'_k)$ the kernel of $\gamma_\Psi^{(k)}$, $\rho_\Psi^{(k)}(x_1, \dots, x_k) = \gamma_\Psi^{(k)}(x_1, \dots, x_k; x_1, \dots, x_k)$.

(i) Prove that $\gamma_\Psi^{(1)}(x; x') = \sum_{\ell=1}^N \overline{\psi_\ell(x)} \psi_\ell(x')$.

(ii) Prove that $\rho_\Psi^{(2)}(x_1, x_2) = \rho_\Psi^{(1)}(x_1) \rho_\Psi^{(1)}(x_2) - |\gamma_\Psi^{(1)}(x_1; x_2)|^2$.

★**Exercise 36 [15 points].** Consider the Hilbert space $\mathcal{H}$ of all (equivalence classes of) measurable functions $\psi : \mathbb{R}^d \to \mathbb{C}$ which are 2π-periodic (i.e., $\psi(\dots, x_j + 2\pi, \dots) = \psi(\dots, x_j, \dots)$ for almost all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\forall j = 1, \dots, d$) and square-integrable on $[-\pi, \pi]^d$, equipped with the scalar product

$$\langle \phi, \psi \rangle \;=\; \int_{[-\pi,\pi]^d} \overline{\phi(x)}\, \psi(x)\, \mathrm{d}x \,.$$

(i) Construct an orthonormal basis of $\mathcal{H}$ made of eigenvectors of the operator $-\Delta$ acting on the functions of $\mathcal{H}$ that are H^2-functions on $[-\pi, \pi]^d$.

(ii) Let $0 \leqslant E_0 \leqslant E_1 \leqslant E_2 \leqslant \cdots$ be the eigenvalues of $-\Delta$ counted with multiplicity. Prove that there exist two constants $a_d, b_d \in (0, \infty)$ such that

$$N(E) \;=\; \#\{E_j \,|\, E_j \leqslant E\} \;\leqslant a_d\, E^{d/2} + b_d \,.$$

(iii) Prove that $\displaystyle \lim_{E \to \infty} \frac{N(E)}{E^{d/2}} \;=\; \mathrm{vol}\big(\{x \in \mathbb{R}^d \,|\, |x| \leqslant 1\}\big)\,.$