HOMEWORK ASSIGNMENT – WEEK 07
Hand-in deadline: Fri 30 May by 12 p.m. in the "MSP" drop box.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 20. For this exercise you will need the following classical theorem by Paley and Wiener:

A measurable function F on $\mathbb{R}$ is the inverse Fourier transform of a function $\widehat{F} \in C_0^\infty(\mathbb{R})$ with support in $[-R, R]$ $(R > 0)$ if and only if the continuation of F to the complex plane is entire analytic and for each $n \in \mathbb{N}$ there exists a constant C_n such that $|F(z)| \leqslant C_n \dfrac{e^{R|\mathfrak{Im} z|}}{(1+|z|)^n}$.

Let $\mathcal{A}$ be a C^*-algebra with unit, $\{\tau_t \,|\, t \in \mathbb{R}\}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$, and $\beta \in \mathbb{R}$. Prove that the following conditions are equivalent:

(1) ω satisfies the (τ_t, β)-KMS condition,

(2) the relation
$$\int_{-\infty}^{+\infty} \mathrm{d}t\, f(t)\, \omega(A\tau_t(B)) \;=\; \int_{-\infty}^{+\infty} \mathrm{d}t\, f(t+\mathrm{i}\beta)\, \omega(\tau_t(B)A)$$
is valid for all $A, B \in \mathcal{A}$ and all f with Fourier transform $\widehat{f} \in C_0^\infty(\mathbb{R})$.

Exercise 21. Let $\mathcal{A}$ be a C^*-algebra with unit, $\{\tau_t \,|\, t \in \mathbb{R}\}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$, and ω be an τ_t-invariant state, such that

(1) τ_t is asymptotically abelian, i.e., $\lim\limits_{t\to\infty} \|[A, \tau_t(B)]\| = 0 \quad \forall A, B \in \mathcal{A}$,

(2) ω has the "cluster property" $\omega(A\tau_t(B)) \xrightarrow{t\to\infty} \omega(A)\omega(B)$ $\forall A, B \in \mathcal{A}$. (This is the typical property shown by KMS states with respect to an asymptotically abelian dynamics, although proving this fact is laborious.)

Consider a state ν that is ω-normal. Prove that ν "returns to equilibrium" in the sense that
$$\nu(\tau_t(A)) \xrightarrow{t\to\infty} \omega(A) \qquad \forall A \in \mathcal{A}\,.$$

Exercise 22. Consider

- the quasi-local UHF algebra $(\mathcal{A}, (\mathcal{A}_\Lambda)_{\Lambda\in\mathcal{F}(\mathbb{Z})})$ associated with an infinite quantum spin system on $\mathbb{Z}$ with one-site Hilbert space $\mathcal{H}_x = \mathbb{C}^n$ ($n \in \mathbb{N}$ and $\mathcal{F}(\mathbb{Z})$ is the collection of the finite subsets of $\mathbb{Z}$),
- the automorphism α that, locally, is defined by

$$\alpha(A) \;=\; \Big(\bigotimes_{x\in\Lambda} U_x^*\Big) A \Big(\bigotimes_{x\in\Lambda} U_x\Big) \qquad \forall A \in \mathcal{A}_\Lambda \qquad (\Lambda \in \mathcal{F}(\mathbb{Z}))\,,$$

  and is then extended by density on the whole $\mathcal{A}$, where each U_x is a unitary on $\mathcal{H}_x$,
- an interaction $\Phi : \mathcal{F}(\mathbb{Z}^d) \to \mathcal{A}$ (recall from class: $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$ $\forall X \in \mathcal{F}(\mathbb{Z})$) that
  - decreases at infinity in the sense that

$$\sup_{x\in\mathbb{Z}} \sum_{\substack{X\in\mathcal{F}(\mathbb{Z})\\ X\ni x}} \mathrm{diam}(X)\|\Phi(X)\| \;<\; \infty \qquad (\mathrm{diam}(\Lambda) := \sup_{x,y\in\Lambda} |x-y|)\,,$$

  - is invariant under α, i.e., $\alpha(\Phi(X)) = \Phi(X)$ $\forall X \in \mathcal{F}(\mathbb{Z})$,
- the corresponding one-parameter strongly continuous group $\{\tau_t \,|\, t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathcal{A}$ constructed, by similar arguments as in Exercises 18 and 19, as the limit of the local dynamics generated by the local Hamiltonians $H_\Lambda := \sum_{X\subset\Lambda} \Phi(X)$; it satisfies $\tau_t \circ \alpha = \alpha \circ \tau_t$ for all $t \in \mathbb{R}$.

Prove that if ω is a (τ_t, β)-KMS state for some $\beta \in (0, +\infty)$, then

$$\omega \circ \alpha \;=\; \omega \qquad \text{on } \mathcal{A}\,.$$

(*Hint:* check the applicability of the theorem discussed in class in Section (iii)d.)

Exercise 23. For each of the 10 questions below answer YES or NO and provide a brief explanation.

23.1 If $X \subset \mathbb{R}$ is closed and the C^*-algebra $\mathcal{A} = C_0(X)$ contains the identity, is then X compact?

❏ YES ❏ NO

23.2 If $X \subset \mathbb{R}$ is compact, does then the C^*-algebra $\mathcal{A} = C_0(X)$ contains the identity?

❏ YES ❏ NO

23.3 Is there a representation π of the C^*-algebra $\mathcal{A} = \mathcal{M}(2, \mathbb{C})$ such that $\|\pi(\sigma_x)\| = 2$?

❏ YES ❏ NO

23.4 Is there a representation π of the C^*-algebra $\mathcal{A} = \mathcal{M}(2, \mathbb{C})$ such that $\|\pi(\sigma_z)\| = \frac{1}{2}$?

❏ YES ❏ NO

23.5 Consider the C^*-algebra $\mathcal{A}$ of 17×17 complex-valued matrices. Is $\omega(A) = \frac{1}{17}\sum_{i,j=1}^{17} A_{ij}$, $A = (A_{ij}) \in \mathcal{A}$, a state over $\mathcal{A}$?

❏ YES ❏ NO

23.6 Consider the C^*-algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{M}(2, \mathbb{C})$ (two-block diagonal matrices with blocks of size 1 and 2 respectively) and the vector $\mathbf{v} = \frac{1}{\sqrt{6}}\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$. Does the map $M \mapsto \langle \mathbf{v}, M\mathbf{v}\rangle$ define a state on $\mathcal{A}$?

❏ YES ❏ NO

23.7 Is the map $M \mapsto \langle \mathbf{v}, M\mathbf{v}\rangle$ defined in the previous question a pure state on $\mathcal{A}$?

❏ YES ❏ NO

23.8 Is the map $\omega : \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h}) \to \mathbb{C}$ defined by

$$\omega(W(f)) \ := \ \begin{cases} 1 & \text{if } f = 0 \\ 0 & \text{if } f \neq 0 \end{cases}$$

and extended by linearity and density on the whole $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$, a state?

❏ YES ❏ NO

23.9 Is the map defined in the previous question a regular state?

❏ YES ❏ NO

23.10 Concerning the global dynamics $\{\tau_t \,|\, t \in \mathbb{R}\}$ constructed in Exercises 18 and 19 on the quasi-local UHF algebra $(\mathcal{A}, (\mathcal{A}_\Lambda)_{\Lambda \in \mathcal{F}(\mathbb{Z}^d)})$ associated with an infinite quantum spin system, is there an element $H = H^* \in \mathcal{A}$ such that $\tau_t(A) = e^{\mathrm{i}tH} A \, e^{-\mathrm{i}tH}$ $\forall A \in \mathcal{A}$?

❏ YES ❏ NO