HOMEWORK ASSIGNMENT – WEEK 04
Hand-in deadline: Thu 8 May by 12 p.m. in the "MSP" drop box.
Rules: Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 9. Let $\mathcal{A}$ be a C^*-algebra with unit and let $\{\alpha_t \,|\, t \in \mathbb{R}\}$ be a weakly continuous one-parameter group of $*$-automorphisms on $\mathcal{A}$. Prove that there exists one state ν on $\mathcal{A}$ which is α_t-invariant, namely, $\nu(\alpha_t(A)) = \nu(A)$ $\forall A \in \mathcal{A}$ and $\forall\, t \in \mathbb{R}$.

Hint: we know that $\mathcal{A}$ has a state ω over $\mathcal{A}$; a priori ω is not invariant but there is a natural operation on ω that yields a candidate invariant state.

Exercise 10. Let $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ be the CAR algebra over a Hilbert space $\mathfrak{h}$ and let $\mathcal{I}$ be a net of closed non-empty subspaces of $\mathfrak{h}$ ordered by inclusion such that

(1) if $M \in \mathcal{I}$ then $\exists\, N \in \mathcal{I}$ such that $M \perp N$ (in the sense of the scalar product in $\mathfrak{h}$),

(2) if $M \perp N$ and $M \perp K$, then $\exists\, L \in \mathcal{I}$ such that $M \perp L$ and $N, K \subset L$

(3) $\mathfrak{h} = \overline{\bigcup_{M \in \mathcal{I}} M}^{\|\,\|}$.

For each $M \in \mathcal{I}$ let $\mathcal{A}_M \subset \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ be the sub-C^*-algebra generated by $\{a(f) \,|\, f \in M\}$. Prove that $(\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h}), \{\mathcal{A}_M\}_{M \in \mathcal{I}})$ is a quasi-local algebra with involutive automorphism σ such that $\sigma(a(f)) = -a(f)$ for all $f \in \mathfrak{h}$.

Exercise 11. Let $d \in \mathbb{N}$. Consider the CAR algebra $\mathcal{A} = \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ over the Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^d, \mathbb{C})$. For every $v \in \mathbb{R}^d$ and every $f \in \mathfrak{h}$ define $(U_v f)(x) := f(x-v)$ for a.e. $x \in \mathbb{R}^d$.

(i) Let τ_v be the $*$-homomorphism defined by

$$\tau_v(a(f)) \;:=\; a(U_v f)\,, \qquad \tau_v(a^*(f)) \;:=\; a^*(U_v f)\,,$$

and extended by linearity on the polynomials generated by $\mathbb{1}$, $a(f)$, and $a^*(f)$, $f \in \mathfrak{h}$. Prove that $\{\tau_v\}_{v \in \mathbb{R}^d}$ extends to a strongly continuous $\mathbb{R}^d$-parameter group of $*$-automorphisms of $\mathcal{A}$.

(ii) Prove that

$$\lim_{\substack{v\in\mathbb{R}^d\\|v|\to\infty}} \big\| \{a(f)^*, \tau_v(a(g))\} \big\| \;=\; 0$$

for any $f, g \in L^2(\mathbb{R}^d)$, where $\{\ ,\ \}$ denotes the anti-commutator in $\mathcal{A}$.

(iii) Prove that if A is an odd element of $\mathcal{A}$, then

$$\lim_{\substack{v\in\mathbb{R}^d\\|v|\to\infty}} \big\| \{A^*, \tau_v(A)\} \big\| \;=\; 0\,.$$

(*Hint:* polynomial approximation.)

(iv) Prove that if A or B is an even element of $\mathcal{A}$, then

$$\lim_{\substack{v\in\mathbb{R}^d\\|v|\to\infty}} \big\| [A, \tau_v(B)] \big\| \;=\; 0\,,$$

where $[\ ,\ ]$ denotes the commutator in $\mathcal{A}$.

Exercise 12. Consider the quasi-local C^*-algebra $\mathcal{A}$ of a one-dimensional infinite chain of spin-$\frac{1}{2}$ systems, where the corresponding net $(\mathcal{A}_\Lambda)_{\Lambda\subset\mathbb{Z}}$ of local algebras is defined by

$$\begin{aligned}\mathcal{A}_\Lambda &:= \bigotimes_{n\in\Lambda} \mathcal{A}_n\,,\\ \mathcal{A}_n &:= \text{matrix algebra generated by } \{\mathbb{1}_n, \sigma_n^x, \sigma_n^y, \sigma_n^z\} \;\cong\; \mathcal{M}(2\times 2, \mathbb{C})\,.\end{aligned}$$

As usual, each $A \in \mathcal{A}_n$ is regarded as $A \in \mathcal{A}$ via the identification $A \equiv \cdots\otimes\mathbb{1}_{n-1}\otimes A\otimes\mathbb{1}_{n+1}\otimes\cdots$. The goal of this exercise is to show that $\mathcal{A}$ admits two *inequivalent* representations $(\mathcal{H}^+, \pi^+)$ and $(\mathcal{H}^-, \pi^-)$, i.e., two representations for which no unitary operator $U : \mathcal{H}^+ \to \mathcal{H}^-$ exists such that $\pi^-(A) = U\pi^+(A)U^*\ \forall A \in \mathcal{A}$. The spaces $\mathcal{H}^\pm$ and the $*$-homomorphisms $\pi^\pm$ are defined as follows.

Target spaces. Given the two countable sets

$$\begin{aligned}S^+ &:= \{s \equiv (s_n)_{n\in\mathbb{Z}} \,|\, s_n \in \{-1, 1\}\ \forall n \in \mathbb{Z}\,, s_n \neq 1 \text{ for at most finitely many } n\text{'s}\},\\ S^- &:= \{s \equiv (s_n)_{n\in\mathbb{Z}} \,|\, s_n \in \{-1, 1\}\ \forall n \in \mathbb{Z}\,, s_n \neq -1 \text{ for at most finitely many } n\text{'s}\},\end{aligned}$$

the Hilbert spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are defined by

$$\mathcal{H}^\pm \;:=\; \ell^2(S^\pm) \;=\; \Big\{f : S^\pm \to \mathbb{C} \,\Big|\, \sum_{s\in S^\pm} |f(s)|^2 < \infty\Big\}\,.$$

Note that since $S^\pm$ is countable, then $\ell^2(S^\pm)$ is separable: in fact, a canonical orthonormal basis is $(f_s)_{s\in S^\pm}$, where $f_s(t) \;:=\; \begin{cases} 1 & \text{if } t = s\\ 0 & \text{if } t \neq s\end{cases}$.

Representations. Clearly, it is enough to define $\pi^\pm$ on the elements $\{\mathbb{1}_n, \sigma_n^x, \sigma_n^y, \sigma_n^z \,|\, n \in \mathbb{Z}\}$ of $\mathcal{A}$. In terms of the "flip of spin n" maps

$$\Theta_n : S^+ \to S^+\,, \qquad (\Theta_n(f))_k \;:=\; \begin{cases} -s_n & \text{if } k = n\\ s_k & \text{if } k \neq n\end{cases}$$

defined for each $n \in \mathbb{Z}$, set

$$\begin{aligned}(\pi^+(\mathbb{1}_n)f)(s) &:= f(s)\\ (\pi^+(\sigma^x_n)f)(s) &:= f(\Theta_n(s))\\ (\pi^+(\sigma^y_n)f)(s) &:= \mathrm{i}\, s_n\, f(\Theta_n(s))\\ (\pi^+(\sigma^z_n)f)(s) &:= s_n\, f(s) \qquad\qquad \forall f \in \mathcal{H}^+\,,\ \forall s \in S^+\,.\end{aligned}$$

π^- is defined on $\mathcal{H}^-$ by precisely the same formulas as above.

(i) Prove that $(\mathcal{H}^+, \pi^+)$ and $(\mathcal{H}^-, \pi^-)$ are two representations of $\mathcal{A}$.

(ii) Prove that both π^+ and π^- are irreducible.

(*Hint:* apply to a generic non-zero $f \in \mathcal{H}^+$ a suitable number of projections $P^{\pm}_n := \frac{1}{2}\pi^+(\mathbb{1}_n \pm \sigma^z_n)$ and of flip operators $\pi^+(\sigma^x_n)$, for a finite number of sites n, to get arbitrarily close to any element of the canonical orthonormal basis of $\mathcal{H}^+$. The same on $\mathcal{H}^-$.)

(iii) For each $N \in \mathbb{N}$ consider the local magnetisation operator $M^z_N := \frac{1}{2N+1}\sum_{n=-N}^{N}\sigma^z_n$. Prove that

$$\pi^{\pm}(M^z_N) \xrightarrow{N\to\infty} \pm\mathbb{1} \qquad \text{weakly in the operator sense}$$

and thus deduce that π^+ and π^- are not unitarily equivalent.

(*Hint:* for every $\psi, \phi \in \mathcal{H}^{\pm}$ compute the limit $\lim\limits_{N\to\infty}\langle\psi, \pi^{\pm}(M^z_N)\phi\rangle_{\mathcal{H}^{\pm}}$.)

(iv) Argue that actually $\mathcal{A}$ admits *infinitely* many non-equivalent representations.