Lecture

Statistical Mathematical Mechanics

Sommersemester 2013

at the

LMU, Munich

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Contents

Chapter 1

Introduction and Motivation

In mathematical statistical physics one studies certain classes of physical systems from a statistical point of view. In particular, one is concerned with

- Equilibrium properties of a macroscopic molecular system,
- Laws of thermodynamics,
- Thermodynamic functions.

" Statistical mechanics, however, does not describe how a system approaches equilibrium, nor does it determine whether a system can ever be found to be in equilibrium. It merely states what the equilibrium situation is for a given system" [7]

We will start with an example how a statistical consideration can lead to the description of a equilibrium in case of a classical system.

Consider a dilute gas with $N >> 1$ molecules each of mass m and contained in a box $\Lambda \subset \mathbb{R}^3$ of volume V. Each molecule is considered as a classical particle having a well-defined position and momentum. We assume that the molecules are distinguishable from each other and they reflected elastically at the walls of Λ.

A (microscopic) state of the gas is given by $3N$ canonical coordinates $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_n) \in \Lambda^N$ and $3N$ canonical momenta $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_N) \in \mathbb{R}^{3N}$ of the N molecules. We put

$$\Gamma := \mathbb{R}^{3N} \times \Lambda^{3N} := \textit{space of all possible states.}$$

A *macroscopic state* of the system (e.g. temperature, pressure, $\cdots$) can be represented by many microscopic states in Γ. So we can interpret a macroscopic state as a system in Γ (so called *ensemble*) of corresponding microscopic states.

1.1 Method of the most probable distribution

With $(p, q, t) \in \mathbb{R}^3 \times \Lambda \times \mathbb{R}_+$ let $f(p, q, t)$ be the *distribution function* of the gas. More precisely, if $\Omega \subset \mathbb{R}^3 \times \Lambda$ and $t > 0$, then

$$\int_\Omega f(p, q, t)\, dpdq = \textit{number of molecules with coordinates in } \Omega \textit{ at time } t. \tag{1.1.1}$$

If a (microscopic) state of the gas is given, then the integrals on the left hand side are uniquely determined for all Ω. However, different states in Γ can have the same distribution, e.g. one may exchange two particles. Since we can distinguish the molecules one obtains different states with the same distribution function. Hence we can identify a distribution function $f(p,q,t)$ with the subset

$$\Gamma_f \subset \Gamma$$

of all states (in a given ensemble) that are distributed according to f, i.e. all states (in the ensemble) that fulfill (1.1.1) for all $\Omega \subset \mathbb{R}^3 \times \Lambda$. The *equilibrium distribution* is the distribution that "maximizes" Γ_f.

Let the ensemble be defined by fixing the energy E of the system [1]. The possible values of $(p,q) \in \mathbb{R}^3 \times \Lambda$ are restricted through the energy condition and we replace the phase space $\mathbb{R}^3 \times \Lambda$ by a sufficiently large box $B_{\mathrm{ps}} = B \times \Lambda$ where $B \subset \mathbb{R}^3$. Then we divide B_{ps} into $K >> 1$ small cells c_i each of volume $\omega = \delta p \delta q$ and number the cells by $c_1, \cdots, c_K$. For a given state in Γ and $i = 1, \cdots, K$ we put

$$\begin{aligned} n_i &= \textit{number of molecules of the state in cell } c_i, \\ \varepsilon_i &:= \frac{p_i^2}{2m} = \textit{energy of a molecule in the } i\textit{th cell,} \end{aligned}$$

which by assumption must fulfill the conditions

$$\begin{cases} (a): \sum_{i=1}^K n_i & = N \\ (b): \sum_{i=1}^K \varepsilon_i n_i & = E. \end{cases}$$

A distribution function $f(p,q)$ is given by the "step function":

$$f(p,q) := \frac{n_i}{\delta p \delta q}, \qquad \text{if} \quad (p,q) \in c_i.$$

For given integers $\{n_i\}_{i=1,\cdots,K}$ with (a) and (b) we will write $\widetilde{\Omega}(n_1, \cdots, n_K)$ for the number of possibilities to distribute the N (distinguishable) particles to K cells $c_1, \cdots, c_K$ such that the cell c_i contains n_i molecules

$$\widetilde{\Omega}(n_1, \cdots, n_K) = \frac{N!}{n_1! \cdots n_K!}.$$

We assume that n_i is large that we can replace $\log n_i!$ by $n_i \log n_i$ due to *Stirling's formula* [2]

$$\log \widetilde{\Omega}(n_1, \cdots, n_K) \sim F(n_1, \cdots, n_K) := N \log N - \sum_{i=1}^K n_i \log n_i.$$

Now we need to maximize $F(n_1, \cdots, n_K)$ under the conditions (a) and (b). We consider $n_1, \cdots, n_K$ as real variables and use the method of *Lagrange multipliers*. Let us define

$$F_\lambda(n_1, \cdots, n_K) := F(n_1, \cdots, n_K) + \lambda_1 \sum_{i=1}^K n_i + \lambda_2 \sum_{i=1}^K \varepsilon_i n_i$$

[1] this ensemble is called *microcanonical ensemble*

[2] Stirling's formula: $\log(n!) = n \log n - n + O(\log n)$ as $n \to \infty$

with $\lambda_1, \lambda_2 \in \mathbb{R}$. A necessary condition of an extreme point of F under the conditions (a) and (b) is given by

$$0 = \frac{\partial F}{\partial n_i} = -(\log n_i + 1) + \lambda_1 + \varepsilon_i \lambda_2, \quad i = 1, \cdots, K.$$

This gives $n_i = Ce^{\varepsilon_i \lambda_2}$ for some $C \in \mathbb{R}$ and therefore $f_i = Ce^{\varepsilon_i \lambda_2}/\delta p \delta q$. One can check that in fact this choice of n_i maximizes F. In the limit $K \to \infty$ we find the distribution function

$$f(p, q) = Ce^{\frac{p^2}{2m}\lambda_2},$$

which actually only depends on p. The requirement of being a density distribution gives

$$N = \int_{\mathbb{R}^3 \times \Lambda} f(p, q)\, dpdq = CV \int_{\mathbb{R}^3} e^{\frac{p^2 \lambda_2}{2m}}\, dp = CV \left(-\frac{2\pi m}{\lambda_2}\right)^{\frac{3}{2}}.$$

If we write $n := N/\text{vol}\,(\Lambda)$ for the *particle density*, then it follows

$$C = n \left(\frac{-\lambda_2}{2\pi m}\right)^{\frac{3}{2}}. \tag{1.1.2}$$

In the next step we calculate λ_2. Note that for the *mean energy* of a molecule we have:

$$\frac{E}{N} = \frac{\int_{\mathbb{R}^3} \frac{p^2}{2m} e^{\frac{p^2 \lambda_2}{2m}}\, dp}{\int_{\mathbb{R}^3} e^{\frac{p^2 \lambda_2}{2m}}\, dp}. \tag{1.1.3}$$

By using the standard integral formula

$$\int_{\mathbb{R}^3} p^2 e^{-ap^2} dp = 4\pi \int_0^\infty x^4 e^{-ax^2} dx = \frac{3\pi^{\frac{3}{2}}}{2a^{\frac{5}{2}}}$$

in (1.1.3) we obtain

$$\frac{E}{N} = -\frac{3}{2\lambda_2} \quad \Longrightarrow \quad \lambda_2 = -\frac{3N}{2E} = -\frac{1}{kT},$$

where $k = 8,6173 \cdot 10^{-5} eV/K$ denotes the *Boltzma constant*, T is the *temperature* and we have used that $E/N = \frac{3}{2}kT$. Inserting the value of λ_2 into (1.1.2) allows us to calculate C:

$$C = n \left(\frac{1}{2\pi mkT}\right)^{\frac{3}{2}}.$$

Conclusion: In the case of a dilute gas and under our simplifying assumptions the "most probable distribution" is the *Maxwell-Boltzmann distribution*:

$$f(p) = n \left(\frac{1}{2\pi mkT}\right)^{\frac{3}{2}} e^{-\frac{p^2}{2mkT}}.$$

Note that we have receive this result by a purely statistical consideration not taking into account the kinematics of the microscopic states.

Chapter 2

C^*-algebras in quantum statistical mechanics

In a classical mechanical system the *observables* are polynomials or more generally elements of the space $C(\Gamma)$ of all real valued continuous functions defined on the phase space Γ. Note that $C(\Gamma)$ has the structure of a commutative algebra under the pointwise multiplication. In the case where Γ is compact or we only admit bounded continuous functions [1], then $C(\Gamma)$ or $C_b(\Gamma)$ are complete normed algebras. The algebra of complex valued continuous functions on the compact space Γ has the structure of a C^*-algebra (see the definition below).

More abstractly, we may consider a physical system as being defined by its *C^*-algebra* $\mathcal{A}$ *of observables.* [2] The *states* of the system correspond to the measurements of the observables. In the abstract mathematical framework states are normalized positive linear functionals on $\mathcal{A}$. We explain these notions now more in detail:

Let us write $\mathbb{R}$ and $\mathbb{C}$ for the field of real and complex numbers, respectively. If $\lambda \in \mathbb{C}$, then we denote by $\overline{\lambda}$ its complex conjugate. Recall the following notions:

I. Let $\mathcal{A}$ be a complex vector space equipped with an *associative* and *distributive* product, i.e. if AB denotes the product of $A, B \in \mathcal{A}$, then it holds:

   (i) $A(BC) = (AB)C$,

   (ii) $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$,

   (iii) $\lambda\gamma(AB) = (\lambda A)(\gamma B)$, where $\lambda, \gamma \in \mathbb{C}$.

   We call $\mathcal{A}$ an *associative algebra* over $\mathbb{C}$.

II. An *involution* $\mathcal{A} \ni A \mapsto A^* \in \mathcal{A}$ of $\mathcal{A}$ is a map such that:

   (iv) $A^{**} = A$,

   (v) $(AB)^* = B^* A^*$,

   (vi) $(\lambda A + \gamma B)^* = \overline{\lambda} A^* + \overline{\gamma} B^*$, where $\lambda, \gamma \in \mathbb{C}$.

III. The algebra $\mathcal{A}$ is called *normed* with norm $\| \cdot \| : \mathcal{A} \to [0, \infty)$ if for all $A, B \in \mathcal{A}$:

[1] we write $C_b(\Gamma)$ for the space of bounded continuous functions on Γ

[2] cf. the Gelfand-Naimark theorem (GN-theorem) below.

(vii) $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$,

(viii) $\|\lambda A\| = |\lambda|\|A\|$ where $\lambda \in \mathbb{C}$,

(ix) $\|A + B\| \leq \|A\| + \|B\|$, (triangle inequality),

(x) $\|AB\| \leq \|A\|\|B\|$, (product inequality).

Definition 2.0.1. (Banach-, B^*- and C^*-algebra)

(a) Let $\mathcal{A}$ be a normed associative algebra which is complete in the norm topology, then $\mathcal{A}$ is called *Banach algebra*.

(b) A Banach algebra with an involution and such that $\|A\| = \|A^*\|$ holds for all $A \in \mathcal{A}$ is called a B^*-algebra.

(c) A C^**-algebra* is a B^*-algebra which for all $A \in \mathcal{A}$ fulfills the norm equality

$$\|AA^*\| = \|A\|^2.$$

The algebra $\mathcal{A}$ is called *abelian* or *commutative* if the product is commutative.

We do not assume that the algebras have a unit. However, if this is the case we call it *unital algebra* and we add the assumption $\|e\| = 1$ where e denotes the unit of the algebra.

Example 2.0.2. (C^*-algebras)

(a) Let H be a complex Hilbert space and $\mathcal{L}(H)$ the algebra of bounded operators on H with the operator norm. The adjoint operation

$$\mathcal{L}(H) \ni A \mapsto A^* \in \mathcal{L}(H)$$

is an involution. Then $\mathcal{L}(H)$ is a C^*-algebra. More general: each closed sub-algebra of $\mathcal{L}(H)$ which is invariant under the involution "$*$" is a C^*-algebra.

(b) According to (a) the space $\mathbb{C}(n)$ of complex $n \times n$-matrices can be interpreted as a C^*-algebra via the identification $\mathbb{C}(n) \cong \mathcal{L}(\mathbb{C}^n)$.

(c) Let X be a compact space, then the space $C(X)$ of continuous complex valued functions on X with the pointwise product, the norm

$$\|f\| := \sup\left\{|f(x)| \;:\; x \in X\right\}$$

and the involution $f^*(x) := \overline{f(x)}$ defines a commutative unital C^*-algebra.

(d) Let X be a locally compact space and $f : X \to \mathbb{C}$ continuous. We say that f *vanishes at infinity* if for each $\varepsilon > 0$ there is a compact set $K \subset X$ such that

$$|f(x)| \leq \varepsilon \qquad \text{for all} \qquad x \in X \setminus K.$$

The space $C_0(X)$ of all continuous functions on X vanishing at infinity with the norm and the involution in (c) is a commutative C^*-algebra which is unital if and only if X is compact.

Exercise 2.0.3. *(a) Show that the spaces in Example 2.0.2 in fact define C^*-algebras.*

(b) The C^*-condition $\|AA^*\| = \|A\|^2$ implies that $\|A^*\| = \|A\|$.

(c) Let $\mathcal{A}$ be a unital Banach algebra. If $\|A^2\| = \|A\|^2$ holds for all $A \in \mathcal{A}$, then $\mathcal{A}$ is commutative.
Hint: *Let $B \in \mathcal{A}$ and consider the function $f(z) := e^{-zA}Be^{zA}$ where $z \in \mathbb{C}$.*

2.1 Abelian C^*-algebras and GN-theorem

In describing a physical system one usually starts with the "geometry" by choosing an appropriate manifold (phase space) and then considering the algebra of observables (continuous functions on the phase space). As a consequence of the GN-theorem one could reverse the procedure: one may start with an abstract characterization of observables by fixing a unital commutative C^*-algebra $\mathcal{A}$ which encodes the relations between physical quantities. The GN-theorem (which will be explained in this section) allows to construct a compact Hausdorff space Γ such that $\mathcal{A}$ can be identified with the C^*-algebra of continuous functions on Γ.

For the moment we do not assume the existence of an involution. Let $\mathcal{A}$ be a unital commutative Banach algebra over $\mathbb{C}$.

Definition 2.1.1. A *multiplicative functional* $m : \mathcal{A} \to \mathbb{C}$ of $\mathcal{A}$ is a linear map that "preserves the multiplication":

$$m(AB) = m(A)m(B),$$

for all $A, B \in \mathcal{A}$. In particular, if $m \neq 0$, then we have $m(I) = 1$ where I denotes the unit of $\mathcal{A}$.

We will show that there is a close relation between multiplicative functionals and maximal ideals on $\mathcal{A}$ which allows us to identify these objects.

Definition 2.1.2. An *ideal* $\mathcal{I}$ of $\mathcal{A}$ is a sub-algebra with $A\mathcal{I} := \{AJ \ : \ J \in \mathcal{I}\} \subset \mathcal{I}$ for all $A \in \mathcal{A}$. The ideal is called *maximal* if there is no proper ideal $\widetilde{\mathcal{I}} \subset \mathcal{A}$ with $\mathcal{I} \subsetneq \widetilde{\mathcal{I}} \subsetneq \mathcal{A}$.

Exercise 2.1.3. *Show the following:*

(a) The closure $\overline{\mathcal{I}}$ of an ideal $\mathcal{I} \subset \mathcal{A}$ is an ideal as well. In particular, maximal ideals are closed.

(b) A proper ideal $\mathcal{I} \subsetneq \mathcal{A}$ contains no invertible elements of $\mathcal{A}$. In particular, it does not contain the unit of $\mathcal{A}$ and the closure of a proper ideal is a proper ideal.

In the following we write $\mathcal{A}^{-1}$ for the group of invertible elements of $\mathcal{A}$. Recall that the *spectrum* $\sigma(A)$ of an element $A \in \mathcal{A}$ is defined by

$$\sigma(A) = \left\{\lambda \in \mathbb{C} \ : \ A - \lambda I \notin \mathcal{A}^{-1}\right\}.$$

As is known the spectrum $\sigma(A)$ is compact and non-empty for all $A \in \mathcal{A}$ (as for a proof see [8]). We call $\rho(A) := \mathbb{C} \setminus \sigma(A)$ the *resolvent set* of A.

Exercise 2.1.4. *Let X be a compact space and let $C(X)$ be the Banach algebra of continuous functions on X (cf. Example 2.0.2, (c)).*

(i) Then $\sigma(f) = f(X)$ for all $f \in \mathcal{A}$.

(ii) Let $\mathcal{B}$ be a unital Banach algebra and $B \in \mathcal{B}$, then $\sigma(B) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|B\|\}$.

Theorem 2.1.5 (Gel'fand-Mazur)**.** *Assume that all elements of $\mathcal{A} \setminus \{0\}$ are invertible in $\mathcal{A}$, i.e $\mathcal{A}^{-1} = \mathcal{A} \setminus \{0\}$. Then $\mathcal{A} = \{\lambda I : \lambda \in \mathbb{C}\} \cong \mathbb{C}$.*

Proof. Let $A \in \mathcal{A}$ and assume that $\lambda \in \sigma(A) \neq \emptyset$. Then $A - \lambda I \notin \mathcal{A}^{-1}$ and by assumption it follows that $A - \lambda I = 0$. Therefore $A = \lambda I$. □

Lemma 2.1.6. *Let $\mathcal{A}$ be a unital commutative Banach algebra. Then* (i) *and* (ii) *are equivalent:*

(i) *$\mathcal{I} \subset \mathcal{A}$ is a maximal ideal*

(ii) *There is a unique multiplicative functional $0 \neq m : \mathcal{A} \to \mathbb{C}$ with*

$$\mathcal{I} = \ker m := \big\{A \in \mathcal{A} : m(A) = 0\big\}.$$

Proof. (i) ⇒ (ii): If $\mathcal{I} \subset \mathcal{A}$ is a maximal ideal, then $\mathcal{I}$ is closed (see Exercise 2.1.3, (a)) and we can consider the quotient space

$$\mathcal{A}/\mathcal{I} = \big\{a + \mathcal{I} : a \in \mathcal{A}\big\} \quad \text{with norm} \quad \|A + \mathcal{J}\|_{\mathcal{A}/\mathcal{I}} := \inf_{J \in \mathcal{I}} \|A + J\|.$$

If we define a product on $\mathcal{A}/\mathcal{I}$ in a natural way via

$$(A + \mathcal{I})(B + \mathcal{I}) := AB + \mathcal{I},$$

then $\mathcal{A}/\mathcal{I}$ becomes a commutative algebra with unit $I + \mathcal{I}$ where I is the unit in $\mathcal{A}$. Note that for all $J_1, J_2 \in \mathcal{I}$:

$$\|(A + \mathcal{I})(B + \mathcal{I})\|_{\mathcal{A}/\mathcal{I}} = \|AB + \mathcal{I}\|_{\mathcal{A}/\mathcal{I}} \leq \|(A + J_1)(B + J_2)\| \leq \|A + J_1\| \, \|B + J_2\|.$$

By taking the infimum over $J_1, J_2 \in \mathcal{I}$ we see that

$$\|(A + \mathcal{I})(B + \mathcal{I})\|_{\mathcal{A}/\mathcal{I}} \leq \|A + \mathcal{I}\|_{\mathcal{A}/\mathcal{I}} \|B + \mathcal{I}\|_{\mathcal{A}/\mathcal{I}}$$

and therefore $\mathcal{A}/\mathcal{I}$ has the structure of a commutative Banach algebra. The natural projection

$$\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I} : \pi(A) := A + \mathcal{I}$$

becomes a surjective algebra homomorphism, i.e. π is linear continuous and

$$\pi(AB) = \pi(A)\pi(B), \quad \text{for all } A, B \in \mathcal{A}.$$

We show that all non-trivial elements $0 \neq \pi(A) \in \mathcal{A}/\mathcal{I}$ are invertible in $\mathcal{A}/\mathcal{I}$. This follows from the following two observations:

(1): The quotient algebra $\mathcal{A}/\mathcal{I}$ contains no proper non-trivial ideal: If $\{0\} \neq \mathcal{Q} \subsetneq \mathcal{A}/\mathcal{I}$ was such an ideal then the pre-image

$$\mathcal{I} \subsetneq \pi^{-1}(\mathcal{Q}) := \big\{A \in \mathcal{A} \;:\; \pi(A) \in \mathcal{Q}\big\} \subsetneq \mathcal{A}$$

would be an ideal in $\mathcal{A}$ which properly contains $\mathcal{I}$. This contradicts the assumption that $\mathcal{I}$ was chosen maximal.

(2): If $0 \neq \pi(A)$ is not invertible in $\mathcal{A}/\mathcal{I}$, then

$$\{0\} \neq \mathcal{Q} := \pi(A)\big(\mathcal{A}/\mathcal{I}\big) := \big\{AB + \mathcal{I} \;:\; B \in \mathcal{A}\big\} \subsetneq \mathcal{A}/\mathcal{I}$$

is a proper non-trivial ideal in $\mathcal{A}/\mathcal{I}$. This would contradict the first observation (1).

From the *Gelfand-Mazur-theorem* (Theorem 2.1.5) one concludes that

$$\mathcal{A}/\mathcal{I} = \big\{\lambda e + \mathcal{I} \;:\; \lambda \in \mathbb{C}\big\} \cong \mathbb{C}$$

and $m : \mathcal{A} \stackrel{\pi}{\to} \mathcal{A}/\mathcal{I} \stackrel{\cong}{\to} \mathbb{C}$ defines a multiplicative functional on $\mathcal{A}$ with $\mathcal{I} = \ker\, m$.

(ii) $\Rightarrow$ (i): Let $0 \neq m : \mathcal{A} \to \mathbb{C}$ be a multiplicative functional with $\mathcal{I} := \ker\, m$, then $\mathcal{I}$ is an ideal of $\mathcal{A}$, in fact, if $A \in \mathcal{A}$ and $B \in \mathcal{I}$, then

$$m(AB) = m(A)m(B) = 0 \quad\Longrightarrow\quad AB \in \mathcal{I} = \ker\, m.$$

Moreover, since $\mathcal{A}/\mathcal{I} = \mathcal{A}/\ker\, m \cong \mathrm{im}\; m = \mathbb{C}$ is complex one-dimensional it follows that $\mathcal{J}$ is maximal. Finally assume that $\ker\, m = \mathcal{I} = \ker\, \tilde{m}$ where $\tilde{m}$ is a multiplicative functional. Then $\tilde{m}$ defines a multiplicative functional on $\mathcal{A}/\ker\, m = \mathbb{C}$ and therefore $m = \alpha\tilde{m}$ with $\alpha \in \mathbb{C}$. Since

$$1 = m(e) = \alpha\tilde{m}(e) = \alpha$$

one concludes that $m = \tilde{m}$ and the statement about uniqueness follows. □

Example 2.1.7. *Let X be a compact space. For each $x \in X$ a maximal ideal $\mathcal{I}_x \subset C(X)$ is given by*

$$\mathcal{I}_x := \big\{f \in C(X) \;:\; f(x) = 0\big\} = \ker\, \delta_x,$$

where $\delta_x : C(X) \longrightarrow \mathbb{C}$ is the multiplicative functional which acts by evaluation in $x \in X$, i.e. $\delta_x(f) = f(x)$ for $f \in C(X)$.

Definition 2.1.8. We denote by $M(\mathcal{A})$ the space of all non-trivial multiplicative functionals on $\mathcal{A}$ and according to Lemma 2.1.6 we call $M(\mathcal{A})$ the *maximal ideal space* or the *Gelfand spectrum* of $\mathcal{A}$. [3]

Consider the topological dual $\mathcal{A}'$ of $\mathcal{A}$:

$$\mathcal{A}' = \big\{\varphi : \mathcal{A} \to \mathbb{C} \;:\; \varphi \text{ is linear and continuous}\big\}.$$

Then $\mathcal{A}'$ is a complete normed space with norm

$$\|\varphi\|_{\mathcal{A}'} := \sup\big\{|\varphi(A)| \;:\; A \in \mathcal{A},\; \|A\| \leq 1\big\}.$$

[3] A connection between the spectrum of elements in A and $M(\mathcal{A})$ will be given in Theorem 2.1.13 below.

Multiplicative functionals on a unital commutative Banach algebra $\mathcal{A}$ are automatically continuous and therefore

$$M(\mathcal{A}) \subset \mathcal{A}'. \tag{2.1.1}$$

More precisely, $M(\mathcal{A})$ is contained in the unit sphere of $\mathcal{A}'$

$$M(\mathcal{A}) \subset S_{\mathcal{A}'} := \big\{\varphi \in \mathcal{A}' \,:\, \|\varphi\|_{\mathcal{A}'} = 1\big\} \subset \mathcal{A}'.$$

This a consequence of the following lemma:

Lemma 2.1.9. *Each multiplicative functional $m \in M(\mathcal{A})$ is continuous with $\|m\|_{\mathcal{A}'} = 1$.*

Proof. Let $m \in M(\mathcal{A})$ and recall ker m is a maximal ideal and in particular ker m is closed in $\mathcal{A}$ (see Exercise 2.1.3, (a)). Therefore m factorizes through the quotient $\mathcal{A}/\ker m$:

$$m : \mathcal{A} \xrightarrow{\ \pi\ } \mathcal{A}/\ker m \xrightarrow{\ \widetilde{m}\ } \mathbb{C}, \quad \text{where} \quad \widetilde{m}\big(A + \ker m\big) := m(A).$$

Since $\mathcal{A}/\ker m$ is one-dimensional it is clear that $\widetilde{m}$ is continuous. The continuity of the natural projection π shows the continuity of $m = \pi \circ \widetilde{m}$.

It remains to show that $\|m\|_{\mathcal{A}'} = 1$: Let $A \in \mathcal{A}$ and assume that $m(A) > \|A\|$. Then $\|m(A)^{-1}A\| < 1$ and we have

$$e - m(A)^{-1}A \in \mathcal{A}^{-1}$$

(geometric series!). If we define $B := (I - m(A)^{-1}A)^{-1} \in \mathcal{A}$, then we obtain the contradiction:

$$\begin{aligned} 1 = m(e) &= m\big(B(e - m(A)^{-1}A)\big) \\ &= m\big(B - m(A)^{-1}BA\big) = m(B) - m(B) = 0. \end{aligned}$$

Therefore, it holds $m(A) \leq \|A\|$ for all $A \in \mathcal{A}$. From $m(e) = 1$ and the definition of $\|\cdot\|_{\mathcal{A}'}$ we have $\|m\|_{\mathcal{A}'} = 1$. □

On the dual $\mathcal{A}'$ of $\mathcal{A}$ we can consider a second topology which in a sense is "weaker" than the norm topology [4] and is called *weak-$*$-topology* or *topology of pointwise convergence*. We explain the construction: On $\mathcal{A}'$ a family of maps E_A parametrized by $A \in \mathcal{A}$ is defined by

$$E_A : \mathcal{A}' \longrightarrow \mathbb{C} : E_A(\varphi) := \varphi(A).$$

The *weak $*$-topology* on $\mathcal{A}'$ is the "roughest topology" such that all the maps E_A with $A \in \mathcal{A}$ are continuous. According to the inclusion $M(\mathcal{A}) \subset \mathcal{A}'$ in (2.1.1) the weak-$*$-topology descends from $\mathcal{A}'$ to the maximal ideal space $M(\mathcal{A})$.

Exercise 2.1.10. *Show that $M(\mathcal{A})$ is weak-$*$-closed in $B^\circ = \{\varphi \in \mathcal{A}' \,:\, \|\varphi\|_{\mathcal{A}'} \leq 1\}$.*

Theorem 2.1.11. *The maximal ideal space $M(\mathcal{A})$ equipped with the weak-$*$-topology is compact.*

Proof. This follows from an abstract result in functional analysis (Banach-Alaoglu theorem) which implies that the ball $B^\circ := \{\varphi \in \mathcal{A}' \,:\, \|\varphi\|_{\mathcal{A}'} \leq 1\}$ is weak-$*$-compact. Note that $M(\mathcal{A})$ is weak-$*$-closed in B° (see Exercise 2.1.10) and according to Lemma 2.1.1 we have the inclusion $M(\mathcal{A}) \subset B^\circ$. Since closed subsets of compacts sets are compact the statement follows. □

[4] roughly speaking, it contains fewer open sets

Consider the space $C(M(\mathcal{A}))$ of all continuous functions on $M(\mathcal{A})$ with respect to the weak-$*$-topology. Note that due to the compactness of $M(\mathcal{A})$ the space $C(M(\mathcal{A}))$ has the structure of a C^*-algebra in the sense of Example 2.0.2, (c).

Definition 2.1.12 (Gelfand-transform)**.** For each $A \in \mathcal{A}$ and $m \in M(\mathcal{A})$ put $\Gamma(A)(m) := m(A)$. [5] The map

$$\Gamma : \mathcal{A} \longrightarrow C\big(M(\mathcal{A})\big) : A \mapsto \Gamma(A) \tag{2.1.2}$$

is well-defined and called *Gelfand transform.*

Let $\mathcal{B}$ be a Banach algebra. We call a linear map $\pi : \mathcal{A} \to \mathcal{B}$

(i) *(algebra) homomorphism*, if π is multiplicative $\pi(AB) = \pi(A)\pi(B)$

(ii) $*$-*homomorphism*, if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and π is a homomorphism with $\pi(A^*) = \pi(A)^*$ for all $A \in \mathcal{A}$

(iii) $*$-*isomorphism*, if π is bijective $*$-homomorphism and isometric, i.e. $\|\pi(A)\| = \|A\|$. [6]

Theorem 2.1.13 (Gelfand)**.** *The Gelfand transform is a continuous homomorphism of algebras with norm 1, i.e.*

$$\|\Gamma\| = \sup\Big\{\|\Gamma(A)\| \ : \ \|A\| \leq 1\Big\} = 1.$$

Moreover, for all $A \in \mathcal{A}$ the spectrum of A fulfills

$$\sigma(A) = \Big\{m(A) \ : \ m \in M(\mathcal{A})\Big\} \tag{2.1.3}$$

and in particular

$$\|\Gamma(A)\| = \sup\big\{|m(A)| \ : \ m \in M(\mathcal{A})\big\} = r(A) := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \text{“}\textit{spectral radius of } A''.$$

Proof. From the definition of the weak-$*$-topology it is clear that $\Gamma(A)$ is a continuous function on $M(\mathcal{A})$ and therefore the Gelfand transform is well-defined. It is clear that Γ is linear and $\Gamma(AB) = \Gamma(A)\Gamma(B)$ follows with $m \in M(\mathcal{A})$ from

$$\Gamma(AB)(m) = m(AB) = m(A)m(B) = \Gamma(A)(m) \cdot \Gamma(B)(m).$$

We show that $\|\Gamma\| = 1$: According to Lemma 2.1.9 we have $|\Gamma(A)(m)| = |m(A)| \leq \|A\|$ for all $A \in \mathcal{A}$ and therefore

$$\|\Gamma(A)\| = \sup\Big\{|\Gamma(A)(m)| \ : \ m \in M(\mathcal{A})\Big\} \leq \|A\|.$$

Since $\Gamma(e) = e \in C(M(\mathcal{A}))$ we see that $\|\Gamma\| = \sup\{\|\Gamma(A)\| \ : \ \|A\| \leq 1\} = 1$.

It remains to show the equality (2.1.3). From this the last assertion clearly follows.

"$\supseteq$": Let $m \in M(\mathcal{A})$, then $A - m(A)e \in \ker m$ and therefore

$$A - m(A)e \notin \mathcal{A}^{-1}.$$

---

[5] In the literature also the notation $\widehat{A}$ is used instead of $\Gamma(A)$

[6] If π is an injective $*$-homomorphism with $\pi(I) = I$, then π already is isometric.

This implies that $m(A) \in \sigma(A)$.

"$\subseteq$": Assume that $\lambda \in \sigma(A)$ with $A \in \mathcal{A}$. Then $A - \lambda e \notin \mathcal{A}^{-1}$ and

$$J_{\lambda,A} := \Big\{ (A - \lambda e)B \ : \ B \in \mathcal{A} \Big\} \subsetneq \mathcal{A}$$

is a proper ideal of $\mathcal{A}$. There is a maximal ideal J with $J_{\lambda,A} \subset J \subsetneq \mathcal{A}$. Let $m \in M(\mathcal{A})$ with $J = \ker m$. Then

$$0 = m(A - \lambda e) = m(A) - \lambda$$

and as a consequence $\lambda = m(A) \in \{m(A) \ : \ m \in M(\mathcal{A})\}$. □

Exercise 2.1.14. *Let $\mathcal{A}$ be a commutative unital Banach algebra which contains nilpotent elements, i.e. there is $A \in \mathcal{A}$ such that $A^n = 0$ for some $n \in \mathbb{N}$.*

(i) *Show that the Gelfand transform $\Gamma : \mathcal{A} \longrightarrow C(M(\mathcal{A}))$ is not injective.*

(ii) *Give an explicit example of a commutative unital Banach algebra that contains nilpotent elements.*

Exercise 2.1.15. *Prove the formula for the spectral radius $r(A) := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ of an operator $A \in \mathcal{A}$ in Theorem 2.1.13.*

Lemma 2.1.16. *Let $\mathcal{A}$ be a unital commutative C^*-algebra and $m \in M(\mathcal{A})$, then m is a $*$-homomorphism, i.e. $m(A^*) = \overline{m(A)}$.*

Proof. First we show that if $A = A^*$, then $m(A)$ is real. If we write $m(A) = a + ib$ with $a, b \in \mathbb{R}$, then we have for all $c \in \mathbb{R}$:

$$\begin{aligned} b^2 + c^2 + 2bc = |b + c|^2 &\leq |a + i(b + c)|^2 \\ &= |m(A + ice)|^2 && (m(e) = 1) \\ &\leq \|A + ice\|^2 \\ &= \|(A + ice)(A^* - ice)\| && (\|BB^*\| = \|B\|^2, \text{ for all } B \in \mathcal{A}) \\ &= \|A^2 + c^2 e\| && (A^* = A) \\ &\leq \|A\|^2 + c^2. \end{aligned}$$

Here we have used $\|m\|_{\mathcal{A}'} = 1$ and the C^*-property of the norm. Hence we have shown that

$$b^2 + 2bc \leq \|A\|^2.$$

Since c is arbitrary we have $b = 0$ and therefore $m(A) = a \in \mathbb{R}$.

Let now $A \in \mathcal{A}$ be arbitrary, then we decompose A in the form $A = A_r + iA_i$ where

$$A_r = \frac{1}{2}\left(A + A^*\right) \quad \text{and} \quad A_i = \frac{1}{2i}\left(A - A^*\right).$$

Since $A_r = A_r^*$ and $A_i = A_i^*$ we obtain from the first part of the proof

$$m(A^*) = m\left(A_r - iA_i\right) = m\left(A_r\right) - im\left(A_i\right) = \overline{m(A)},$$

and the assertion is proven. □

The proof of the GN-theorem requires the *Stone Weierstrass theorem* which we recall next:

Theorem 2.1.17 (Stone-Weierstrass)**.** *Let X be a compact space and let $C(X)$ be the algebra of complex valued continuous functions on X. Assume that $\mathcal{A} \subset C(X)$ is a sub-algebra with the following properties:*

(i) *$\mathcal{A}$ contains all constant functions and if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$.*

(ii) *$\mathcal{A}$ separates the points of X, i.e. for $x \neq y \in X$ there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.*

Then the inclusion $\mathcal{A} \subset C(X)$ is dense.

Now we can state and prove the *Gelfand-Naimark theorem* (GN-theorem). Roughly speaking it says that all unital commutative C^*-algebras can be identified with an algebra of continuous functions as in Example 2.0.2, (c).

Theorem 2.1.18 (Gelfand-Naimark)**.** *Let $\mathcal{A}$ be a unital commutative C^*-algebra. Then the Gelfand transform $\Gamma : \mathcal{A} \to C(M(\mathcal{A}))$ is a $*$-isomorphism.*

Proof. **1. Step:** *Show that $\Gamma(A^*) = \Gamma(A)^*$ for $A \in \mathcal{A}$:*

Let $m \in M(\mathcal{A})$. According to the definition of the Gelfand transform and Lemma 2.1.16 we have

$$\Gamma(A^*)(m) = m(A^*) = \overline{m(A)} = \overline{\Gamma(A)(m)} = \Gamma(A)^*(m).$$

2. Step: *Show that $\|\Gamma(A)\| = \|A\|$ for all $A \in \mathcal{A}$, i.e. Γ is an isometry:*

Let $B = B^* \in \mathcal{A}$ be self-adjoint, then it follows from the C^*-property of the norm that

$$\|B^2\| = \|BB^*\| = \|B\|^2.$$

Inductively, we have $\|B^{2^n}\| = \|B\|^{2^n}$ for all $n \in \mathbb{N}$ and we obtain for the spectral radius of B:

$$r(B) = \lim_{n\to\infty} \|B^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|B^{2^n}\|^{\frac{1}{2^n}} = \|B\|. \tag{2.1.4}$$

In particular, we put $B = A^*A$ with $A \in \mathcal{A}$. Then we conclude from Theorem 2.1.13, the first step and (2.1.4) that

$$\|\Gamma(A)\|^2 = \|\Gamma(A)\Gamma(A)^*\| = \|\Gamma(AA^*)\| = r(AA^*) = \|AA^*\| = \|A\|^2.$$

Since the Gelfand transform Γ is an isometry it is clearly injective and it also follows that the range

$$\Gamma(\mathcal{A}) \subset C(M(\mathcal{A})) \tag{2.1.5}$$

is closed. In order to prove the equality $\Gamma(\mathcal{A}) = C(M(\mathcal{A}))$ it therefore is sufficient to show that the inclusion (2.1.5) is dense. Note that $\Gamma(A)$ fulfills the following properties:

(i) Since $\Gamma(\lambda e) = \lambda e$ for all $\lambda \in \mathbb{C}$ we conclude that the range $\Gamma(\mathcal{A})$ is a subalgebra of $C(M(\mathcal{A}))$ which contains the constant functions. Also $M(\mathcal{A})$ is weak-$*$- compact.

(ii) $\Gamma(\mathcal{A})$ is invariant under complex conjugation since $\overline{\Gamma(A)} = \Gamma(A)^* = \Gamma(A^*)$.

(iii) $\Gamma(\mathcal{A})$ separates points of $M(\mathcal{A})$, i.e. for any pair $m_1 \neq m_2 \in M(\mathcal{A})$ there is $A \in \mathcal{A}$ with

$$\Gamma(A)(m_1) = m_1(A) \neq m_2(A) = \Gamma(A)(m_2).$$

Therefore the density of the inclusion (2.1.5) is a consequence of the *Stone-Weierstrass theorem*, (Theorem 2.1.17). □

Exercise 2.1.19. *Let $\mathcal{A}$ be a commutative unital Banach algebra. The space*

$$\text{Rad } (\mathcal{A}) := \bigcap \{\mathcal{I} \subset \mathcal{A} \,:\, \mathcal{I} \textit{ is a maximal ideal }\}$$

is an ideal of $\mathcal{A}$ itself and is called the radical of $\mathcal{A}$.

(i) *If $A \in \mathcal{A}$ is nilpotent, then $A \in \text{Rad } (\mathcal{A})$.*

(ii) *Calculate the radical of a commutative unital C^*-algebra $\mathcal{A}$. Show that a commutative unital C^*-algbera contains no nilpotent elements.*

Exercise 2.1.20. *Let X be a compact space, then the maximal ideal space $M(C(X))$ can be identified with X via the map* [7]

$$\Delta : X \longrightarrow M(C(X)) : x \mapsto \delta_x,$$

where $\delta_x(f) := f(x)$ for all $f \in C(X)$ (cf. Example 2.1.7). Show that the map Δ is surjective.

Hint: *Assume that there is $m \in M(C(X)) \setminus \Delta(X)$. By using the compactness of X construct $f \in C(X)$ with $f > 0$ and $m(f) = 0$.*

More precisely, for each $x \in X$ there is $f_x \in C(X)$ such that $f_x(x) \neq 0$ and $m(f_x) = 0$. Put

$$U_x := \{y \in X \,:\, f_x(y) \neq 0\}.$$

Then $\{U_x\}_{x \in X}$ defines an open covering of X and since X is compact we may pass to a sub-cover $\{U_{x_j}\}_{j=1}^N$. Consider the function

$$h = \sum_{j=1}^{N} f_{x_j} f_{x_j}^* = \sum_{j=1}^{N} |f_{x_j}|^2.$$

Then $h \in C(X)$ and $h > 0$ on X and $m(h) = \sum_{j=1}^N |m(f_{x_j})|^2 = 0$. Hence $h \in \ker m$ is invertible in $C(X)$, which gives a contradiction.

2.2 Fock-space, CCR and CAR- algebras

(Robert Helling)

[7] More precisely, Δ is a homeomorphism, i.e. a continuous bijective map with continuous inverse.

2.2.1 CAR-Algebra

Let h be a pre-Hilbert space with completion $\overline{h}$.

Definition 2.2.1 (CAR-algebra)**.** The (unique up to $*$-isomorphisms) algebra $\mathcal{A}(h)$ generated by element $a(f)$ where $f \in h$ with the properties (i)-(iii) below is called *CAR-algebra.*

(i) $h \ni f \mapsto a(f)$ is anti-linear

(ii) $\{a(f), a(g)\} = 0$, with $f, g \in h$

(iii) $\{a(f), a(g)^*\} = \langle f, g\rangle \mathrm{id}$, with $f, g \in h$

2.2.2 CCR-Algbera

Let H be a real Hilbert space with a non-degenerate symplectic bilinear form $\sigma : H \times H \to \mathbb{R}$, i.e. σ is anti-symmetric.

$$\sigma(f, g) = -\sigma(g, f), \quad \text{for all } \ f, g \in H.$$

Definition 2.2.2 (CCR-algebra)**.** The (unique up to $*$-isomorphisms) algebra $\mathcal{A}(H)$ generated by Weyl-operators $W(f)$ where $f \in H$ with the properties (i)-(ii) is called *CCR-algebra.*

(i) $W(-f) = W(f)^*$ for all $f \in H$,

(ii) $W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)}W(f + g)$ for all $f, g \in H$.

2.3 Quasi-local Algebras

We introduce the notion of *quasi-local algebras.* These are classes of C^*-algebras that are used to describe infinite systems of statistical mechanics. We start with the definition.

A directed set $I = (I, \prec)$ [8] is said to possess an *orthogonality relation* $\perp$ if the following properties hold:

(a) if $\alpha \in I$, then there is $\beta \in I$ with $\alpha \perp \beta$.

(b) if $\alpha \prec \beta$ and $\beta \perp \gamma$, then $\alpha \perp \gamma$.

(c) if $\alpha \perp \beta$ and $\alpha \perp \gamma$, then there is $\delta \in I$ such that $\alpha \perp \delta$ and $\gamma, \beta \prec \delta$.

Example 2.3.1. The following are intrinsic examples for a directed set with an orthogonality relation:

1. Let $I :=$*bounded open subsets of* $\mathbb{R}^n$ or $I :=$*finite subsets of* $\mathbb{Z}^n$ directed by inclusion:

$$A \prec B :\Longleftrightarrow A \subset B \quad \text{and} \quad A \perp B :\Longleftrightarrow A \cap B = \emptyset.$$

In (c) choose $\delta = \beta \cup \gamma$

[8] "directed" means that the binary relation "$\prec$" is reflexive and transitive. In addition to each pair $\alpha, \beta \in I$ there is an "upper bound" $\gamma \in I$, i.e. $\alpha, \beta \prec \gamma$.

2. Let H be a vector space over $\mathbb{R}$ with a non-degenerated symplectic bilinear form [9] b and I :=*set of linear subspaces of* H directed by inclusion as in 1. Put

$$L \perp G :\Longleftrightarrow b(\ell, g) = 0 \quad \text{for all } \ell \in L \text{ and } g \in G.$$

In (c) choose $\delta = \text{span}\{\beta, \gamma\}$.

We also assume an abstract versions of the "union" or "span" in 1. and 2. of the above example. Let $\alpha, \beta \in I$, then we assume existence of a *least upper bound* denoted by $\alpha \vee \beta \in I$ with

(d) $\alpha \prec \alpha \vee \beta$ and $\beta \prec \alpha \vee \beta$.

(e) if $\alpha \prec \gamma$ and $\beta \prec \gamma$ then $\alpha \vee \beta \prec \gamma$.

Let $\mathcal{A}$ be a C^*-algebra equipped with an involutive automorphism σ, i.e. $\sigma^2 = \text{id}$. Given $A \in \mathcal{A}$ we can define its *even part* A^e and *odd part* A^o with respect to σ:

$$A^e = \frac{1}{2}\{A + \sigma(A)\} \quad \text{and} \quad A^o = \frac{1}{2}\{A - \sigma(A)\}.$$

such that $A = A^e + A^o$. Clearly it holds $\sigma(A^e) = A^e$ and $\sigma(A^o) = -A^o$. Moreover,

$$\mathcal{A}^e := \left\{A^e \ : \ A \in \mathcal{A}\right\} = C^*\textit{-subalgebra of } \mathcal{A}.$$
$$\mathcal{A}^o := \left\{A^o \ : \ A \in \mathcal{A}\right\} = \textit{Banach space.}$$

Definition 2.3.2 (quasi-local algebra)**.** Let I be a directed index set with an orthogonality relation. A *quasi-local algebra* is a C^*-algebra $\mathcal{A}$ with an involutive automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ and a net $\{\mathcal{A}\}_{\alpha \in I}$ of C^*-sub-algebras such that the following properties hold:

(a) if $\beta \prec \alpha$, then $\mathcal{A}_\beta \subset \mathcal{A}_\alpha$.

(b) all algebras $\mathcal{A}_\alpha$ have the common identity $e \in \mathcal{A}$.

(c) $\bigcup_{\alpha \in I} \mathcal{A}_\alpha$ is dense in $\mathcal{A}$ (with respect to the norm topology).

(d) Each $\mathcal{A}_\alpha$ for $\alpha \in I$ is invariant under σ, i.e. $\sigma(\mathcal{A}_\alpha) = \mathcal{A}_\alpha$.

(e) With the commutator $[\cdot, \cdot]$ and the anti-commutator $\{\cdot, \cdot\}$ on $\mathcal{A}$ and with $\alpha, \beta \in I$ such that $\alpha \perp \beta$ it holds:

- $[\mathcal{A}^e_\alpha, \mathcal{A}^e_\beta] = \{0\}$,
- $[\mathcal{A}^e_\alpha, \mathcal{A}^o_\beta] = \{0\}$,
- $\{\mathcal{A}^o_\alpha, \mathcal{A}^o_\beta\} = \{0\}$.

Remark 2.3.3. We may choose $\sigma = \text{id}$, then property (d) simply reduces to

[9] symplectic form means:

(i) skew-symmetric: $b(u, v) = -b(v, u)$ for all $u, v \in H$

(ii) totally isotropic: $b(v, v) = 0$ for all $v \in H$

(iii) non-degenerate: $b(u, \cdot) \equiv 0$ implies that $u = 0$

$$[\mathcal{A}_\alpha, \mathcal{A}_\beta] = 0 \qquad \text{for all} \qquad \alpha, \beta \in I \ \text{ with } \ \alpha \perp \beta.$$

If I is the set of bounded open subsets of $\mathbb{R}^n$ as in Example 2.3.1 1., then $\mathcal{A}_\alpha$ can be interpreted as the observables for a sub-system localized in $\alpha \subset \mathbb{R}^n$.

The corresponding quasi-local algebra describes the observables of the infinite system. The condition

$$[\mathcal{A}_\alpha, \mathcal{A}_\beta] = 0, \qquad \alpha \perp \beta$$

states that observations become independent if $\alpha \cap \beta = \emptyset$.

We now give some explicit examples for quasi-local algebras that play a role in statistical mechanics:

Example 2.3.4. (quasi-local algebras)

1. Let the index set $I := \{\Lambda \subset \mathbb{Z}^n \ : \ \Lambda \text{ is finite } \}$ be directed by inclusion and define the orthogonality relation $\Lambda_1 \perp \Lambda_2 :\Longleftrightarrow \Lambda_1 \cap \Lambda_2 = \emptyset$ for all $\Lambda_1, \Lambda_2 \in I$.

   Let $\Lambda \in I$ and assign to each $x \in \Lambda$ a finite dimensional Hilbert space H_x. Consider the tensor product Hilbert spaces H_Λ and a corresponding C^*-algebra $\mathcal{A}_\Lambda$:

   $$H_\Lambda := \bigotimes_{x \in \Lambda} H_x \quad \text{ and } \quad \mathcal{A}_\Lambda := \mathcal{L}(H_\Lambda) = \textit{bounded operators on } H_\Lambda.$$

   The family of algebras $\{\mathcal{A}_\Lambda\}_{\Lambda \in I}$ is increasing: if $\Lambda_1 \cap \Lambda_2 = \emptyset$ then $H_{\Lambda_1 \cup \Lambda_2} = H_{\Lambda_1} \otimes H_{\Lambda_2}$ and it holds

   $$\mathcal{A}_{\Lambda_1} \cong \mathcal{A}_{\Lambda_1} \otimes \mathrm{id}_{\Lambda_2} = \mathcal{L}(H_{\Lambda_1}) \otimes \mathrm{id}_{\Lambda_2} \subset \mathcal{L}(H_{\Lambda_1} \otimes H_{\Lambda_2}) = \mathcal{A}_{\Lambda_1 \cup \Lambda_2}.$$

   A quasi local algebra $\mathcal{A}$ with $\sigma = \mathrm{id}$ is defined by the minimal norm completion of the normed algebra

   $$\bigcup_{\Lambda \in I} \mathcal{A}_\Lambda.$$

   If $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $A_j \in \mathcal{A}_{\Lambda_j}$ where $j = 1, 2$, then property (e) follows from

   $$\begin{aligned} [A_1, A_2] &= A_1 A_2 - A_2 A_1 \\ &= \big(A_1 \otimes \mathrm{id}\big)\big(\mathrm{id} \otimes A_2\big) - \big(\mathrm{id} \otimes A_2\big)\big(A_1 \otimes \mathrm{id}\big) \\ &= A_1 \otimes A_2 - A_1 \otimes A_2 = 0. \end{aligned}$$

   Algebras of the above type where the index set I is countable frequently are called *UHF-algebras* [10]. They play a role in the study of *quantum spin systems*.

2. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and consider an index set

   $$I \subset \big\{ M \subset H \ : \ M \text{ is a closed non-empty subspace} \big\}$$

   which should be directed by inclusion and such that

   $$\bigcup_{M \in I} M \subset H$$

[10] UHF means "uniformly hyper-finite"

is norm dense. Assume that usual orthogonality $\perp$ with respect to the inner product defines an orthogonality relation on I in the previous sense.

Let $\mathcal{A}_{\mathrm{CAR}}(H)$ be the *CAR-algebra* over H generated by $\{a(f) : f \in H\}$ with the conditions in Definition 2.2.1. For each $M \in I$ put

$$\mathcal{A}_{\mathrm{CAR}}(M) := C^*\textit{-algebra generated by } a(f) \textit{with } f \in M.$$

Define an involutive automorphism σ on $\mathcal{A}_{\mathrm{CAR}}(H)$ via the requirement $\sigma(a(f)) = -a(f)$, for all $f \in H$. Then

$$\Big(\mathcal{A}_{\mathrm{CAR}}(H), \{\mathcal{A}_{\mathrm{CAR}}(M)\}_{M\in I}\Big)$$

defines a quasi-local algebra (proof see [2] vol II, Proposition 5.2.6).

Exercise 2.3.5. *With the notation in 2. and $M \in I$ let us write*

$$P^o(M) := \textit{odd polynomials in elements } a(f) \textit{ and } a(g) \textit{ where } f, g \in M$$
$$P^e(M) := \textit{even polynomials in elements } a(f) \textit{ and } a(g) \textit{ where } f, g \in M.$$

Show that

(a) $P^e(M) \subset \mathcal{A}^e_{\mathrm{CAR}}(M)$ and $P^o(M) \subset \mathcal{A}^o_{\mathrm{CAR}}(M)$.

(b) the conditions (e) in Definition 2.3.2 are fulfilled if we replace there $\mathcal{A}^e_{\mathrm{CAR}}(M)$ by polynomial $P^e(M)$ and $\mathcal{A}^o_{\mathrm{CAR}}(M)$ by polynomial $P^o(M)$.

3. Let H be a vector space over $\mathbb{R}$ equipped with a non-degenerated symplectic bilinear form $b : H \times H \to \mathbb{R}$. Define the index set

$$I := \big\{M \subset H \ : \ M \text{ is a subspace}\big\}$$

ordered by inclusion and with the orthogonality relation $\perp$ in Example 2.3.1, 2.:

$$M \perp N :\Longleftrightarrow \ b(\ell, g) = 0 \quad \text{for all } \ell \in M \text{ and } g \in N.$$

In particular, it holds

$$H = \bigcup_{M\in I} M.$$

Let $\mathcal{A}_{\mathrm{CCR}}(H)$ be the *CCR-algbera* over H generated by Weyl-operators $\{W(f) \ : \ f \in H\}$ with the conditions in Definition 2.2.2. For $M \in I$ put

$$\mathcal{A}_{\mathrm{CCR}}(M) := C^*\textit{-algbera generated by } W(f) \textit{ with } f \in M.$$

With the involutive automorphism $\sigma = \mathrm{id}$

$$\Big(\mathcal{A}_{\mathrm{CCR}}(H), \{\mathcal{A}_{\mathrm{CCR}}(M)\}_{M\in I}\Big)$$

defines a quasi-local algebra (proof, see [2] vol. II, Proposition 5.2.10), e,g if $\sigma(f, g) = 0$, then

$$W(f)W(g) = W(f + g) = W(g)W(f).$$

Remark 2.3.6. The Examples 2.3.4 2. and 3. have different features. Whereas one always has equality $\mathcal{A}_{\mathrm{CAR}}(h) = \mathcal{A}_{\mathrm{CAR}}(H)$ for any dense subset h of the Hilbert space H it can be shown that

$$\mathcal{A}_{\mathrm{CCR}}(H_1) \cong \mathcal{A}_{\mathrm{CCR}}(H_2)$$

for $H_1 \subset H_2$ exactly holds in the case where $H_1 = H_2$.

2.4 States, representations and Gelfand-Segal construction

Let $\mathcal{A}$ be a C^*-algebra. To simplify the proofs we assume that $\mathcal{A}$ is unital with unit $e \in \mathcal{A}$. However, most of the results here are also true in general and in the proofs one may use so called *approximate units* which always exist (or an extension to a unital algebra).

We start with some remarks on *self-adjoint functional calculus.* Let $A = A^* \in \mathcal{A}$ be selfadjoint. Consider the commutative C^*-algebra $\mathcal{A}_A$ which is generated by A and the unit $e \in \mathcal{A}$. According to EXERCISE 8 there is an isometric $*$-isomorphism

$$\pi : \mathcal{A}_A \longrightarrow C\big(\sigma(A)\big),$$

where $C\big(\sigma(A)\big)$ denotes the C^*-algebra of continuous functions on the spectrum $\sigma(A)$ of A and such that $\pi \circ p(A) = p$ for all polynomials. Given $f \in C(\sigma(A))$ we define

$$f(A) := \pi^{-1}(f) \in \mathcal{A}_A \subset \mathcal{A}. \tag{2.4.1}$$

Hence we have for $f, g \in C(\sigma(A))$:

$$(fg)(A) = f(A)g(A) \qquad \text{and} \qquad f(A)^* = \overline{f}(A) \tag{2.4.2}$$

by using the fact that π^{-1} is a $*$-isomorphism.

Exercise 2.4.1. *Let $A \in \mathcal{A}$ be selfadjoint, i.e. $A = A^*$. Show that $\sigma(A) \subset \mathbb{R}$.*

Definition 2.4.2. An element $A \in \mathcal{A}$ is called *positive* if it is self-adjoint and $\sigma(A) \subset [0, \infty)$.

If $A \in \mathcal{A}$ is positive then we write $A \geq 0$ and by $A \geq B$ we mean that $A - B \geq 0$.

Exercise 2.4.3. *Let $A \in \mathcal{A}$ be self-adjoint, i.e. $A = A^*$ with $\|A\| \leq 2$. Then $A \geq 0$ if and only if $\|e - A\| \leq 1$.*

Exercise 2.4.4. *A subset $\mathcal{C} \subset \mathcal{A}$ is called a "cone" if $\mathcal{C}$ is invariant under multiplications with $\lambda \in (0, \infty)$. Show*

(1) *What are the positive elements of $\mathcal{A} = C(X)$ where X is a compact Hausdorff space?*

(2) *The positive elements of a C^*-algebra form a closed convex cone.*

Hint: *Use the characterization of positivity in Exercise 2.4.3.*

(3) *If $A, B \in \mathcal{A}$ are positive, then $A + B$ is positive.*

(4) *Elements of the form AA^* are positive.*

Exercise 2.4.5. *Let $A \in \mathcal{A}$ be positive. Show that there exist a positive element $B \in \mathcal{A}$ such that $A = B^2$. We write $B = A^{\frac{1}{2}}$.*

Hint: *Use the above self-adjoint functional calculus.*

2.4.1 Positive functional and states

We write $\mathcal{A}^*$ [11] for the *topological dual* of $\mathcal{A}$ consisting of all continuous linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ and with norm

$$\|\varphi\|_{\mathcal{A}^*} := \sup\big\{|\varphi(A)| \ : \ A \in \mathcal{A} \text{ and } \|A\| = 1\big\}.$$

Example 2.4.6. Let X be a compact Hausdorff-space and let $\mathcal{A} = C(X)$. Then $\mathcal{A}^*$ can be identified with the space of all complex Borel measures on X.

A linear functional φ is called *positive* if $\varphi(A^*A) \geq 0$ holds for all $A \in \mathcal{A}$. (Here we do not assume continuity of φ explicitely, it will be a consequence of positivity)

Definition 2.4.7 (state)**.** A positive functional $\varphi \in \mathcal{A}^*$ with norm $\|\varphi\| = 1$ is called *state*. We denote the set of all states in $\mathcal{A}^*$ by $E_{\mathcal{A}}$.

Exercise 2.4.8. *Each element $A \in \mathcal{A}$ with $\|A\| \leq 1$ can be decomposed in the form*

$$A = B_0 - B_1 + i(B_2 - B_3), \tag{2.4.3}$$

where $B_j \in \mathcal{A}$ with $B_j \geq 0$ and $\|B_j\| \leq 1$ for $j = 0, \cdots, 3$.

Proof. First, decompose A in real and imaginary part:

$$A_r = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_i := \frac{1}{2i}(A - A^*)$$

both are selfadjoint, i.e. $A_r = A_r^*$ and $A_i = A_i^*$. We further decompose $A_r = A_{r,+} - A_{r,-}$ and $A_i = A_{i,+} - A_{i,-}$ into their "positive" and "negative parts"

$$A_{r,\pm} = \frac{1}{2}(|A_r| \pm A_r) := f_\pm(A_r) \quad \text{and} \quad A_{i,\pm} = \frac{1}{2}(|A_i| \pm A_i) = f_\pm(A_i).$$

where $f_\pm = (|x| \pm x)/2$ maps $\sigma(A_r)$ and $\sigma(A_i)$ to $[0,\infty)$. Using the relation (2.4.2) and taking square root of $f_\pm$ the first assertion follows. □

We show some simple properties of positive functionals. Note that the proof makes neither use of the closedness of $\mathcal{A}$ nor of the C^*-property of the norm.

Lemma 2.4.9. *Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be positive and linear, then we have for all $A, B \in \mathcal{A}$:*

(1) $\varphi(A^*B) = \overline{\varphi(B^*A)}$. *In particular, with $B = e$ one has $\varphi(A^*) = \overline{\varphi(A)}$.*

(2) $|\varphi(A^*B)|^2 \leq \varphi(A^*A)\varphi(B^*B)$, *(Cauchy-Schwarz inequality),*

Proof. For all $\lambda \in \mathbb{C}$ it follows from the positivity of φ:

$$0 \leq \varphi\Big((\lambda A + B)^*(\lambda A + B)\Big) = |\lambda|^2\varphi(A^*A) + \overline{\lambda}\varphi(A^*B) + \lambda\varphi(B^*A) + \varphi(B^*B)$$

(1): Taking the imaginary part of both sides gives:

$$0 = \overline{\lambda}\big[\varphi(A^*B) - \overline{\varphi(B^*A)}\big] - \lambda\big[\overline{\varphi(A^*B)} - \varphi(B^*A)\big] = 2i\mathrm{Im}\,\Big[\lambda\big[\,\varphi(A^*B) - \overline{\varphi(B^*A)}\,\big]\Big].$$

[11] we change the notation to $\mathcal{A}^*$ since $\mathcal{A}'$ usually means the commutant of $\mathcal{A}$

Since this is true for all $\lambda \in \mathbb{C}$, we conclude (1).

(2): Using (1) in the above inequality gives

$$0 \leq |\lambda|^2 \varphi(A^*A) + \overline{\lambda}\varphi(A^*B) + \lambda\overline{\varphi(A^*B)} + \varphi(B^*B).$$

If $\varphi(A^*A) = 0$ then we conclude that $\varphi(A^*B) = 0$ and (2) follows trivially. Otherwise we choose

$$\lambda := -\varphi(A^*B)/\varphi(A^*A),$$

which implies (2) again. □

Proposition 2.4.10. *Let φ be a positive linear functional on a unital C^*-algebra $\mathcal{A}$, then φ is continuous with $\varphi(e) = \|\varphi\|$ (e is the unit in $\mathcal{A}$).*

Proof. Assume that φ is unbounded and consider

$$M := \sup\Big\{\varphi(A) \;:\; A \geq 0,\ \|A\| \leq 1\Big\} \in \mathbb{R}_+ \cup \{\infty\}. \tag{2.4.4}$$

Assume that $M < \infty$ and let $A \in \mathcal{A}$ with $\|A\| \leq 1$. According to Exercise 2.4.8 we can decompose A in the form

$$A = (B_0 - B_1) + i(B_2 - B_3)$$

where $B_j \geq 0$ and $\|B_j\| \leq 1$. Hence, by the triangle inequality

$$|\varphi(A)| \leq \sum_{j=0}^{3} \big|\varphi(B_j)\big| \leq 4M < \infty$$

which contradicts the assumption that φ is unbounded. Hence $M = \infty$ and we can choose a sequence $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ with $\|A_j\| \leq 1$ and

$$\varphi(A_j) > 2^j, \qquad j \in \mathbb{N}.$$

Consider the partial sums $S_m := \sum_{j=0}^{m} 2^{-j} A_j \in \mathcal{A}$ where $m \in \mathbb{N}$. Then

$$S := \lim_{m\to\infty} S_m \in \mathcal{A}$$

exists and is positive (according to Exercise 2.4.3, (2)) and $S_m \leq S$ for all m. [12] We have for all $m \in \mathbb{N}$:

$$\infty > \varphi(S) \geq \varphi(S_m) = \sum_{j=0}^{m} \underbrace{2^{-j}\varphi(A_j)}_{>1} > m + 1,$$

which is a contradiction. Hence φ must be bounded.

It remains to show that $\|\varphi\| = \varphi(e)$. Since $\|e\| = 1$ we have $\varphi(e) \leq \|\varphi\|$ and the Cauchy-Schwarz inequality (Lemma 2.4.9) shows:

$$|\varphi(A)|^2 = |\varphi(Ae)|^2 \leq \varphi(AA^*)\varphi(e) \leq \|\varphi\|\|AA^*\|\varphi(e) = \|\varphi\|\|A\|^2\varphi(e).$$

Dividing both sides by $\|A\|^2$ and taking the supremum over $0 \neq A \in \mathcal{A}$ on the right hand side gives

$$\|\varphi\|^2 \leq \|\varphi\|\varphi(e).$$

Hence $\|\varphi\| \leq \varphi(e)$ and we have proven equality $\|\varphi\| = \varphi(e)$. □

[12] The positive elements of a C^*-algebra form a closed convex cone.

Corollary 2.4.11. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\varphi_1, \varphi_2 \in \mathcal{A}^*$ be positive functionals. Then*

(i) *the sum $\varphi_1 + \varphi_2 \in \mathcal{A}^*$ is positive with norm $\|\varphi_1 + \varphi_2\|_{\mathcal{A}^*} = \|\varphi_1\|_{\mathcal{A}^*} + \|\varphi_2\|_{\mathcal{A}^*}$,*

(ii) *the states over $\mathcal{A}$ form a convex subset of $\mathcal{A}^*$.*

Proof. (i): It is clear that $\varphi_1 + \varphi_2$ is positive. Moreover, it follows from Proposition 2.4.10 that

$$\|\varphi_1 + \varphi_2\|_{\mathcal{A}^*} = (\varphi_1 + \varphi_2)(e) = \varphi_1(e) + \varphi_2(e) = \|\varphi_1\|_{\mathcal{A}^*} + \|\varphi_2\|_{\mathcal{A}^*}.$$

(ii): let $\lambda \in [0, 1]$ and assume that $\varphi_1, \varphi_2 \in \mathcal{A}^*$ are states, i.e. $\|\varphi_1\|_{\mathcal{A}^*} = \|\varphi_2\|_{\mathcal{A}^*} = 1$, then

$$\|\lambda\varphi_1 + (1 - \lambda)\varphi_2\|_{\mathcal{A}^*} = \lambda\|\varphi_1\|_{\mathcal{A}^*} + (1 - \lambda)\|\varphi_2\|_{\mathcal{A}^*} = 1,$$

where we have used the property (i). The convexity follows. □

We can define a partial ordering on $\mathcal{A}^*$ using the notion of "positivity".

Definition 2.4.12. Let $\varphi_1, \varphi_2 \in \mathcal{A}^*$ be positive, then we write $\varphi_1 \geq \varphi_2$ if $\varphi_1 - \varphi_2$ is positive. In this case one says "φ_1 majorizes φ_2.

Assume that $\varphi_1, \varphi_2 \in \mathcal{A}^*$ are states and fix $\lambda \in [0, 1]$. According to Corollary 2.4.11 we know that $\varphi := \lambda\varphi_1 + (1 - \lambda)\varphi_2$ is a stated with

$$\varphi \geq \lambda\varphi_1 \quad \text{and} \quad \varphi \geq (1 - \lambda)\varphi_2.$$

States that cannot be expressed as a non-trivial convex combination of two other states will play a special role.

Definition 2.4.13. A state $\varphi \in \mathcal{A}^*$ is called *pure* if the only positive linear functionals that are majorized by φ have the form $\lambda\varphi$ with $\lambda \in [0, 1]$. We write $P_{\mathcal{A}} \subset E_{\mathcal{A}}$ for the set of pure states.

The pure states are the so called *extreme points* of $E_{\mathcal{A}}$. If K is a subset of a vector space X, then $\rho \in K$ is called *extreme point* of K if it cannot be expressed in the form

$$\rho = \alpha\rho_1 + (1 - \alpha)\rho_2, \qquad \text{with} \quad \alpha \in (0, 1) \quad \text{and} \quad \rho_1, \rho_2 \in K.$$

In this framework the following is an important result:

Theorem 2.4.14 (Krein-Milman [13])**.** *Let X be a topological vector space on which the dual X^* separates points. If K is a non-empty compact convex set in X, then K is the closed convex hull of its extreme points. In particular, the set of extreme points is non-empty.*

Exercise 2.4.15. *Let $\mathcal{A}$ be a unital C^*-algebra. Then the set of states is a weak-*-compact convex subset of $\mathcal{A}^*$.*

[13] Mark Krein (1907-1989) russian mathematician, David Milman (1912-1982) russian/israeli mathematician

2.4.2 Star-homomorphisms

Before introducing the important concept of representations we start with some general observations on $*$-homomorphism between C^*-algebras.

Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras with units $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Consider a $*$-homomorphism

$$\pi : \mathcal{A} \longrightarrow \mathcal{B} \tag{2.4.5}$$

We assume that If $\pi(e_{\mathcal{A}}) = e_{\mathcal{B}}$. Otherwise we replace $\mathcal{B}$ by the C^*-subalgebra $\widetilde{\mathcal{B}} \subset \mathcal{B}$ defined by

$$\widetilde{\mathcal{B}} := \pi(e_{\mathcal{A}})\mathcal{B}\pi(e_{\mathcal{A}}) = \Big\{\pi(e_{\mathcal{A}})B\pi(e_{\mathcal{A}}) \; : \; B \in \mathcal{B}\Big\}$$

with the same norm as $\mathcal{B}$ and the unit

$$e_{\widetilde{\mathcal{B}}} := \pi(e_{\mathcal{A}})e_{\mathcal{B}}\pi(e_{\mathcal{A}}) = \pi(e_{\mathcal{A}}).$$

Assume that $A \in \mathcal{A}$ and $\lambda \in \rho_{\mathcal{A}}(A)$= *"resolvent set of* $\mathcal{A}$", i.e. $A - \lambda e_{\mathcal{A}} \in \mathcal{A}^{-1}$. Then

$$\pi(A) - \lambda e_{\mathcal{B}} = \pi\big(A - \lambda e_{\mathcal{A}}\big) \in \mathcal{B}^{-1}.$$

The inverse is given by $\pi\big((A - \lambda e_{\mathcal{A}})^{-1}\big)$ and therefore $\lambda \in \rho_{\mathcal{B}}(\pi(A))$. In particular, we have for all $A \in \mathcal{A}$:

$$\sigma_{\mathcal{B}}\big(\pi(A)\big) \subset \sigma_{\mathcal{A}}(A). \tag{2.4.6}$$

Here $\sigma_{\mathcal{A}}(\cdot)$ and $\sigma_{\mathcal{B}}(\cdot)$ denote the spectrum in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Proposition 2.4.16. *The $*$-homomorphism π is (automatically) continuous and contractive, i.e. $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$ and note that $\pi(AA^*) \in \mathcal{B}$ is self-adjoint. It follows from the inclusion (2.4.6) and the property

$$\|C\| = r(C) = \textit{spectral radius of } C$$

for all self-adjoint elements C of a C^*-algebra that

$$\begin{aligned}\|\pi(A)\|^2 = \|\pi(A)\pi(A)^*\| = \|\pi(AA^*)\| &= \sup\big\{\lambda \in \sigma_{\mathcal{B}}\big(\pi(AA^*)\big)\big\} \\ &\leq \sup\big\{\lambda \in \sigma_{\mathcal{A}}\big(AA^*\big)\big\} = \|AA^*\| = \|A\|^2.\end{aligned}$$

By taking the square root on both sides the assertion follows. □

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative unital C^*-algebras with maximal ideal spaces $M(\mathcal{A})$ and $M(\mathcal{B})$, respectively (interpreted as multiplicative functionals). Let $\pi : \mathcal{A} \to \mathcal{B}$ be an injective $*$-homomorphism which maps the unit $e_{\mathcal{A}}$ in $\mathcal{A}$ to the unit $e_{\mathcal{B}}$ in $\mathcal{B}$. Then π induces a map

$$\pi^t : M(\mathcal{B}) \to M(\mathcal{A}) : m \mapsto \pi^t(m) := m \circ \pi, \tag{2.4.7}$$

which is continuous with respect to the weak-$*$-topology. Since $M(\mathcal{B})$ is weak-$*$-compact, it follows that the range $\pi^t(M(\mathcal{A}))$ is compact in $M(\mathcal{B})$ and, in particular, closed.

Lemma 2.4.17. *Under the above assumption it follows that the map π^t in (2.4.7) is surjective, i.e. $\pi^t(M(\mathcal{B})) = M(\mathcal{A})$.*

Proof. Assume that $X := M(\mathcal{A}) \setminus \pi^t(M(\mathcal{B})) \neq \emptyset$ and let $m_0 \in X$. Since X is open we can choose two non-trivial functions $f, g \in C(M(\mathcal{A}))$ with $f \cdot g \equiv 0$ and

$$f(m) \equiv 1 \quad \text{for all } m \in \pi^t(M(\mathcal{B})). \tag{2.4.8}$$

Consider the Gelfand transform

$$\Gamma : \mathcal{A} \longrightarrow C(M(\mathcal{A})) \ni f, g,$$

which is a $*$-isomorphism. Define $A := \Gamma^{-1}(f) \in \mathcal{A}$ and $0 \neq B = \Gamma^{-1}(g) \in \mathcal{A}$. Then we have

$$AB = \Gamma^{-1}(f)\Gamma^{-1}(g) = \Gamma^{-1}(f \cdot g) = 0 \tag{2.4.9}$$

and for all $m \in M(\mathcal{B})$ it follows from (2.4.7)

$$m\big(\pi(A)\big) = \pi^t(m)(A) = \Gamma(A)(\pi^t(m)) = f\big(\pi^t(m)\big) = 1.$$

Therefore $\pi(A)$ does not belong to any maximal ideal of $\mathcal{B}$ and as a consequence must be invertible. Applying the inverse to both sides of

$$\pi(A)\pi(B) = \pi(AB) = \pi(0) = 0$$

gives $\pi(B) = 0$ and by injectivity $B = 0$ which is a contradiction. □

Corollary 2.4.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras (not necessarily commutative) and assume that $\pi : \mathcal{A} \to \mathcal{B}$ an injective $*$-homomorphism. Then π is isometric, i.e.*

$$\|\pi(A)\| = \|A\|, \qquad \textit{for all} \quad A \in \mathcal{A}.$$

Proof. First assume that $A = A^* \in \mathcal{A}$ is self-adjoint. Without restriction we can assume that $\pi(e_{\mathcal{A}}) = e_{\mathcal{B}}$. Denote by $\mathcal{A}_A$ and $\mathcal{B}_{\pi(A)}$ the commutative C^*-subalgebras of $\mathcal{A}$ and $\mathcal{B}$ generated by A and $\pi(A)$, respectively. According to Lemma 2.4.17 we have

$$\begin{aligned} \|\pi(A)\| = r(\pi(A)) &= \sup\big\{m(\pi(A)) \ : \ m \in M(\mathcal{B})\big\} \\ &= \sup\big\{\pi^t(m)(A) \ : \ m \in M(\mathcal{B})\big\} \\ &= \sup\big\{\widetilde{m}(A) \ : \ \widetilde{m} \in \pi^t(M(\mathcal{B}))\big\} \\ &= \sup\big\{\widetilde{m}(A) \ : \ \widetilde{m} \in M(\mathcal{A})\big\} = r(A) = \|A\|. \end{aligned}$$

For general $A \in \mathcal{A}$ this observation implies

$$\|A\|^2 = \|A^*A\| = \|\pi(AA^*)\| = \|\pi(A)\pi(A)^*\| = \|\pi(A)\|^2$$

and the assertion follows by taking the square root. □

Corollary 2.4.19. *Let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism, then the range $\pi(A)$ is a C^*-algebra and in particular closed.*

Proof. According to Corollary 2.4.18 the induced map

$$\tilde{\pi} : \mathcal{A}/\text{ker } \pi \longrightarrow \mathcal{B}$$

is an injective isometric $*$-homomorphism. Since the quotient $\mathcal{A}/\text{ker}$ has the structure of a unital C^*-algebra the assertion follows. □

Exercise 2.4.20. *Assume that $\pi(A) > 0$ for all $A \in \mathcal{A}$ with $A > 0$. Show that π is isometric.*

2.4.3 Representations

In order to show the connection between the abstract C^*-algebras and Hilbert space operators we introduce the concept of representation. A link between representations and states is then given by the *Gelfand-Naimark-Segal construction* which we explain in this section.

Consider a complex Hilbert space H and choose $\mathcal{B} := \mathcal{L}(H)$. Let

$$\pi : \mathcal{A} \longrightarrow \mathcal{L}(H)$$

be a $*$-homomorphism.

Definition 2.4.21. The pair (H, π) is called a *representation* of the C^*-algebra $\mathcal{A}$.

(i) A representation is called *faithful* if π is even a $*$-isomorphism (then it is isometric!).

(ii) Two representations π and ρ of $\mathcal{A}$ on H_1 and H_2, respectively, are called (*unitarily*) *equivalent*, if there is a unitary operator $U : H_1 \to H_2$ with

$$U\pi(x)U^* = \rho(x), \quad \text{for all } x \in \mathcal{A}.$$

We will often identify unitarily equivalent representations.

Let $\mathcal{M} \subset \mathcal{L}(H)$ be a set of bounded operator on H. A vector $\Omega \in H$ is called *cyclic* for $\mathcal{M}$ if the inclusion $\{A\Omega \,:\, A \in \mathcal{M}\} \subset H$ is dense. We define

Definition 2.4.22. A *"cyclic representation"* of a C^*-algebra $\mathcal{A}$ by definition is a triple (H, π, Ω), where (H, π) is a representation of $\mathcal{A}$ and $\Omega \in H$ is cyclic for $\pi(\mathcal{A})$.

Exercise 2.4.23. *Let (H, π, Ω) be a cyclic representation. Then (H, π) is non-degenerate in the sense that*

$$\{f \in H \,:\, \pi(A)f = 0 \ \text{for all } A \in \mathcal{A}\} = 0$$

Definition 2.4.24 (Commutant)**.** The *commutant* $\mathcal{M}'$ of $\mathcal{M}$ is defined by

$$\mathcal{M}' := \Big\{C \in \mathcal{L}(H) \,:\, [C, M] = CM - MC = 0 \ \text{ for all } M \in \mathcal{M}\Big\}.$$

A subspace $G \subset H$ is said to be *invariant under* $\mathcal{M}$ if

$$T(G) \subset G, \qquad \text{for all } \ T \in \mathcal{M}.$$

We call $\mathcal{M}$ *irreducible* if the only closed invariant subspaces $G \subset H$ of $\mathcal{M}$ are $G = \{0\}$ and $G = H$. There are some relation between these notions. Without a proof we mention:

Proposition 2.4.25. *Let $\mathcal{M} \subset \mathcal{L}(H)$ be a "self-adjoint" subset, i.e. $M \in \mathcal{M}$ implies that $M^* \in \mathcal{M}$. Then* (i) *and* (ii) *are equivalent*

(i) *$\mathcal{M}$ is irreducible*

(ii) $\mathcal{M}' = \{\lambda \cdot \mathrm{id} \,:\, \lambda \in \mathbb{C}\}$,

Proof. See [8]. □

Exercise 2.4.26. *Proof* (ii) $\Longrightarrow$ (i) *of Proposition 2.4.25.*

Definition 2.4.27. A representation (H, π) of a C^*-algebra $\mathcal{A}$ is called *irreducible* if the set $\pi(\mathcal{A}) \subset \mathcal{L}(H)$ is irreducible.

Exercise 2.4.28. *If (H, π) is an irreducible representation of the C^*-algebra $\mathcal{A}$, then each vector $\xi \in H$ is cyclic or $\pi(\mathcal{A}) = \{0\}$ and $H = \mathbb{C}$.*

2.4.4 The GNS-construction

The GNS construction was discovered independently by Gelfand/Naimark and by I. Segal. It provides a method to construct representations of C^*-algebras with the help of positive linear functionals.

Let $\mathcal{A}$ be a C^*-algebra with positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, i.e. $\varphi \in \mathcal{A}^*$. We put a pre-inner product on $\mathcal{A}$ by

$$\big\langle A, B \big\rangle_\varphi := \varphi(A^* B)$$

(see Lemma 2.4.9). We define

$$\mathcal{N}_\varphi := \big\{ N \in \mathcal{A} \;:\; \varphi(N^* N) = 0 \big\}.$$

Lemma 2.4.29. *$\mathcal{N}_\varphi$ is a closed left ideal of $\mathcal{A}$, i.e. $AN \in \mathcal{N}_\varphi$ for all $A \in \mathcal{A}$ and $N \in \mathcal{N}_\varphi$.*

Proof. As a preparation for the proof we show that for all $A, B \in \mathcal{A}$ we have

$$\varphi(A^* B^* B A) \leq \|B\|^2 \varphi(A^* A). \tag{2.4.10}$$

In fact, since $\sigma(B^* B) \subset [0, \|B\|^2]$ it follows that $B^* B - \|B\|^2 e \leq 0$ and therefore

$$\|B\|^2 A^* A - A^* B^* B A = A^* \underbrace{\big(\|B\|^2 e - B^* B\big)}_{\text{is of form } C^* C \geq 0} A = (CA)^* C A \geq 0. \tag{2.4.11}$$

Since φ is positive we find that (2.4.10) holds. Let $N \in \mathcal{N}_\varphi$ and $A \in \mathcal{A}$ then $AN \in \mathcal{N}_\varphi$ follows by applying (2.4.10) from:

$$0 \leq \varphi\big((AN)^*(AN)\big) = \varphi(N^* A^* A N) \leq \underbrace{\varphi(N^* N)}_{=0} \|A^* A\| = 0.$$

By using the Cauchy-Schwarz inequality again one easily shows that $\mathcal{N}_\varphi$ is a linear space and closedness follows from the continuity of φ. □

According to Lemma 2.4.29 we can consider the quotient algebra

$$\mathcal{A}/\mathcal{N}_\varphi = \big\{ \widehat{A} := A + \mathcal{N}_\varphi \;:\; A \in \mathcal{A} \big\}$$

with the inner product (for simplicity we use the same notation as before):

$$\big\langle \widehat{A}, \widehat{B} \big\rangle_\varphi := \varphi(A^* B). \tag{2.4.12}$$

Exercise 2.4.30. *Check that the inner-product (2.4.12) is well-defined on the quotient $\mathcal{A}/\mathcal{N}_\varphi$.*

Definition 2.4.31. We write H_φ for the Hilbert space completion of $(\mathcal{A}/\mathcal{N}_\varphi, \langle \cdot, \cdot \rangle_\varphi)$ which then is a Hilbert space.

Our next aim is to define a representation of $\mathcal{A}$ on H_φ. The quotient $\mathcal{A}/\mathcal{N}_\varphi$ can be identified with a closed subspace of H_φ. For any given $A \in \mathcal{A}$ we define $\pi_\varphi(A) : \mathcal{A}/\mathcal{N}_\varphi \to \mathcal{A}/\mathcal{N}_\varphi$ by

$$\pi_\varphi(A)(B + \mathcal{N}_\varphi) := AB + \mathcal{N}_\varphi. \tag{2.4.13}$$

Since $\mathcal{N}_\varphi$ is a left ideal in $\mathcal{A}$ it is clear that $\pi_\varphi(A)$ is well-defined. We show that $\pi_\varphi(A)$ is continuous on $\mathcal{A}/\mathcal{N}_\varphi$ with respect to the norm $\|\cdot\|_\varphi$ induced by the inner-product $\langle\cdot,\cdot\rangle_\varphi$.

$$\begin{aligned}\|\pi_\varphi(A)(\widehat{B})\|_\varphi^2 &= \|AB+\mathcal{N}_\varphi\|_\varphi^2\\ &= \varphi\big((AB)^*AB\big)\\ &= \varphi(B^*A^*AB)\\ &\le \|A^*A\|\varphi(B^*B) = \|A\|^2\|\widehat{B}\|_\varphi^2.\end{aligned}$$

The inequality follows from (2.4.10). Hence $\pi_\varphi(A)$ extends to a bounded operator on the completion H_φ with

$$\|\pi_\varphi(A)\|_\varphi \le \|A\|$$

and clearly the assignment

$$\pi_\varphi : \mathcal{A} \longrightarrow \mathcal{L}(H_\varphi) : A \mapsto \pi_\varphi(A)$$

gives a representation of $\mathcal{A}$ on H_φ (in particular we have $\pi_\varphi(A_1)\pi_\varphi(A_2) = \pi_\varphi(A_1A_2)$.)

Definition 2.4.32 (GNS-representation)**.** (H_φ, π_φ) is called *GNS-representation* associated with φ.

We show that the GNS-representation is cyclic. Put $\xi_\varphi := \pi_\varphi(e_\mathcal{A}) = e_\mathcal{A} + \mathcal{N}_\varphi = \widehat{e_\mathcal{A}} \in H_\varphi$. Then we have for all $A \in \mathcal{A}$

$$\big\langle \xi_\varphi, \pi_\varphi(A)\xi_\varphi\big\rangle_\varphi = \big\langle \widehat{e_\mathcal{A}}, \widehat{A}\big\rangle_\varphi = \varphi(e_\mathcal{A}^*A) = \varphi(A) \tag{2.4.14}$$

and due to Proposition 2.4.10 we find

$$\|\xi_\varphi\|_\varphi^2 = \varphi(e_\mathcal{A}) = \|\varphi\|_{\mathcal{A}^*}.$$

Definition 2.4.33. States of the form $\pi(A) = \langle\Omega, \pi(A)\Omega\rangle$ where (H,π) is a representation of a C^*-algebra $\mathcal{A}$ and $\Omega \in H$ are called *vector states.*

Proposition 2.4.34. *The triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$ defines a cyclic representation of $\mathcal{A}$.*

Proof. By definition we need to show that

$$\big\{\pi_\varphi(A)\xi_\varphi \ : \ A\in\mathcal{A}\big\} = \big\{A+\mathcal{N}_\varphi \ : \ A\in\mathcal{A}\big\} = \mathcal{A}/\mathcal{N}_\varphi$$

is dense in H_φ. But this is clear since H_φ is the completion of $\mathcal{A}/\mathcal{N}_\varphi$. □

Exercise 2.4.35. *Let $\mathcal{A} := \mathbb{C}^{n\times n}$ ="C^*-algebra of $n\times n$ complex matrices". On $\mathcal{A}$ consider the trace functional $\varphi_{\mathrm{tr}} : \mathcal{A}\to\mathbb{C}$ defined by the usual matrix trace*

$$\varphi_{\mathrm{tr}}(A) = \mathrm{trace}(A), \qquad A\in\mathcal{A}.$$

(a) *Show that φ_{tr} is a positive linear functional on $\mathcal{A}$.*

(b) *Give an explicit description of the GNS-representation $(H_{\varphi_{\mathrm{tr}}}, \pi_{\varphi_{\mathrm{tr}}}, \xi_{\varphi_{\mathrm{tr}}})$.*

The GNS-construction gives is a relation between the notions "pure state" and "irreducible representation".

Theorem 2.4.36. *Let $\mathcal{A}$ be a unital C^*-algebra with a state φ and corresponding GNS-representation $(H_\varphi, \pi_\varphi, \xi_\varphi)$. Then* (a) *and* (b) *are equivalent:*

(a) *the representation (H_φ, π_φ) is irreducible*

(b) *φ is a pure state (extreme point of $E_\mathcal{A}$).*

Proof. (a) $\Longrightarrow$ (b): Assume that (a) holds and φ is not a pure state. Then we find a state ω not of the form $\omega = \lambda\varphi$ where $\lambda \in [0,1]$ with $\omega \leq \varphi$. The Cauchy-Schwarz inequality implies for all $A, B \in \mathcal{A}$:

$$\begin{aligned} |\omega(B^*A)|^2 &\leq \omega(B^*B)\omega(A^*A) \\ &\leq \varphi(B^*B)\varphi(A^*A) \\ &= \|\widehat{B}\|_\varphi^2 \|\widehat{A}\|_\varphi^2 \\ &= \|\pi_\varphi(B)\xi_\varphi\|_\varphi^2 \|\pi_\varphi(A)\xi_\varphi\|_\varphi^2. \end{aligned}$$

Therefore the assignment

$$\mathcal{A}/\mathcal{N}_\varphi \times \mathcal{A}/\mathcal{N}_\varphi \longrightarrow \mathbb{C} : \big(\pi_\varphi(B)\xi_\varphi, \pi_\varphi(A)\xi_\varphi\big) \mapsto \omega(B^*A)$$

is continuous w.r.t. to $\|\cdot\|_\varphi$ and extends to a bounded bilinear form $S : H_\varphi \times H_\varphi \longrightarrow \mathbb{C}$. Hence we can choose a bounded operator $T \in \mathcal{L}(H_\varphi)$ such that

$$\big\langle \pi_\varphi(B)\xi_\varphi, T\pi_\varphi(A)\xi_\varphi \big\rangle_\varphi = S\big(\pi_\varphi(B)\xi_\varphi, \pi_\varphi(A)\xi_\varphi\big) = \omega(B^*A).$$

If there was $\lambda \in \mathbb{R}$ such that $T = \lambda \cdot \mathrm{id}$ then we had

$$\lambda\varphi(A^*A) = \big\langle \pi_\varphi(A)\xi_\varphi, \lambda\pi_\varphi(A)\xi_\varphi \big\rangle_\varphi = \omega(A^*A),$$

with $\lambda \in [0,1]$ which contradicts our above assumption. Fix $A, B, C \in \mathcal{A}$, then we have

$$\begin{aligned} \big\langle \pi_\varphi(B)\xi_\varphi, T \overbrace{\pi_\varphi(C)\pi_\varphi(A)}^{=\pi_\varphi(CA)} \xi_\varphi \big\rangle_\varphi &= \omega(B^*CA) \\ &= \omega\big((C^*B)^*A\big) \\ &= \big\langle \pi_\varphi(C^*B)\xi_\varphi, T\pi_\varphi(A)\xi_\varphi \big\rangle_\varphi \\ &= \big\langle \pi_\varphi(B)\xi_\varphi, \pi_\varphi(C)T\pi_\varphi(A)\xi_\varphi \big\rangle_\varphi. \end{aligned}$$

Since A and B were chosen arbitrarily we conclude that T commutes with all elements in $\pi_\varphi(\mathcal{A}) \subset \mathcal{L}(H_\varphi)$. In other words $\lambda \cdot \mathrm{id} \neq T$ is in the commutant $\pi_\varphi(\mathcal{A})' \subset \mathcal{L}(H_\varphi)$ and according to Proposition 2.4.25, (i) $\Longrightarrow$ (ii) the representation π_φ cannot be irreducible. Contradiction.

(b) $\Longrightarrow$ (a): No proof here (requires the notation of spectral projections). ☐

As for the uniqueness of the GNS-construction up one can say the following

Exercise 2.4.37. *With the above notation let $(\widetilde{H}, \tilde{\pi}, \tilde{\xi})$ be another cyclic representation of the unital C^*-algebra $\mathcal{A}$ such that $\varphi(A) = \langle \widetilde{\xi}, \widetilde{\pi}(A)\widetilde{\xi} \rangle_{\widetilde{H}}$ for all $A \in \mathcal{A}$, cf. (2.4.14).*

(a) *Show that there is a unitary operator* $U : \widetilde{H} \to H$ *which sets up a unitary equivalence between* π_φ *and* $\widetilde{\pi}$.

The following result (which we state without a proof) sometimes also is called the Gelfand-Naimark theorem (cf. Theorem 2.1.18).

Theorem 2.4.38 (Gelfand-Naimark)**.** *If* $\mathcal{A}$ *is a* C^**-algebra, then* $\mathcal{A}$ *has a faithful representation, i.e.* $\mathcal{A}$ *is isometrically isomorphic to a concrete* C^**-algbera of operators on a Hilbert space* H*. If* $\mathcal{A}$ *is separable, then* H *may be chosen separable.*

2.4.5 The GNS-construction for a matrix algebra

(see Robert's lecture)

Chapter 3

Equilibrium States and KMS condition

(see Robert's lecture)

Chapter 4

Ising model in $2d$

Consider a square lattice $\Lambda \subset \mathbb{R}^2$ with n rows and n columns, i.e we have $N = n^2$ lattice points.

(i) The *spin variable* is a function $\Lambda \ni p \mapsto s_p \in \{\pm 1\}$.

(ii) A *configuration of the system* is given by

$$S = \big\{ s_p \; : \; p \in \Lambda \big\}.$$

(iii) The *energy* in the configuration state S has the form

$$E_I(S) = -\sum_{\langle pq \rangle} \epsilon_{pq} s_p s_q - B \sum_{p=1}^{N} s_p. \tag{4.0.1}$$

Here

$$\begin{aligned} \langle pq \rangle = \langle qp \rangle &:= \text{direct neighbors in } \Lambda, \\ B &= \text{exterior magnetic field}, \\ \epsilon_{pq} &= \text{interaction energy between } p \text{ and } q. \end{aligned}$$

The *partition function* is given by

$$Q_I(B,T) = \sum_S e^{-\beta E_I(S)}, \qquad \text{with} \quad \beta = \frac{1}{kT}. \tag{4.0.2}$$

The sum is taken over all $2^N = |\{S = (s_1, \cdots, s_N) : s_p = \pm 1\}|$ configurations S. The *Helmholtz free energy* has the form

$$A_I(B,T) = -kT \log Q_I(B,T) = -\beta^{-1} \log Q_I(B,T). \tag{4.0.3}$$

Goal of this section: Calculate the thermodynamical limit for the two dimensional Ising model ("Onsager solution" for the Ising model)

$$\lim_{N\to\infty} \frac{1}{N} \log Q_I(B,T), \qquad N = n^2. \tag{4.0.4}$$

and observe a *phase transition.* [1]

[1] This is the only non-trivial example of a model, in which the phase transition can be calculated mathematically exact.

Simplifications:

(1) Assume that $\epsilon_{pq} = \epsilon > 0$ (ferromagnetism) is *independent* of the pair $\langle pq \rangle$.

(2) Pose *boundary conditions*: add one column and one row to the right and to the bottom which has the same configuration as the first column and the first row, respectively.

Some notation: With $\alpha \in \{1, \cdots, n+1\}$ we write $R_\alpha = (s_{\alpha,1}, \cdots, s_{\alpha,n})$ for the spin coordinates of the α-th row of Λ. It follows from (2) that

$$R_1 = R_{n+1} \quad \text{and} \quad s_{\alpha,1} = s_{\alpha,n+1}, \quad \text{for } \alpha = 1, \cdots n.$$

- *Interaction energies:*

$$E_I(R_\alpha, R_{\alpha+1}) = -\epsilon \sum_{k=1}^{n} s_{\alpha,k} s_{\alpha+1,k} \qquad \text{(between neighboring rows)}$$
$$E_I(R_\alpha) = -\epsilon \sum_{k=1}^{n} s_{\alpha,k} s_{\alpha,k+1} - B \sum_{k=1}^{n} s_{\alpha,k} \quad \text{(within the } \alpha\text{-th row).}$$

If the configuration S of the system is determined by the rows $R_1, \cdots, R_n$, then we can write the energy $E_I(S)$ in (4.0.1) as

$$E_I(S) = \sum_{\alpha=1}^{n} \Big[E_I\big(R_\alpha, R_{\alpha+1}\big) + E_I(R_\alpha) \Big].$$

The partition functions takes the form

$$Q_I(B,T) = \sum_{R_1} \cdots \sum_{R_n} \exp \left\{ -\beta \sum_{\alpha=1}^{n} \Big[E_I\big(R_\alpha, R_{\alpha+1}\big) + E_I(R_\alpha) \Big] \right\}.$$

Strategy: Express this complicated sum in form of a "matrix trace" using the periodic boundary conditions with respect to the rows (i.e. $R_1 = R_{n+1}$).

Consider the set

$$\mathcal{R} = \big\{ (s_1, \cdots, s_n) \ : \ s_p = \pm 1 \big\} = \text{“\textit{possible configurations of the row R}”}, \qquad |\mathcal{R}| = 2^n.$$

Fix an order of $\mathcal{R}$ and define a matrix $P \in \mathcal{M}_{2^n}(\mathbb{R})$ having the entries

$$\big\langle R|P|R' \big\rangle := e^{-\beta[E_I(R,R') + E(R)]}, \qquad R, R' \in \mathcal{R}.$$

We can rewrite $Q_I(B,T)$ in the form:

$$Q_I(B,T) = \sum_{R_1} \cdots \sum_{R_n} \big\langle R_1|P|R_2 \big\rangle \big\langle R_2|P|R_3 \big\rangle \cdots \big\langle R_n|P|R_1 \big\rangle$$
$$= \sum_{R_1} \big\langle R_1|P^n|R_1 \big\rangle = \text{Trace } P^n.$$

Assume that P can be diagonalized with eigenvalues $\{\lambda_1(n), \cdots, \lambda_{2^n}(n)\}$ (counted with multiplicities). Then P^n has the eigenvalues $\{\lambda_1(n)^n, \cdots, \lambda_{2^n}(n)^n\}$. Therefore

$$Q_I(B,T) = \text{Trace } P^n = \sum_{i=1}^{2^n} \lambda_i(n)^n. \tag{4.0.5}$$

Observation: Assume that $\lambda_i(n)$ for all i grow of the order e^n as $n \to \infty$ and let $\lambda_{\max}(n)$ denote the largest eigenvalue for fixed n. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_{\max}(n) = c \in \mathbb{R}.$$

Then we obtain from (4.0.5) that

$$\begin{aligned}
\frac{1}{n} \log \lambda_{\max}(n) &= \frac{1}{n^2} \log \lambda_{\max}(n)^n \\
&\leq \frac{1}{n^2} \log \sum_{i=1}^{2^n} \lambda_i(n)^n = \frac{1}{N} \log Q_I(B,T) \\
&\leq \frac{1}{n^2} \log \left(2^n \lambda_{\max}(n)^n\right) \\
&= \frac{1}{n} \log \lambda_{\max}(n) + \frac{1}{n} \log 2.
\end{aligned}$$

This shows

$$\lim_{N\to\infty} \frac{1}{N} \log Q_I(B,T) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_{\max}(n).$$

Therefore, in order to calculate the limit (4.0.4) we will study the eigenvalues (in particular the largest one) of the matrices P as a function of n.

4.1 A decomposition of the transfer matrix

Let $R = (s_1, \cdots, s_n) \in \mathcal{R}$ and $R' = (s'_1, \cdots, s'_n) \in \mathcal{R}$ be two configuration of rows. The entries of P are:

$$\begin{aligned}
\langle R|P|R'\rangle &= e^{-\beta[E_I(R,R')+E_I(R)]} \\
&= \exp\left\{\beta\epsilon \sum_{k=1}^{n} s_k s'_k + \beta\epsilon \sum_{k=1}^{n} s_k s_{k+1} + \beta B \sum_{k=1}^{n} s_k\right\} \\
&= \prod_{k=1}^{n} e^{\beta B s_k} \cdot \prod_{k=1}^{n} e^{\beta\epsilon s_k s_{k+1}} \cdot \prod_{k=1}^{n} e^{\beta\epsilon s_k s'_k}.
\end{aligned} \tag{4.1.1}$$

define the matrix $Q_1 = (\langle R|Q_1|R'\rangle)_{R,R'\in\mathcal{R}} \in \mathcal{M}_{2^n}(\mathbb{R}) = 2^n \times 2^n$-*real matrices* by

$$\langle R|Q_1|R'\rangle = \prod_{k=1}^{n} e^{\beta\epsilon s_k s'_k}$$

Let Q_2 and Q_3 be the diagonal matrices

$$\langle R|Q_2|R'\rangle = \begin{cases} 0, & \text{if } R \neq R' \\ \prod_{k=1}^{n} e^{\beta\epsilon s_k s_{k+1}}, & \text{if } R = R' \end{cases} \tag{4.1.2}$$

$$\langle R|Q_3|R'\rangle = \begin{cases} 0, & \text{if } R \neq R' \\ \prod_{k=1}^{n} e^{\beta B s_k}, & \text{if } R = R'. \end{cases} \tag{4.1.3}$$

Note that $Q_3 = Id$ in the case where $B = 0$ (we will assume this later on in order to further simplify things).

Lemma 4.1.1. *The matrix P decomposes into a product $P = Q_3 Q_2 Q_1$.*

Proof. This follows from (4.1.1), the well-known formula for the matrix multiplication

$$\langle R|Q_3Q_2Q_1|R'\rangle = \sum_{\tilde{R},\tilde{\tilde{R}}} \langle R|Q_3|\tilde{R}\rangle\langle\tilde{R}|Q_2|\tilde{\tilde{R}}\rangle\langle\tilde{\tilde{R}}|Q_1|R'\rangle$$

together with the definition of the diagonal matrices Q_2 and Q_3. □

Next: Find an expression of Q_1 which we can handle more easily. We need some preparations:

Definition 4.1.2. Let $A_1, \cdots, A_k \in \mathcal{M}_m(\mathbb{C})$ with $k \in \mathbb{N}$ with entries $\langle i|A_l|j\rangle$ for $l = 1, \cdots, k$. Define the *tensor product* $A_1 \otimes A_2 \otimes \cdots \otimes A_k \in \mathcal{M}_{m^k}(\mathbb{C})$ by [2]

$$\big\langle (i_1, \cdots, i_k)|A_1 \otimes \cdots \otimes A_k|(j_1, \cdots, j_k)\big\rangle := \prod_{l=1}^{k} \langle i_l|A_l|j_l\rangle,$$

where $(i_1, \cdots, i_k), (j_1, \cdots, j_k) \in \{1, \cdots, m\}^k$.

Lemma 4.1.3. *Tensor products of matrices $A_l, B_l \in \mathcal{M}_m(\mathbb{C})$ multiply as follows:*

$$\Big(A_1 \otimes \cdots \otimes A_k\Big) \cdot \Big(B_1 \otimes \cdots \otimes B_k\Big) = (A_1 \cdot B_1) \otimes (A_2 \cdot B_2) \otimes \cdots \otimes (A_k \cdot B_k),$$

where "$\cdot$" denotes the usual matrix multiplication.

Proof. Exercise 24, homework 06. □

Let now $m = 2$ and consider in particular the matrix

$$A := \begin{pmatrix} e^{\beta\epsilon} & e^{-\beta\epsilon} \\ e^{-\beta\epsilon} & e^{\beta\epsilon} \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

Lemma 4.1.4. *Q_1 can be expressed as n-fold tensor product of A, i.e. $Q_1 = A \otimes \cdots \otimes A$.*

[2] We can put the tuples $(i_1, \cdots i_k)$ in lexicographical order

Proof. With $R = (s_1, \cdots, s_n), R' = (s'_1, \cdots, s'_n) \in \mathcal{R}$ we have from Lemma 4.1.3

$$\langle R|\underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}|R'\rangle = \prod_{k=1}^{n} \langle s_k|A|s'_k\rangle = \prod_{k=1}^{n} e^{\beta\epsilon s_k s'_k} = \langle R|Q_1|R'\rangle.$$

The assertion follows from the definition of Q_1. □

Recall that the *Pauli matrices* X, Y, Z are defined by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $I \in \mathcal{M}_2(\mathbb{C})$ denote the identity matrix.

Lemma 4.1.5. *With $\theta \in \mathbb{R}$ we have $e^{\theta X} = \cosh(\theta)I + X\sinh(\theta)$. Moreover, A and X are related via*

$$\sqrt{2\sinh(2\epsilon\beta)}\, e^{\theta X} = A, \qquad \textit{where} \qquad \tanh\theta = e^{-2\beta\epsilon}.$$

Proof. Straightforward calculation, (see Exercise 24, (iii) of the homework assignment 6). □

For $\alpha = 1, \cdots, n$ we now put:

$$\begin{aligned} X_\alpha &= I \otimes \cdots \otimes X \otimes \cdots \otimes I \in \mathcal{M}_{2^n}(\mathbb{C}), \\ Y_\alpha &= I \otimes \cdots \otimes Y \otimes \cdots \otimes I \in \mathcal{M}_{2^n}(\mathbb{C}), \\ Z_\alpha &= I \otimes \cdots \otimes Z \otimes \cdots \otimes I \in \mathcal{M}_{2^n}(\mathbb{C}), \end{aligned}$$

(each tensor product has n factors and X, Y, Z are located at the α-th position).

Lemma 4.1.6. *With $\theta > 0$ such that $\tanh\theta = e^{-2\beta\epsilon}$ it holds*

$$Q_1 = A \otimes \cdots \otimes A = \Big[2\sinh(2\epsilon\beta)\Big]^{\frac{n}{2}} e^{\theta(X_1+X_2+\cdots+X_n)}. \tag{4.1.4}$$

Proof. From Lemma 4.1.4 and Lemma 4.1.5 it follows that

$$Q_1 = A \otimes \cdots \otimes A = \Big[2\sinh(2\epsilon\beta)\Big]^{\frac{n}{2}} e^{\theta X} \otimes \cdots \otimes e^{\theta X}.$$

It remains to show that $e^{\theta X} \otimes \cdots \otimes e^{\theta X} = e^{\theta(X_1+X_2+\cdots+X_n)}$. It is clear that $e^{\theta X_\alpha} = I \otimes \cdots \otimes e^{\theta X} \otimes \cdots \otimes$ where the $e^{\theta X}$ is at the α's position. Hence it follows from Lemma 4.1.3 that

$$e^{\theta X} \otimes \cdots \otimes e^{\theta X} = e^{\theta X_1} e^{\theta X_2} \cdots e^{\theta X_n}.$$

Since the matrices X_α and $X_{\alpha'}$ commute for $\alpha \neq \alpha'$ we see that the right hand side of the last equality coincides with $e^{\theta(X_1+X_2+\cdots+X_n)}$. □

Instead of Q_1 we further examine the matrix that appears on the right hand side of (4.1.4) and we set

$$\widetilde{Q}_1 := e^{\theta(X_1+\cdots+X_n)} \in \mathcal{M}_{2^n}(\mathbb{C}). \tag{4.1.5}$$

The diagonal matrices Q_2 and Q_3 in (4.1.2) and (4.1.3) can be expressed in terms of Z_α as well.

Lemma 4.1.7. *Let $Z_{n+1} := Z_1$, then*

$$(a): \; Q_2 = \prod_{\alpha=1}^{n} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}},$$
$$(b): \; Q_3 = \prod_{\alpha=1}^{n} e^{\beta B Z_\alpha}.$$

Proof. (a): Since $Z^2 = 1$ we find for fixed $\alpha \in \{1, \cdots, n\}$ that

$$e^{\beta\epsilon Z_\alpha Z_{\alpha+1}} = \cosh(\beta\epsilon)I + \sinh(\beta\epsilon)Z_\alpha Z_{\alpha+1}.$$

Therefore $e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}$ is diagonal with

$$\begin{aligned}&\big\langle (s_1, \cdots, s_n)|e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}|(s'_1, \cdots, s'_n)\big\rangle = \\ &\quad =\delta_{s_1,s'_1}\cdots\delta_{s_n,s'_n}\begin{cases}\cosh(\beta\epsilon) + \sinh(\beta\epsilon) = e^{\beta\epsilon}, & \text{if } \operatorname{sgn}(s_\alpha) = \operatorname{sgn}(s_{\alpha+1})\\ \cosh(\beta\epsilon) - \sinh(\beta\epsilon) = e^{-\beta\epsilon}, & \text{if } \operatorname{sgn}(s_\alpha) \neq \operatorname{sgn}(s_{\alpha+1})\end{cases}\\ &\quad =\delta_{s_1,s'_1}\cdots\delta_{s_n,s'_n}e^{\beta\epsilon s_\alpha s_{\alpha+1}}.\end{aligned}$$

Now, (a) follows from the definition (4.1.2).
(b): Follows by a similar argument from $e^{\beta B Z_\alpha} = \cosh(\beta B)I + \sinh(\beta B)Z_\alpha$ and (4.1.3). □

Summarizing these calculation we have

Proposition 4.1.8. *Let $\theta > 0$ with $\tanh\theta = e^{-2\beta\epsilon}$. Then the matrix $P \in \mathcal{M}_{2^n}(\mathbb{R})$ decomposes in the form*

$$P = Q_3Q_2Q_1 = \Big[2\sinh(2\epsilon\beta)\Big]^{\frac{n}{2}} \prod_{\alpha=1}^{n} e^{\beta B Z_\alpha} \prod_{\alpha=1}^{n} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}} e^{\theta(X_1+\cdots+X_n)}.$$

4.2 On spin representations of rotations

Consider the following $2n$ matrices in $\mathcal{M}_{2^n}(\mathbb{C})$:

$$\Gamma_{2\alpha} = X_1X_2\cdots X_{\alpha-1}Y_\alpha \quad \text{and} \quad \Gamma_{2\alpha-1} = X_1X_2\cdots X_{\alpha-1}Z_\alpha. \tag{4.2.1}$$

where $\alpha = 1, \cdots, n$. From Lemma 4.1.3 check that $X_\alpha, Y_\alpha, Z_\alpha$ fulfill the relations

I. $\alpha \neq \beta$: then $\big[X_\alpha, X_\beta\big] = \big[Y_\alpha, Y_\beta\big] = \big[Z_\alpha, Z_\beta\big] = 0$ and $[X_\alpha, Y_\beta] = [X_\alpha, Z_\beta] = [Y_\alpha, Z_\beta] = 0$.

II. For fixed $\alpha \in \{1, \cdots, n\}$ the matrices $Z_\alpha, Y_\alpha, Z_\alpha$ are involutive and anti-commute, i.e. it holds $Z_\alpha^2 = Y_\alpha^2 = Z_\alpha^2 = I$ and

$$\big\{X_\alpha, Y_\alpha\big\} = X_\alpha Y_\alpha + Y_\alpha X_\alpha = \big\{Y_\alpha, Z_\alpha\big\} = \big\{X_\alpha, Z_\alpha\big\} = 0$$

Proposition 4.2.1. *The matrices $\Gamma_\nu \in \mathcal{M}_{2^n}(\mathbb{C})$ with $\nu = 1, \cdots, 2n$ fullfil*

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\delta_{\mu,\nu}\mathrm{I}, \qquad \mu, \nu = 1, \cdots, 2n. \tag{4.2.2}$$

Proof. We only check one case. Let $\mu < \nu$, then

$$\Gamma_{2\nu}\Gamma_{2\mu} = X_1 \cdots X_{\nu-1}Y_\nu X_1 \cdots X_{\mu-1}Y_\mu = X_\mu X_{\mu+1} \cdots X_{\nu-1}Y_\nu Y_\mu$$
$$\Gamma_{2\mu}\Gamma_{2\nu} = X_1 \cdots X_{\mu-1}Y_\mu X_1 \cdots X_{\nu-1}Y_\nu = Y_\nu Y_\mu X_\mu \cdots X_{\nu-1} = -\Gamma_{2\nu}\Gamma_{2\mu}.$$

In the case where $\nu = \mu$ the right hand sides of both of the above equations give the identity since $Y_\nu^2 = 1$. □

Consider any system $\{\widetilde{\Gamma}_\nu \,:\, \nu = 1, \cdots, 2n\} \subset \mathcal{M}_{2^n}(\mathbb{C})$ of matrices that fullfil

$$\widetilde{\Gamma}_\nu\widetilde{\Gamma}_\mu + \widetilde{\Gamma}_\mu\widetilde{\Gamma}_\nu = 2\delta_{\nu,\mu}I, \qquad \nu, \mu = 1, \cdots, 2n. \tag{4.2.3}$$

In the following we write

- $O(m)$:= *group of orthogonal elements in* $\mathcal{M}_m(\mathbb{R})$, i.e $\omega \in O(m) :\Longleftrightarrow \omega\omega^t = I$.
- $\mathrm{GL}(\mathbb{C}, m)$:= *group of invertible matrices in* $\mathcal{M}_m(\mathbb{C})$.

Lemma 4.2.2. *Let $S \in \mathrm{GL}(\mathbb{C}, 2^n)$, then it holds*

(i) *The system $\{\Gamma_\nu^S := S\,\widetilde{\Gamma}_\nu S^{-1} \,:\, \nu = 1, \cdots, 2n\}$ fulfills the anti-commutator relations (4.2.3).*

(ii) *There is $T \in \mathrm{GL}(\mathbb{C}, 2^n)$ such that $T\Gamma_\nu T^{-1} = \widetilde{\Gamma}_\nu$ for $\nu = 1, \cdots, 2n$.*

(iii) *Let $\omega = (\omega_{\mu\nu}) \in O(2n)$ and define*

$$\Gamma'_\mu := \sum_{\ell=1}^{2n} \omega_{\mu\ell}\widetilde{\Gamma}_\ell, \qquad (\mu = 1, \cdots, 2n).$$

Then the system $\{\Gamma'_\mu \,:\, \mu = 1, \cdots, 2n\}$ fulfills the anti-commutator relations (4.2.3).

Proof. (Homework 7) (i) is an easy calculations and we omit the proof of (ii). The statement (iii) is obtained as follows:

$$\begin{aligned}
\Gamma'_\mu\Gamma'_\nu + \Gamma'_\nu\Gamma'_\mu &= \sum_{i,\ell=1}^{2n} \omega_{\mu\ell}\omega_{\nu i}\Big\{\widetilde{\Gamma}_\ell\widetilde{\Gamma}_i + \widetilde{\Gamma}_i\widetilde{\Gamma}_\ell\Big\} \\
&= 2\sum_{i,\ell=1}^{2n} \omega_{\mu\ell}\omega_{\nu i}\delta_{i,\ell}I \\
&= 2\sum_{\ell=1}^{2n} \omega_{\mu\ell}\omega_{\nu\ell}I = 2\delta_{\mu,\nu}I,
\end{aligned}$$

where in the last equality we have used the orthogonality of $\omega = (\omega_{\mu\nu}) \in O(2^n)$. □

Let $\omega = (\omega_{\mu\nu}) \in O(2n)$ and Γ_α be the matrices defined in (4.2.1). By combining Lemma 4.2.2 (ii) and (iii) we conclude that there is $S(\omega) \in \mathrm{GL}(\mathbb{C}, 2^n)$ with

$$\sum_{\ell=1}^{2n} \omega_{\mu\ell}\Gamma_\ell = S(\omega)\Gamma_\mu S(\omega)^{-1}. \tag{4.2.4}$$

Definition 4.2.3. If $\omega = (\omega_{\mu,\nu}) \in O(2n)$ and $S(\omega) \in O(2^n)$ are related via (4.2.4), then we call $S(\omega)$ a *spin representation* of the "rotation" ω. In this case we write $\omega \leftrightarrow S(\omega)$.

Remark 4.2.4. Let $\omega_1, \omega_2 \in O(2n)$ with spin representations $S(\omega_1)$ and $S(\omega_2)$. Then

$$S(\omega_1\omega_2) = S(\omega_1)S(\omega_2)$$

is a spin representation of $\omega_1\omega_2$. In particular, if ω_1 and ω_2 are commuting rotations, then the spin representations $S(\omega_1)$ and $S(\omega_2)$ commute, as well.

Now we specialize the previous observation to rotations in the α-β-plane

$$\omega(\alpha\beta|\theta) \in O(2n), \qquad \alpha \neq \beta \in \{1, \cdots, 2n\}.$$

with angular $\theta \in [0, 2\pi)$ where $\omega(\alpha\beta|\theta)$ acts on the standard basis $[e_i := (\delta_{i,\ell})_{\ell=1}^n : i = 1, \cdots, 2n]$ of $\mathbb{R}^{2n}$ as

$$\begin{cases} \omega(\alpha\beta|\theta)e_i = e_i, & \text{if } i \notin \{\alpha, \beta\} \\ \omega(\alpha\beta|\theta)e_\alpha = e_\alpha \cos\theta + e_\beta \sin\theta \\ \omega(\alpha\beta|\theta)e_\beta = -e_\alpha \sin\theta + e_\beta \cos\theta. \end{cases}$$

We can also admit "complex angles" θ in the definition of $\omega(\mu\nu|\theta)$. Let $\theta_1, \theta_2 \in \mathbb{C}$, then

(i) $\omega(\mu\nu|\theta_1)\omega(\mu\nu|\theta_2) = \omega(\mu\nu|\theta_1 + \theta_2)$. In particular: $\omega(\mu\nu|\theta)^{-1} = \omega(\mu\nu| - \theta)$,

(ii) $\omega(\mu\nu|\theta_1) = \omega(\nu\mu| - \theta_1)$.

We calculate a spin representation of $\omega(\alpha\beta|\theta)$:

Lemma 4.2.5. *With $\alpha \neq \beta \in \{1, \cdots, 2n\}$ it holds*

$$\omega(\alpha\beta|\theta) \longleftrightarrow e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}.$$

Proof. (Homework 7) Since $\alpha \neq \beta$ we have from the anti-commutator relation

$$\left(\Gamma_\alpha\Gamma_\beta\right)^2 = \Gamma_\alpha\Gamma_\beta\Gamma_\alpha\Gamma_\beta = -\Gamma_\alpha^2\Gamma_\beta^2 = -I$$

and therefore

$$e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta} = \cos\frac{\theta}{2} - \Gamma_\alpha\Gamma_\beta \sin\frac{\theta}{2}. \tag{4.2.5}$$

Clearly, $e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}$ has the inverse $e^{\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}$. If $\lambda \notin \{\alpha, \beta\}$ then $[\Gamma_\lambda, e^{\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}] = 0$ and therefore

$$e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}\Gamma_\lambda e^{\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta} = \Gamma_\lambda.$$

Moreover, assume that $\lambda = \alpha$, then it follows from (4.2.5) that

$$\begin{aligned} e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}\Gamma_\alpha e^{\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta} &= \left(\cos\frac{\theta}{2} - \Gamma_\alpha\Gamma_\beta \sin\frac{\theta}{2}\right)\Gamma_\alpha\left(\cos\frac{\theta}{2} + \Gamma_\alpha\Gamma_\beta \sin\frac{\theta}{2}\right) \\ &= \left(\cos\frac{\theta}{2} - \Gamma_\alpha\Gamma_\beta \sin\frac{\theta}{2}\right)\left(\Gamma_\alpha \cos\frac{\theta}{2} + \Gamma_\beta \sin\frac{\theta}{2}\right) \\ &= \Gamma_\alpha\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + 2\Gamma_\beta \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ &= \Gamma_\alpha \cos\theta + \Gamma_\beta \sin\theta. \end{aligned}$$

Here we have used $\Gamma_\beta = -\Gamma_\alpha\Gamma_\beta\Gamma_\alpha$. If $\lambda = \beta$, then the relation

$$e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}\Gamma_\beta e^{\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta} = -\Gamma_\alpha \sin\theta + \Gamma_\beta \cos\theta$$

is obtained by a similar calculation. □

The importance of the previous lemma lies in the fact the eigenvalues of $\omega(\alpha\beta|\theta)$ and its spin representation $e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}$ are related. Clearly the set of eigenvalues of $\omega(\alpha\beta|\theta)$ are given by $\{1, e^{-i\theta}, e^{i\theta}\}$ where the eigenvalue 1 has the multiplicity $2n-2$.

Lemma 4.2.6. *The spin representation $e^{-\frac{\theta}{2}\Gamma_\alpha\Gamma_\beta}$ of $\omega(\alpha\beta|\theta)$ has the eigenvalues $\{e^{i\frac{\theta}{2}}, e^{-i\frac{\theta}{2}}\}$. Each eigenvalue has the multiplicity 2^{n-1}.*

Proof. If we replace Γ_ℓ in the definition (4.2.4) of $S(\omega)$ by another family $\widetilde{\Gamma}_\ell = L^{-1}\Gamma_\ell L$ of matrices that fulfill the anti-commutator relation (cf. Lemma 4.2.2, (ii)), then a spin representation $S(\omega)$ transforms to a spin representation $\widetilde{S}(\omega) = L^{-1}S(\omega)L$ with respect to $\widetilde{\Gamma}_\ell$. In particular, the eigenvalues of $S(\omega)$ and $\widetilde{S}(\omega)$ are the same.

We pass to a new system $\widetilde{\Gamma}_\ell$ of matrices obeying the anti-commutator relations by exchanging the role of X, Y and Z in the definition (4.2.1) of Γ_ℓ.

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y$$

Without restriction we choose $\widetilde{\Gamma}_\alpha = Z_1X_2$ and $\widetilde{\Gamma}_\beta = Z_1Y_2$. Then it follows

$$\widetilde{\Gamma}_\alpha\widetilde{\Gamma}_\beta = Z_1X_2Z_1Y_2 = X_2Y_2 = iZ_2 = I \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes I \otimes \cdots \otimes I.$$

Therefore it holds

$$\begin{aligned} e^{-\frac{\theta}{2}\widetilde{\Gamma}_\alpha\widetilde{\Gamma}_\beta} &= \cos\frac{\theta}{2} - \widetilde{\Gamma}_\alpha\widetilde{\Gamma}_\beta \sin\frac{\theta}{2} \\ &= I \otimes \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} \otimes I \otimes \cdots \otimes I. \end{aligned} \tag{4.2.6}$$

It follows that $e^{-\frac{\theta}{2}\widetilde{\Gamma}_\alpha\widetilde{\Gamma}_\beta}$ has the matrix elements

$$\left\langle (s_1, \cdots, s_n) | e^{-\frac{\theta}{2}\widetilde{\Gamma}_\alpha\widetilde{\Gamma}_\beta} | (s'_1, \cdots, s'_n) \right\rangle = e^{-i\frac{\theta}{2}s_2} \prod_{k=1}^{n} \delta_{s_k, s'_k}.$$

This means that $e^{-\frac{\theta}{2}\widetilde{\Gamma}_\alpha\widetilde{\Gamma}_\beta}$ is a diagonal matrix with eigenvalues $e^{-i\frac{\theta}{2}}$ (if $s_2 = 1$) and eigenvalues $e^{i\frac{\theta}{2}}$ (if $s_2 = -1$), respectively, and each of multiplicity 2^{n-1}. □

Assume that $a, b, c, d \in \{1, \cdots, 2n\}$ are pairwise distinct. Then the matrices $\Gamma_a\Gamma_b$ and $\Gamma_c\Gamma_d$ commute Hence the matrices

$$e^{\frac{\theta_1}{2}\Gamma_a\Gamma_b} \qquad \text{and} \qquad e^{\frac{\theta_1}{2}\Gamma_c\Gamma_d}, \qquad \theta_1, \theta_2 \in \mathbb{C}$$

commute and can be diagonalized simultaneously. It follows from Lemma 4.2.6 that:

Corollary 4.2.7. *Let π be a permutation of $\{1, \cdots, 2n\}$ and fix $\theta_1, \cdots, \theta_n \in \mathbb{C}$, then the 2^n eigenvalues of*

$$\prod_{j=1}^{n} e^{-\frac{\theta_j}{2}\Gamma_{\pi_{2j-1}}\Gamma_{\pi_{2j}}} = \exp\left\{-\frac{\theta_j}{2}\sum_{j=1}^{n}\Gamma_{\pi_{2j-1}}\Gamma_{\pi_{2j}}\right\} \in \mathcal{M}_{2^n}(\mathbb{C}) \tag{4.2.7}$$

are given by $e^{\frac{i}{2}(\pm\theta_1\pm\theta_2\pm\cdots\pm\theta_n)}$ where the sings $+$ and $-$ are chosen independently. Note that (4.2.7) is the spin representation of a product of commuting rotations.

Remark 4.2.8. The set of eigenvalues must be invariant under all possible reflections $\theta_j \to -\theta_j$ (which can be seen as a change of the set $\{\Gamma_\alpha\}_\alpha$ to another one with the same anti-commutation relation). Therefore, all possible combinations of signs must appear in the set $e^{\frac{i}{2}(\pm\theta_1\pm\theta_2\pm\cdots\pm\theta_n)}$ of eigenvalues.

4.3 The Onsager solution for $B = 0$

We calculate the *Onsager solution* for the Ising model when $B = 0$, (i.e. no exterior magnetic field). Recall that

$$Q_I(B, T) = \text{Trace } P^n$$

and according to Proposition 4.1.8 we in the case where $B = 0$ that $Q_3 = I$ and therefore

$$P = Q_2Q_1 = \Big[2\sinh(2\epsilon\beta)\Big]^{\frac{n}{2}} \underbrace{\prod_{\alpha=1}^{n} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}}_{=\widetilde{Q}_2} \overbrace{e^{\theta(X_1+\cdots+X_n)}}^{=Q_1} \in \mathbb{R}(2^n),$$

with $Z_{n+1} := Z_1$ and $\theta > 0$ such that $\tanh\theta = e^{-2\beta\epsilon}$. We have argued that

$$\lim_{N\to\infty}\frac{1}{N}\log Q_I(B, T) = \lim_{n\to\infty}\frac{1}{n}\log\lambda_{\max}(n), \qquad (N = n^2), \tag{4.3.1}$$

where $\lambda_{\max}(n)$ denotes the largest eigenvalue of P. Let $V := \widetilde{Q}_2Q_1$ and assume that V has only positive eigenvalues. Let $\Lambda = \Lambda(n)$ be the largest eigenvalue of V. Then we have

$$\lambda_{\max}(n) = \Big[2\sinh(2\epsilon\beta)\Big]^{\frac{n}{2}}\Lambda(n)$$

and it follows from (4.3.1) that

$$\lim_{N\to\infty}\frac{1}{N}\log Q_I(0, T) = \frac{1}{2}\log\big[2\sinh(2\epsilon\beta)\big] + \lim_{n\to\infty}\frac{1}{n}\log\Lambda(n). \tag{4.3.2}$$

Next aim: *Justify the above assumptions and to calculate the limits on both sides of (4.3.2).*

First, we rewrite V and Q_j for $j = 1, 2$: for $\alpha = 1, \cdots, n$ one has

$$\Gamma_{2\alpha}\Gamma_{2\alpha-1} = X_1X_2\cdots X_{\alpha-1}Y_\alpha X_1X_2\cdots X_{\alpha-1}Z_\alpha = Y_\alpha Z_\alpha = iX_\alpha.$$

Therefore

$$Q_1 = e^{\theta(X_1+\cdots+X_n)} = \prod_{\alpha=1}^{n} e^{\theta X_\alpha} = \prod_{\alpha=1}^{n} e^{-i\theta\Gamma_{2\alpha}\Gamma_{2\alpha-1}} \tag{4.3.3}$$

and similarly

$$\Gamma_{2\alpha+1}\Gamma_{2\alpha} = X_1 \cdots X_\alpha Z_{\alpha+1} X_1 \cdots X_{\alpha-1} Y_\alpha = X_\alpha Y_\alpha Z_{\alpha+1} = i Z_\alpha Z_{\alpha+1}$$
$$\Gamma_1 \Gamma_{2n} = Z_1 X_1 \cdots X_{n-1} Y_n = Z_1 \underbrace{Y_n X_n}_{=-iZ_n} X_1 \cdots X_{n-1} X_n = -i Z_1 Z_n (X_1 \cdots X_n).$$

If we define

$$U := X_1 X_2 \cdots X_n \in \mathcal{M}_{2^n}(\mathbb{C}),$$

with $U^2 = I$ then we have $i\Gamma_1\Gamma_{2n}U = Z_n Z_1$ and find the following representation of Q_2 from these relations [3]

$$\widetilde{Q}_2 = e^{\beta\epsilon Z_n Z_1} \left[\prod_{\alpha=1}^{n-1} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}} \right] = e^{i\beta\epsilon\Gamma_1\Gamma_{2n}U} \prod_{\alpha=1}^{n-1} e^{-i\beta\epsilon\Gamma_{2\alpha+1}\Gamma_{2\alpha}}. \tag{4.3.4}$$

Lemma 4.3.1. *The matrix* $V = \widetilde{Q}_2 Q_1$ *can be expressed in the form*

$$V = e^{i\beta\epsilon\Gamma_1\Gamma_{2n}U} \left[\prod_{\alpha=1}^{n-1} e^{-i\beta\epsilon\Gamma_{2\alpha+1}\Gamma_{2\alpha}} \right] \left[\prod_{\alpha=1}^{n} e^{-i\theta\Gamma_{2\alpha}\Gamma_{2\alpha-1}} \right], \tag{4.3.5}$$

where Γ_α *were defined in (4.2.1). Here* $\theta > 0$ *is the solution of the equation* $\tanh\theta = e^{-2\epsilon\beta}$.

We want to get rid of the matrix U which appears in the exponent of the first factor of V and just work with products of spin representation. In the next step we further decompose V. First we collect some properties of the matrix U:

Lemma 4.3.2. *The matrix* $U = X_1 \cdots X_n \in \mathcal{M}_{2^n}(\mathbb{C})$ *satisfies:*

(i) $U = X \otimes X \otimes \cdots \otimes X = i^n \Gamma_1 \Gamma_2 \cdots \Gamma_{2n}$,

(ii) U *has the eigenvalues* ± 1 *each of multiplicity* 2^{n-1},

(iii) $U^2 = I$, $(I - U)U = -(I - U)$ *and* $(I + U)U = I + U$,

(iv) *If* $a \neq b \in \{1, \cdots 2n\}$, *then* $\Gamma_a \Gamma_b$ *commutes with* U.

(v) *Let* ω *be an orthogonal transformation with spin representation* $S(\omega)$, *then we have*

$$S(\omega) U S(\omega)^{-1} = \det(\omega)\, U.$$

Proof. The first equation in (i) follows from the definition of X_α and Lemma 4.1.3, the second equation is a consequence of

$$U = X_1 \cdots X_n = (iZ_1Y_1)(iZ_2Y_2) \cdots (iZ_nY_n) = i^n \Gamma_1 \cdots \Gamma_{2n}. \tag{4.3.6}$$

Note that by (i) the matrix $Z \otimes \cdots \otimes Z$ is a diagonal form of U and $Z \in \mathcal{M}_2(\mathbb{C})$ has the eigenvalues ± 1. The equations in (iii) immediately follow from the definition of U and (iv) is obtained as follows from (i):

$$\begin{aligned} \Gamma_a \Gamma_b U &= i^n \Gamma_a \Gamma_b \Gamma_1 \cdots \Gamma_{2n} \\ &= i^n (-1)^{2n-1} \Gamma_a \Gamma_1 \cdots \Gamma_{2n} \Gamma_b \\ &= i^n (-1)^{4n-2} \Gamma_1 \cdots \Gamma_{2n} \Gamma_a \Gamma_b = U \Gamma_a \Gamma_b. \end{aligned}$$

□

[3] Recall that all matrices $e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}$ are diagonal and hence commute

Consider the factor $e^{i\beta\epsilon\Gamma_1\Gamma_{2n}U}$ which appears in the representation (4.3.5) of V. Lemma 4.3.2, (iv) implies $(i\Gamma_1\Gamma_{2n}U)^2 = -U^2(\Gamma_1\Gamma_{2n})^2 = I$ which means that

$$e^{i\beta\epsilon\Gamma_1\Gamma_{2n}U} = \cosh(\beta\epsilon) + iU\Gamma_1\Gamma_{2n}\sinh(\beta\epsilon).$$

From the relations in Lemma 4.3.2, (iii) we see that

$$\begin{aligned} e^{i\beta\epsilon\Gamma_1\Gamma_{2n}U} &= \left[\frac{1}{2}(I+U) + \frac{1}{2}(I-U)\right]\big[\cosh(\beta\epsilon) + i\Gamma_1\Gamma_{2n}\sinh(\beta\epsilon)\big] \\ &= \frac{1}{2}(I+U)\big[\cosh(\beta\epsilon) + i\Gamma_1\Gamma_{2n}\sinh(\beta\epsilon)\big] + \\ &\qquad + \frac{1}{2}(I-U)\big[\cosh(\beta\epsilon) - i\Gamma_1\Gamma_{2n}\sinh(\beta\epsilon)\big] \\ &= \frac{1}{2}(I+U)e^{i\epsilon\beta\Gamma_1\Gamma_{2n}} + \frac{1}{2}(I-U)e^{-i\epsilon\beta\Gamma_1\Gamma_{2n}}. \end{aligned}$$

If we plug this result into the representation of V in Lemma 4.3.1 then we obtain

$$V = \frac{1}{2}(I+U)V^+ + \frac{1}{2}(I-U)V^-, \tag{4.3.7}$$

where $V^\pm \in \mathcal{M}_{2^n}(\mathbb{C})$ are defined by

$$V^\pm := e^{\pm i\beta\epsilon\Gamma_1\Gamma_{2n}}\left[\prod_{\alpha=1}^{n-1} e^{-i\beta\epsilon\Gamma_{2\alpha+1}\Gamma_{2\alpha}}\right]\left[\prod_{\alpha=1}^{n} e^{-i\theta\Gamma_{2\alpha}\Gamma_{2\alpha-1}}\right] \tag{4.3.8}$$

and $\tanh\theta = e^{-2\epsilon\beta}$. Note that the matrices $V^\pm$ have the form of a product of spin representation from the last section and therefore they are easier to handle than V.

Lemma 4.3.3. *The matrices U, V^+ and V^- pairwise commute. In particular, they can be diagonalized simultaneously.*

Proof. First we show that U commutes with V^+ and V^-. Let $a \neq b \in \{1, \cdots, 2n\}$ then we see from Lemma 4.3.2, (iv) that $\Gamma_a\Gamma_b$ and U commute. Now $[U, V^+] = [U, V^-] = 0$ follows from the form of $V^\pm$. Note that by a similar reason U also commutes with V. Since $(I+U)/2$ and $(I-U)/2$ are projection onto complementary spaces we find:

$$\begin{aligned} V^+V^- &= \frac{1}{4}(I+U)V(I-U)V = \frac{1}{4}V(I+U)(I-U)V = 0, \\ V^-V^+ &= \frac{1}{4}(I-U)V(I+U)V = \frac{1}{4}V(I-U)(I+U)V = 0. \end{aligned}$$

In particular, it follows that V^+ and V^- commute. □

Consider the orthogonal matrix $g \in \mathcal{M}_{2^n}(\mathbb{C})$ defined by

$$g = 2^{-\frac{n}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = g^{-1}.$$

Then gUg is diagonal, more precisely:

$$gUg^{-1} = g\big(X \otimes X \otimes \cdots \otimes X\big)g^{-1} = Z \otimes Z \otimes \cdots \otimes Z = Z_1 Z_2 \cdots Z_n \tag{4.3.9}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we can choose an orthogonal matrix $o \in \mathcal{M}_{2^n}(\mathbb{R})$ that permutes all eigenvalues "+1" of gUg^{-1} to the upper left corner and all eigenvalues "−1" to the lower right corner. If we define

$$R := og \qquad \text{and} \qquad \widetilde{U} := RUR^{-1},$$

then $R \in \mathcal{M}_{2^n}(\mathbb{C})$ is orthogonal and

$$(og)U(og)^{-1} = RUR^{-1} = \widetilde{U} = \begin{pmatrix} I_{2^{n-1}} & 0 \\ 0 & -I_{2^{n-1}} \end{pmatrix}. \tag{4.3.10}$$

We put $\widetilde{V}^{\pm} = RV^{\pm}R^{-1}$ and conjugate the decomposition (4.3.7) by R:

$$RVR^{-1} =: \widetilde{V} = \frac{1}{2}(I + \widetilde{U})\widetilde{V}^{+} + \frac{1}{2}(I - \widetilde{U})\widetilde{V}^{-}.$$

It follows from Lemma 4.3.3 that $\widetilde{U}$, $\widetilde{V}^{+}$ and $\widetilde{V}^{-}$ are pairwise commuting:

$$\widetilde{V}^{+}\widetilde{U} = \begin{pmatrix} \widetilde{V}^{+}_{11} & -\widetilde{V}^{+}_{12} \\ \widetilde{V}^{+}_{21} & -\widetilde{V}^{+}_{22} \end{pmatrix} = \widetilde{U}\widetilde{V}^{+} = \begin{pmatrix} \widetilde{V}^{+}_{11} & \widetilde{V}^{+}_{12} \\ -\widetilde{V}^{+}_{21} & -\widetilde{V}^{+}_{22} \end{pmatrix}.$$

We find that $\widetilde{V}^{+}_{12} = \widetilde{V}^{+}_{21} = 0$ and (by the analogous calculation for $\widetilde{V}^{-}$) we have

$$\widetilde{V}^{\pm} = \begin{pmatrix} \widetilde{V}^{\pm}_{11} & 0 \\ 0 & \widetilde{V}^{\pm}_{22} \end{pmatrix}, \qquad \text{with} \quad \widetilde{V}^{\pm}_{11}, \widetilde{V}^{\pm}_{22} \in \mathcal{M}_{2^{n-1}}(\mathbb{C}).$$

Therefore we find that

$$\frac{1}{2}(I + \widetilde{U})\widetilde{V}^{+} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{V}^{+}_{11} & 0 \\ 0 & \widetilde{V}^{+}_{22} \end{pmatrix} = \begin{pmatrix} \widetilde{V}^{+}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.3.11}$$

$$\frac{1}{2}(I - \widetilde{U})\widetilde{V}^{-} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \widetilde{V}^{-}_{11} & 0 \\ 0 & \widetilde{V}^{-}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{V}^{-}_{22} \end{pmatrix}, \tag{4.3.12}$$

which shows that $\widetilde{V}$ has the following matrix representation:

$$\widetilde{V} = \frac{1}{2}(I + \widetilde{U})\widetilde{V}^{+} + \frac{1}{2}(I - \widetilde{U})\widetilde{V}^{-} = \begin{pmatrix} \widetilde{V}^{+}_{11} & 0 \\ 0 & \widetilde{V}^{-}_{22} \end{pmatrix}.$$

We are aiming to find the eigenvalues of V. We have

$$\Big\{\text{eigenvalues of } V\Big\} = \Big\{\text{eigenvalues of } \widetilde{V}\Big\} = \Big\{\text{eigenvalues of } \widetilde{V}^{+}_{11}\Big\} \cup \Big\{\text{eigenvalues of } \widetilde{V}^{-}_{22}\Big\}.$$

Moreover, we know

$$\big\{\text{eigenvalues of } \widetilde{V}^{+}_{11}\big\} \subset \big\{\text{ eigenvalues of } \widetilde{V}^{+}\big\} = \big\{\text{eigenvalues of } V^{+}\big\}$$
$$\big\{\text{eigenvalues of } \widetilde{V}^{-}_{22}\big\} \subset \big\{\text{ eigenvalues of } \widetilde{V}^{-}\big\} = \big\{\text{eigenvalues of } V^{-}\big\}.$$

In other words:

Lemma 4.3.4. *The union of the eigenvalues of V^+ and V^- contains all eigenvalues of V.*

In the next step we calculate the eigenvalues of V^+ and V^-. Consider the matrices

$$\Omega^{\pm} := \omega(1, 2n| \pm 2i\beta\epsilon) \left[\prod_{\alpha=1}^{n-1} \omega\big(2\alpha+1, 2\alpha| - 2i\beta\epsilon\big)\right] \left[\prod_{\alpha=1}^{n} \omega\big(2\alpha, 2\alpha-1| - 2i\theta\big)\right] \in \mathcal{M}_{2n}(\mathbb{C}). \tag{4.3.13}$$

Then $V^{\pm} = S(\Omega^{\pm})$ is a "spin representation" of $\Omega^{\pm}$. [4] Define

$$\Delta := \prod_{\alpha=1}^{n} \omega(2\alpha, 2\alpha-1| - i\theta) \in \mathcal{M}_{2n}(\mathbb{C}). \tag{4.3.14}$$

Note that $\omega(\mu\nu|\theta_1)\omega(\mu\nu|\theta_2) = \omega(\mu\nu|\theta_1+\theta_2)$ and $\omega(\mu\nu|\theta)^{-1} = \omega(\mu,\nu| - \theta)$. Therefore:

$$\left[\prod_{\alpha=1}^{n} \omega\big(2\alpha, 2\alpha-1| - 2i\theta\big)\right] \Delta^{-1} = \Delta. \tag{4.3.15}$$

The eigenvalues of $\Omega^{\pm}$ coincide with the eigenvalues of

$$\begin{aligned}\omega^{\pm} :&= \Delta\Omega^{\pm}\Delta^{-1} \\ &= \Delta \underbrace{\omega\big(1, 2n| \pm 2i\beta\epsilon\big) \left[\prod_{\alpha=1}^{n-1} \omega\big(2\alpha, 2\alpha+1|2i\beta\epsilon\big)\right]}_{=:\chi^{\pm}} \Delta,\end{aligned}$$

where in the second equation we have used $\omega(\mu\nu|\theta) = \omega(\nu\mu| - \theta)$. We express Δ and $\chi^{\pm}$ in matrix form. Consider $J, K \in \mathcal{M}_2(\mathbb{C})$ defined by:

$$J := \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix}, \qquad \text{and} \qquad K := \begin{pmatrix} \cosh(2\beta\epsilon) & i\sinh(2\beta\epsilon) \\ -i\sinh(2\beta\epsilon) & \cosh(2\beta\epsilon) \end{pmatrix}.$$

If $n = 1$ we have $\omega(2, 1|i\theta) = J$ and for general $n \in \mathbb{N}$ the above definition show:

$$\Delta = \begin{pmatrix} J & \mathbf{0} & \dots & \\ \mathbf{0} & J & & \\ \vdots & & \ddots & \\ \vdots & & & J \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{C}), \qquad \text{where} \quad \mathbf{0} \in \mathcal{M}_2(\mathbb{C}),$$

$$\chi^{\pm} = \begin{pmatrix} \cosh(2\beta\epsilon) & 0 & \dots & 0 & \pm i\sinh(2\beta\epsilon) \\ 0 & & & & 0 \\ \vdots & & \mathbf{K} & & \vdots \\ 0 & & & & \\ \mp i\sinh(2\beta\epsilon) & 0 & \dots & 0 & \cosh(2\beta\epsilon) \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{C}), \quad \text{where}$$

$$\mathbf{K} := \begin{pmatrix} K & \mathbf{0} & \dots & \\ \mathbf{0} & K & & \\ \vdots & & \ddots & \\ \vdots & & & K \end{pmatrix} \in \mathcal{M}_{2n-2}(\mathbb{C})$$

[4] Recall that $\omega(\mu\nu|\theta)$ is the rotation in the $\mu-\nu$-plane around the angle θ.

Lemma 4.3.5. *The matrix $\omega^{\pm} = \Delta\chi^{\pm}\Delta$ has the form*

$$\omega^{\pm} = \begin{pmatrix} A & B & 0 & 0 & \dots & 0 & \mp B^* \\ B^* & A & B & 0 & & 0 & 0 \\ 0 & B^* & A & B & & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & 0 & & & & A & B \\ \mp B & 0 & & & & B^* & A \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{C}), \tag{4.3.16}$$

where the matrices $A, B \in \mathcal{M}_2(\mathbb{C})$ are given by

$$A := \cosh(2\beta\epsilon) \begin{pmatrix} \cosh(2\theta) & -i\sinh(2\theta) \\ i\sinh(2\theta) & \cosh(2\theta) \end{pmatrix}$$

$$B := \sinh(2\beta\epsilon) \begin{pmatrix} -\frac{1}{2}\sinh(2\theta) & -i\sinh^2\theta \\ i\cosh^2\theta & -\frac{1}{2}\sinh(2\theta) \end{pmatrix}.$$

Moreover, write $B^* = \overline{B}^T$ for the Hermitian adjoint matrix to B.

Proof. (Homework 8) From the above matrix representation one easily sees that $\chi^{\pm}$ has the form

$$\chi^{\pm} = \begin{pmatrix} \tilde{A} & \tilde{B} & 0 & 0 & \dots & 0 & \mp \tilde{B}^* \\ \tilde{B}^* & \tilde{A} & \tilde{B} & 0 & & 0 & 0 \\ 0 & \tilde{B}^* & \tilde{A} & \tilde{B} & & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & 0 & & & & \tilde{A} & \tilde{B} \\ \mp \tilde{B} & 0 & & & & \tilde{B}^* & \tilde{A} \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{C}),$$

where $\tilde{A}, \tilde{B} \in \mathcal{M}_2(\mathbb{C})$ are defined by

$$\tilde{A} := \begin{pmatrix} \cosh(2\beta\epsilon) & 0 \\ 0 & \cosh(2\beta\epsilon) \end{pmatrix}, \quad \text{and} \quad \tilde{B} := \begin{pmatrix} 0 & 0 \\ i\sinh(2\beta\epsilon) & 0 \end{pmatrix}$$

Now the assertion follows from $J\tilde{A}J = A$ and $J\tilde{B}J = B$ together with $J^* = J$. □

We use the matrix representation of $\omega^{\pm}$ to determine the eigenvalues and make the following *Ansatz* for an eigenvector ψ of $\omega^{\pm}$:

$$\psi = \begin{pmatrix} zu \\ z^2u \\ \vdots \\ z^nu \end{pmatrix} \in \mathbb{C}^{2n}, \quad \text{where} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2, \quad z \in \mathbb{C}.$$

It follows from Lemma 4.3.5 that the equation $\omega^{\pm}\psi = \lambda\psi$ is equivalent to the system of equations:

$$(\mathbf{E}): \begin{cases} E_1 : \big(zA + z^2 B \mp z^n B^*\big)u & = z\lambda u \\ E_j : \big(z^j A + z^{j+1} B + z^{j-1} B^*\big)u & = z^j \lambda u, \qquad (j = 2, \cdots, n-1) \\ E_n : \big(z^n A \mp zB + z^{n-1} B^*\big)u & = z^n \lambda u. \end{cases}$$

Note that the equations E_j for $j = 2, \cdots, n-1$ and $z \neq 0$ all are equivalent to

$$(A + zB + z^{-1}B^*)u = \lambda u.$$

Hence the system $(\mathbf{E})$ of equations is equivalent to the three equations:

$$(\widetilde{\mathbf{E}}): \begin{cases} \big(A + zB \mp z^{n-1}B^*\big)u & = \lambda u \\ \big(A + zB + z^{-1}B^*\big)u & = \lambda u \\ \big(A \mp z^{1-n}B + z^{-1}B^*\big)u & = \lambda u. \end{cases}$$

We look for solutions among all $z \in \mathbb{C}$ with $z^n = \pm 1$ where we choose the $-$-sign for ω^+ and the $+$-sign for ω^-. Then $(\widetilde{\mathbf{E}})$ reduces to a single equation, namely

$$\big(A + zB + z^{-1}B^*\big)u = \lambda u.$$

Note that the matrix $A + zB + z^{-1}B^*$ is self-adjoint if $|z| = 1$ and therefore has only real eigenvalues.

<u>The case of ω^+:</u> The n solutions to the equation $z^n = -1$ are given by

$$S_- := \Big\{ z_k := e^{\frac{i\pi k}{n}} \ : \ k = 1, 3, \cdots, 2n-1 \Big\}.$$

A set $\{\lambda_{2l-1,1}, \lambda_{2l-1,2} : l = 1, \cdots, n\}$ of $2n$ eigenvalues for ω^+ can be determined by the solutions of the n equations

$$\Big(A + z_k B + \frac{1}{z_k} B^*\Big)u = \lambda_{k,j} u, \tag{4.3.17}$$

where $k = 1, 3, \cdots, 2n-1$ and $j = 1, 2$.

Lemma 4.3.6. *With the previous notation we have for $k = 1, 3, \cdots, 2n-1$:*

(i) $\det\big(A + z_k B + z_k^{-1} B^*\big) = 1$,

(ii) $C(\beta, \epsilon) \geq \mathrm{Trace}\big(A + z_k B + z_k^{-1} B^*\big) \geq 0$, *where $C(\beta,\epsilon)$ is independent of k and n.*

In particular, the eigenvalues of $\lambda_{k,1}$ and $\lambda_{k,2}$ of $A + z_k B + z_k^{-1}B^*$ are positive and $\lambda_{k,1} = \lambda_{k,2}^{-1}$.

Proof. (Homework 08)
(i): From the explicit form of A and B in Lemma 4.3.5 one checks that for all k:

$$\begin{aligned} \det\big(A + z_k B + z_k^{-1}B^*\big) = &\left[\cosh(2\beta\epsilon)\cosh(2\theta) - \frac{z_k + z_k^{-1}}{2}\sinh(2\beta\epsilon)\sinh(2\theta)\right]^2 \\ &- \Big(\cosh(2\beta\epsilon)\sinh(2\theta) - z_k\sinh(2\beta\epsilon)\sinh^2\theta - z_k^{-1}\sinh(2\beta\epsilon)\cosh^2\theta\Big)\times \\ &\times\Big(\cosh(2\beta\epsilon)\sinh(2\theta) - z_k\sinh(2\beta\epsilon)\cosh^2\theta - z_k^{-1}\sinh(2\beta\epsilon)\sinh^2(\theta)\Big) = 1. \end{aligned}$$

Moreover,

$$\begin{aligned}\mathrm{Trace}\big(A + z_k B + z_k^{-1} B^*\big) &= 2\cosh(2\beta\epsilon)\cosh(2\theta) - 2\cos\left(\frac{\pi k}{n}\right)\sinh(2\beta\epsilon)\sinh(2\theta)\\ &\geq 2\cosh(2\beta\epsilon - 2\theta) \geq 0.\end{aligned}$$

This shows the second inequality in (ii). The first one follows from the uniform estimate $|\cos(\frac{\pi k}{n})| \leq 1$. □

Using the last lemma we define for $k = 1, 3, \cdots, 2n-1$:

$$\lambda_{k,1} := e^{\gamma_k} \qquad \text{and} \qquad \lambda_{k,2} := e^{-\gamma_k}, \quad (\gamma_k \geq 0).$$

One obtains

$$\begin{aligned}\cosh(\gamma_k) = &= \frac{1}{2}\Big\{e^{\gamma_k} + e^{-\gamma_k}\Big\}\\ &= \frac{1}{2}\mathrm{Trace}\big(A + z_k B + z_k^{-1} B^*\big)\\ &= \cosh(2\beta\epsilon)\cosh(2\theta) - \cos\left(\frac{\pi k}{n}\right)\sinh(2\beta\epsilon)\sinh(2\theta).\end{aligned} \tag{4.3.18}$$

Lemma 4.3.7. *The eigenvalues E_{ω^+} of ω^+ are given by:*

$$E_{\omega^+} = \Big\{e^{\pm\gamma_k} \;:\; k = 1, 3, \cdots, 2n-1 \text{ and } \gamma_k > 0 \text{ is solution of } (4.3.18)\Big\}. \tag{4.3.19}$$

In particular, ω^+ can be expressed as a product of n commuting rotations.

<u>The case ω^-:</u> The n solutions to the equation $z^n = 1$ are given by

$$S_+ := \big\{z_k := e^{\frac{i\pi k}{n}} \;:\; k = 0, 2, \cdots, 2n-2\big\}.$$

Now we determine eigenvalues $\{\lambda_{2\ell,1}, \lambda_{2\ell,2} \;:\; \ell = 0, \cdots, n-1\}$ of ω^- as the solutions of the n equations

$$\big(A + z_k B + z_k^{-1} B^*\big)\, u = \lambda_{k,j} u$$

where $k = 0, 2, \cdots, 2n-2$ and $j = 1, 2$. In the same way as before we find $\lambda_{k,1} = e^{\gamma_k}$ and $\lambda_{k,2} = e^{-\gamma_k}$ with $\gamma_k > 0$ which is a solution of (4.3.18).

Lemma 4.3.8. *The eigenvalues E_{ω^-} of ω^- are given by:*

$$E_{\omega^-} = \Big\{e^{\pm\gamma_k} \;:\; k = 0, 2, \cdots, 2n-2 \text{ and } \gamma_k > 0 \text{ is solution of } (4.3.18)\Big\}. \tag{4.3.20}$$

In particular, ω^- can be expressed as a product of n commuting rotations.

Now we observe some relations between these eigenvalues:

Lemma 4.3.9. *For $k = 0, \cdots, 2n$ let $\gamma_k \geq 0$ be the solution of (4.3.18), then it holds*

(i) $\gamma_k = \gamma_{2n-k}$,

(ii) $0 < \gamma_0 < \gamma_1 < \cdots < \gamma_n$.

Proof. Property (i) follows from $\cos(\pi k/n) = \cos(\pi(2n-k)/n)$. In order to see (ii) we take the derivative on both sides of (4.3.18) with respect to k:

$$\frac{\partial \gamma_k}{\partial k} = \sin\left(\frac{\pi k}{n}\right) \frac{\pi \sinh(2\beta\epsilon)\sinh(2\theta)}{n \sinh \gamma_k}.$$

Since we assume that $\gamma_k > 0$ it follows that the right hand side is positive if $0 \le k \le n$. □

Since $V^\pm = S(\Omega^\pm)$ and the matrix $\Omega^\pm$ has the same eigenvalues as $\omega^\pm = \Delta\Omega^\pm\Delta^{-1}$ it follows from Corollary 4.2.7 together with (4.3.19) and (4.3.20)

Proposition 4.3.10. *The eigenvalues of $V^\pm$ are given by*

$$\text{eigenvalues of } V^+ := \left\{ e^{\frac{1}{2}(\pm\gamma_1 \pm \gamma_3 \pm \cdots \pm \gamma_{2n-1})} \;:\; \gamma_k \text{ solution of } (4.3.18) \right\}, \tag{4.3.21}$$

$$\text{eigenvalues of } V^- := \left\{ e^{\frac{1}{2}(\pm\gamma_0 \pm \gamma_2 \pm \cdots \pm \gamma_{2n-2})} \;:\; \gamma_k \text{ solution of } (4.3.18) \right\}. \tag{4.3.22}$$

All eigenvalues grow at most of order e^n as $n \to \infty$. [5]

Proof. The second statement follows from the trace estimate from above in Lemma 4.3.6, (ii) since

$$\begin{aligned}
\big|\pm\gamma_1 \pm \gamma_3 \pm \cdots \pm \gamma_{2n-1}\big| &\le \gamma_1 + \gamma_3 + \cdots + \gamma_{2n-1} \\
&\le \sum_{l=1}^{n} \log \lambda_{2l-1,1} \\
&\le \sum_{l=1}^{n} \lambda_{2l-1,1} \\
&\le \sum_{l=1}^{n} \mathrm{Trace}\big(A + z_l B + z_l^{-1} B^*\big) \le n \cdot C(\epsilon, \beta).
\end{aligned}$$

The right hand side growth linearly in $n \in \mathbb{N}$. □

We return to the task of studying the eigenvalues of V. Recall that

$$\left\{ \text{ eigenvalues of } V \right\} \subset \left\{ \text{eigenvalues of } V^+ \right\} \cup \left\{ \text{ eigenvalues of } V^- \right\}.$$

Moreover, with the notation in (4.3.11) and (4.3.12) we had

$$\frac{1}{2}(I + \widetilde{U})\widetilde{V}^+ = \begin{pmatrix} \widetilde{V}_{11}^+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \frac{1}{2}(I - \widetilde{U})\widetilde{V}^- = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{V}_{22}^- \end{pmatrix}.$$

[5] This last statement was necessary to justify the previous relation

$$\lim_{N\to\infty} \log Q_I(B,T) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_{\max}(n), \qquad N = n^2.$$

Let $R = og \in \mathcal{M}_{2^n}(\mathbb{C})$ be the orthogonal matrix defined in (4.3.10) and consider the following system of anti-commuting matrices

$$\mathbf{\Gamma} := \left\{ \widetilde{\Gamma}_\nu := R\Gamma_\nu R^{-1} \ : \ \nu = 1, \cdots, 2n \right\}.$$

Let $\omega \in \mathcal{M}_{2n}(\mathbb{R})$ be orthogonal, then we write $\widetilde{S}(\omega)$ for the spin representation of ω with respect to the system $\mathbf{\Gamma}$. If $\omega = \omega(\alpha\beta|\theta)$, then we have:

$$\widetilde{S}(\omega(\alpha\beta|\theta)) = e^{-\frac{\theta}{2}\widetilde{\Gamma}_\alpha \widetilde{\Gamma}_\beta}.$$

Note that for $j = 1, \cdots, n$:

$$\widetilde{\Gamma}_{2j-1}\widetilde{\Gamma}_{2j} = og\Gamma_{2j-1}\Gamma_{2j}(og)^{-1} = (og)Z_jY_j(og)^{-1} \tag{4.3.23}$$

$$= iog \left[I \otimes \cdots \otimes \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes \cdots \otimes I \right] go^{-1} \tag{4.3.24}$$

$$= -ioZ_jo^{-1}. \tag{4.3.25}$$

Lemma 4.3.11. *There are orthogonal matrices $T_\pm \in \mathcal{M}_{2n}(\mathbb{R})$ such that*

$$T_+\Omega^+T_+^{-1} = \omega(1,2|i\gamma_1)\omega(3,4|i\gamma_3)\cdots\omega(2n-1,2n|i\gamma_{2n-1}), \tag{4.3.26}$$

$$T_-\Omega^-T_-^{-1} = \omega(1,2|i\gamma_0)\omega(3,4|i\gamma_2)\cdots\omega(2n-1,2n|i\gamma_{2n-2}). \tag{4.3.27}$$

Proof. Follows from Lemma 4.3.7 and Lemma 4.3.8. □

We now shown that spin representations $\widetilde{S}(T_\pm)$ bring $(I - \widetilde{U})\widetilde{V}^\pm$ into diagonal form. Since V^+ and V^- are treated in the same way, we only give the arguments in the case of V^+. We know from Lemma 4.3.2, (v) that

$$\widetilde{S}(T_+)\tilde{U}\widetilde{S}(T_+)^{-1} = \det(T_+)\tilde{U} = \pm\tilde{U}.$$

With $\widetilde{V}^+ = RV^+R^{-1} = RS(\Omega^+)R^{-1} = \widetilde{S}(\Omega^+)$ it follows that

$$\widetilde{S}(T_+)\left\{\frac{1}{2}(I+\widetilde{U})\widetilde{V}^+\right\}\widetilde{S}(T_+)^{-1} = \frac{1}{2}(I \pm \widetilde{U})\widetilde{S}(T_+)\widetilde{V}^+\widetilde{S}(T_+)^{-1} \tag{4.3.28}$$

$$= \frac{1}{2}(I \pm \tilde{U})\widetilde{S}\Big(T_+\Omega^+T_+^{-1}\Big) = (*),$$

Since by Lemma 4.3.11 conjugation by T_+ transforms Ω^+ to a product of commuting rotations we obtain from Lemma 4.2.5 that

$$(*) = \frac{1}{2}(I \pm \tilde{U})\prod_{j=1}^{n} e^{-i\frac{\gamma_{2j-1}}{2}\widetilde{\Gamma}_{2j-1}\widetilde{\Gamma}_{2j}} = \frac{1}{2}o\left(I \pm Z_1Z_2\cdots Z_n\right)\left\{\prod_{j=1}^{n} e^{-\frac{1}{2}\gamma_{2j-1}Z_j}\right\}o^{-1} = V_D.$$

Here we have used (4.3.9) and (4.3.23). The matrices V_D and $o^{-1}V_Do$ [6] are diagonal and so we have diagonalized

$$\frac{1}{2}(I+\tilde{U})\tilde{V}^+ = \begin{pmatrix} \widetilde{V}_{11}^+ & 0 \\ 0 & 0 \end{pmatrix}.$$

[6] the matrix $o^{-1}V_Do$ arises from V_D by permuting the elements in the diagonal

Clearly the eigenvalues of $\widetilde{V}_{11}^{+}$ coincide with the non-zero eigenvalues of

$$o^{-1}V_D o = \frac{1}{2}\left(I \pm Z_1 Z_2 \cdots Z_n\right)\prod_{j=1}^{n} e^{-\frac{1}{2}\gamma_{2j-1}Z_j}. \tag{4.3.29}$$

Let $(s_1, \cdots, s_n)$ be an eigenvector of the right hand side of (4.3.29) and assume that the plus-sign appears in front of the product $Z_1Z_2\cdots Z_n$. Then the corresponding eigenvalue of $o^{-1}V_D o$ is non-zero if the equation

$$Z_j(s_1, \cdots, s_n) = -1 \tag{4.3.30}$$

only holds for an *even number* of $j \in \{1, \cdots, n\}$. If the minus sign appears in front of $Z_1Z_2\cdots Z_n$, then the eigenvalues is non-zero if (4.3.30) only holds for an *odd number* of j:

Corollary 4.3.12. *The largest eigenvalue $\Lambda_n(A)$ of $A \in \{\widetilde{V}_{11}^{+}, \widetilde{V}_{22}^{-}\}$ fulfills*

$$\Lambda_n(\widetilde{V}_{11}^{+}) = e^{\frac{1}{2}(\pm\gamma_1+\gamma_3+\cdots+\gamma_{2n-1})} \qquad \text{and} \qquad \Lambda_n(\widetilde{V}_{22}^{-}) = e^{\frac{1}{2}(\pm\gamma_0+\gamma_2+\cdots+\gamma_{2n-2})}.$$

Proof. We only treat $A = \widetilde{V}_{11}^{+}$. Then the lemma directly follows from the last observation and Lemma 4.3.9 which implies that $\gamma_1 = \min\{\gamma_{2j-1}\, j = 1, \cdots n\}$. □

Since we have the asymptotic equalities

$$\lim_{n\to\infty}\frac{-\gamma_1+\gamma_3+\gamma_5+\cdots}{2} = \lim_{n\to\infty}\frac{\gamma_1+\gamma_3+\gamma_5+\cdots}{2} =: \ell_+$$
$$\lim_{n\to\infty}\frac{-\gamma_0+\gamma_2+\gamma_4+\cdots}{2} = \lim_{n\to\infty}\frac{\gamma_0+\gamma_2+\gamma_4+\cdots}{2} =: \ell_-,$$

and since $\ell_+ \geq \ell_-$ (again by Lemma 4.3.9) we finally obtain that

$$\mathcal{L} := \lim_{n\to\infty}\frac{1}{n}\log\Lambda(n) = \lim_{n\to\infty}\frac{1}{2n}\big(\gamma_1+\gamma_3+\cdots+\gamma_{2n-1}\big),$$

where $\Lambda(n)$ denotes the largest eigenvalue of $V = Q_2Q_1$.

The limit $\lim_{n\to\infty}\frac{1}{n}\log\Lambda(n)$

Recall that

$$\lim_{N\to\infty}\frac{1}{N}\log Q_I(0,T) = \frac{1}{2}\log\big[2\sinh(2\epsilon\beta)\big] + \mathcal{L}.$$

Next step: We determine an integral representation of $\mathcal{L}$.

We define a function $\gamma : [0, 2\pi] \to \mathbb{R}$ as the positive solution of the equation

$$\cosh\gamma(x) = \cosh(2\beta\epsilon)\cosh(2\theta) - \cos(x)\sinh(2\beta\epsilon)\sinh(2\theta). \tag{4.3.31}$$

In particular it follows from the definition of γ_ℓ in (4.3.18) that

$$\gamma\left(\frac{\pi}{n}(2k-1)\right) = \gamma_{2k-1}.$$

Approximation of the integral of $\gamma(x)$ by Riemann sums gives the relation

$$\int_0^{2\pi}\gamma(x)dx = \lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}\gamma_{2k-1}.$$

Hence we can express the above limit $\mathcal{L}$ in form of an integral:

$$\mathcal{L} = \frac{1}{4\pi}\int_0^{2\pi}\gamma(x)dx = \frac{1}{2\pi}\int_0^{\pi}\gamma(x)dx. \tag{4.3.32}$$

In the last equality we have used $\gamma(x) = \gamma(2\pi - x)$ for $x \in [0, \pi]$.

Remove the parameter θ from the definition of $\gamma(x)$:

Recall that $\theta > 0$ was defined through the relation $\tanh\theta = e^{-2\beta\epsilon}$ which shows that

$$\begin{aligned}\frac{1}{\sinh(2\beta\epsilon)} &= \frac{2}{e^{2\beta\epsilon} - e^{-2\beta\epsilon}} = \frac{2e^{-2\beta\epsilon}}{1 - e^{-4\beta\epsilon}} \\ &= \frac{2\tanh\theta}{1 - \tanh^2\theta} = 2\sinh\theta\cosh\theta = \sinh(2\theta),\end{aligned} \tag{4.3.33}$$

where we use $(1 - \tanh^2\theta)^{-1} = \cosh^2$ and from (4.3.33):

$$\begin{aligned}\cosh(2\theta) &= \sqrt{\sinh^2(2\theta) + 1} = \sqrt{\frac{1}{\sinh^2(2\beta\epsilon)} + 1} \\ &= \frac{1}{\sinh(2\beta\epsilon)}\sqrt{1 + \sinh^2(2\beta\epsilon)} = \frac{\cosh(2\beta\epsilon)}{\sinh(2\beta\epsilon)} = \coth(2\beta\epsilon).\end{aligned} \tag{4.3.34}$$

We insert the identities (4.3.33) and (4.3.34) into the equation (4.3.31):

$$\cosh\gamma(x) = \cosh(2\beta\epsilon)\coth(2\beta\epsilon) - \cos x. \tag{4.3.35}$$

In the following calculation we need the identity [7]

Lemma 4.3.13. *Let $z \in \mathbb{R}$, then:*

$$|z| = \frac{1}{\pi}\int_0^{\pi}\log\left(2\cosh z - 2\cos t\right)dt. \tag{4.3.36}$$

Combining (4.3.35) and (4.3.36) leads to an integral representation of $\gamma(x)$:

$$\begin{aligned}\gamma(x) &= \frac{1}{\pi}\int_0^{\pi}\log\Big(2\cosh\gamma(x) - 2\cos t\Big)\,dt \\ &= \frac{1}{\pi}\int_0^{\pi}\log\Big(2\cosh(2\beta\epsilon)\coth(2\beta\epsilon) - 2\cos x - 2\cos t\Big)\,dt.\end{aligned}$$

From the last identity and (4.3.32) we obtain

$$\mathcal{L} = \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\log\Big(2\cosh(2\beta\epsilon)\coth(2\beta\epsilon) - 2(\cos x + \cos t)\Big)\,dtdx.$$

[7] this identity follows immediately from an integral formula in [3], p. 942;

$$\int_0^{\pi}\log(a \pm b\cos x)dx = \pi\log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right), \qquad (a \geq b).$$

The above integration is taken over the square $[0, \pi] \times [0, \pi]$ however, we can as well integrate over the dotted rectangle in the picture in *Exercise 30, Homework assignment 08* without changing the value of the integral. In fact, consider the two maps $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_1(x,t) := (t, -x)^T \qquad \text{and} \qquad F_2(x,t) := (2\pi - t, x)^T.$$

Then F_j maps the triangles D and F to complementary parts of the triangle A (notation with respect to the picture in Exercise 30) and both transformation leave the above integrand unchanged because of

$$\cos(x) + \cos(t) = \cos(t) + \cos(-x) = \cos(2\pi - t) + \cos(x).$$

The square $[0, \pi] \times [0, \pi]$ is mapped to the dotted rectangle $\mathcal{R}$ by the linear transformation

$$A := \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}, \qquad \text{with} \qquad \det A = 1.$$

We put $D := \cosh(2\beta\epsilon)\coth(2\beta\epsilon)$, then it follows from the transformation rule of the integral and the above observation that

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2\pi^2}\int_{\mathcal{R}} \log\Big[2D - 2(\cos x + \cos t)\Big]\, dtdx \\
&= \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \log\left[2D - 2\cos\left(x - \frac{t}{2}\right) - 2\cos\left(x + \frac{t}{2}\right)\right]\, dtdx \\
&= \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \log\left[2D - 4\cos(x)\cos\left(\frac{t}{2}\right)\right]\, dtdx.
\end{aligned}$$

Next, we decompose the integrand as

$$\log\left[2D - 4\cos(x)\cos\left(\frac{t}{2}\right)\right] = \log\left[2\cos\left(\frac{t}{2}\right)\right] + \log\left[\frac{D}{\cos\left(\frac{t}{2}\right)} - 2\cos(x)\right]$$

and then use the identity (4.3.36) again:

$$\frac{1}{\pi}\int_0^\pi \log\left[\frac{D}{\cos\left(\frac{t}{2}\right)} - 2\cos(x)\right] dx = \cosh^{-1}\left(\frac{D}{2\cosh\left(\frac{t}{2}\right)}\right).$$

Thus we obtain

$$\mathcal{L} = \frac{1}{2\pi}\int_0^\pi \log\left[2\cos\left(\frac{t}{2}\right)\right]\, dt + \frac{1}{2\pi}\int_0^\pi \cosh^{-1}\left(\frac{D}{2\cosh\left(\frac{t}{2}\right)}\right)\, dt.$$

Applying the relation $\cosh^{-1} x = \log(x + \sqrt{x^2 + 1})$ and using the abbreviation

$$\kappa := \frac{2}{D} = \frac{2\sinh(2\beta\epsilon)}{\cosh^2(2\beta\epsilon)} = 4\frac{e^{2\beta\epsilon} - e^{-2\beta\epsilon}}{(e^{2\beta\epsilon} + e^{-2\beta\epsilon})^2}$$

we can therefore write

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2\pi}\int_0^{\pi} \log\left[D + \sqrt{D^2 + 4\cos^2\left(\frac{t}{2}\right)}\right] dt \\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \log\left[D(1+\sqrt{1-\kappa^2\cos^2 s})\right] ds \\
&= \frac{1}{2\pi}\int_0^{\pi} \log\left[D(1+\sqrt{1-\kappa^2\sin^2 s})\right] ds \\
&= \frac{1}{2}\log\Bigg(\underbrace{\frac{2\cosh^2(2\beta\epsilon)}{\sinh(2\beta\epsilon)}}_{=2D}\Bigg) + \frac{1}{2\pi}\int_0^{\pi}\log\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2 s}\right) ds.
\end{aligned}$$

Lemma 4.3.14. *The limit $\mathcal{L}$ has the integral representation*

$$\mathcal{L} = \frac{1}{2}\log\left(\frac{2\cosh^2(2\beta\epsilon)}{\sinh(2\beta\epsilon)}\right) + \frac{1}{2\pi}\int_0^{\pi}\log\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2 s}\right) ds.$$

where

$$\kappa = 4\frac{e^{2\beta\epsilon} - e^{-2\beta\epsilon}}{(e^{2\beta\epsilon} + e^{-2\beta\epsilon})^2}.$$

Proof. Homework. □

We summarize our results

Theorem 4.3.15. *Let $T > 0$ and $\beta = 1/(kT)$, then we have the limit*

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\log Q_I(0,T) &= \frac{1}{2}\log\left[2\sinh(2\epsilon\beta)\right] + \mathcal{L} \\
&= \log\left[2\cosh(2\beta\epsilon)\right] + \frac{1}{2\pi}\int_0^{\pi}\log\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2 s}\right) ds,
\end{aligned}$$

where

$$\kappa = 4\frac{e^{2\beta\epsilon} - e^{-2\beta\epsilon}}{(e^{2\beta\epsilon} + e^{-2\beta\epsilon})^2}.$$

4.4 Thermodynamical functions and physical interpretation

In order to write down the thermodynamical functions we use the notion of *elliptic integrals*.

Definition 4.4.1. The *complete elliptic integral $K_1(\kappa)$ of the first kind* and *$E_1(\kappa)$ of the second type* are defined by:

$$\begin{aligned}
K_1(\kappa) &= \int_0^{\frac{\pi}{2}} \frac{ds}{\sqrt{1-\kappa^2\sin^2 s}} = \frac{1}{2}\int_0^{\pi}\frac{ds}{\Delta}, \qquad \text{where} \quad \Delta := \sqrt{1-\kappa^2\sin^2 s} \\
E_1(\kappa) &= \int_0^{\frac{\pi}{2}} \sqrt{1-\kappa^2\sin^2 s}\, ds.
\end{aligned}$$

Lemma 4.4.2. *One has the following asymptotic behaviour if* $\kappa \to 1$*:*

(i) $\lim_{\kappa\to 1}\{K_1(\kappa) - \log \frac{4}{\sqrt{1-\kappa^2}}\} = 0$,

(ii) $\lim_{\kappa\to 1} E_1(\kappa) = 1$,

Proof. Homework 09. □

Together they fullfil the differential equation

$$\frac{dK_1}{d\kappa}(\kappa) = \frac{E_1(\kappa)}{\kappa(1-\kappa^2)} - \frac{K_1(\kappa)}{\kappa}. \tag{4.4.1}$$

From Theorem 4.3.15 and in the case where $B = 0$ we obtain the thermodynamical functions:

Helmholtz free energy per spin:

$$\begin{aligned} a_I(0,T) &= -\lim_{N\to\infty} \frac{1}{\beta N} \log Q_I(0,T) \\ &= -\beta^{-1} \log\left(2\cosh(2\beta\epsilon)\right) - \frac{1}{2\pi\beta}\int_0^\pi \log\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2 s}\right) ds. \end{aligned}$$

Internal energy per spin: is obtained by

$$\begin{aligned} u_I(0,T) &= \frac{d}{d\beta}\Big[\beta a_I(0,T)\Big] \\ &= -2\epsilon\tanh(2\beta\epsilon) + \frac{\kappa}{2\pi}\frac{d\kappa}{d\beta}\int_0^\pi \frac{\sin^2 s}{(1+\Delta)\Delta} ds, \end{aligned} \tag{4.4.2}$$

where $\Delta := \sqrt{1-\kappa^2\sin^2 s}$. We can rewrite the integral on the right hand side. Consider the relation

$$\begin{aligned} \frac{\kappa^2\sin^2 s}{(1+\Delta)\Delta} &= -\frac{1-\kappa^2\sin^2 s}{(1+\sqrt{1-\kappa^2\sin^2})\sqrt{1-\kappa^2\sin^2 s}} + \frac{1}{(1+\sqrt{1-\kappa^2\sin^2 s})\sqrt{1-\kappa^2\sin^2 s}} \\ &= -\frac{\Delta}{1+\Delta} - \frac{1}{1+\Delta} + \frac{1}{\Delta} = -1 + \frac{1}{\Delta}. \end{aligned}$$

Therefore we have

$$\int_0^\pi \frac{\sin^2 s}{(1+\Delta)\Delta} ds = -\frac{\pi}{\kappa^2} + \frac{1}{\kappa^2}\int_0^\pi \frac{ds}{\Delta} = -\frac{\pi}{\kappa^2} + \frac{2}{\kappa^2} K_1(\kappa).$$

Here we have used the notation of elliptic integrals. We also calculate the expression $\kappa^{-1} d\kappa/d\beta$:

$$\frac{1}{\kappa}\frac{d\kappa}{d\beta} = \frac{\cosh^2(2\beta\epsilon)}{\sinh(2\beta\epsilon)}\frac{d}{d\beta}\left(\frac{\sinh(2\beta\epsilon)}{\cosh^2(2\beta\epsilon)}\right) = 2\epsilon\coth(2\beta\epsilon) - 4\epsilon\tanh(2\beta\epsilon).$$

Plugging this relations into (4.4.2) gives

$$\begin{aligned} u_I(0,T) &= -2\epsilon\tanh(2\beta\epsilon) + \frac{1}{2\pi}\left(\frac{1}{\kappa}\frac{d\kappa}{d\beta}\right)[-\pi + 2K_1(\kappa)] \\ &= -2\epsilon\tanh(2\beta\epsilon) + \big[\epsilon\coth(2\beta\epsilon) - 2\epsilon\tanh(2\beta\epsilon)\Big[-1 + \frac{2}{\pi}K_1(\kappa)\Big] \\ &= -\epsilon\coth(2\beta\epsilon) + \frac{2\epsilon}{\pi}K_1(\kappa)\Big[\coth(2\beta\epsilon) - 2\tanh(2\beta\epsilon)\Big] \\ &= -\epsilon\coth(2\beta\epsilon)\Big[1 - \frac{2}{\pi}K_1(\kappa) + \frac{4}{\pi}\tanh^2(2\beta\epsilon)K_1(\kappa)\Big]. \end{aligned}$$

If we define the function

$$\kappa' = \kappa'(\epsilon\beta) := 2\tanh^2(2\beta\epsilon) - 1,$$

then we have

Lemma 4.4.3. *The inner energy per spin is given by*

$$u_I(0,T) = -\epsilon\coth(2\beta\epsilon)\Big[1 + \kappa'\frac{2}{\pi}K_1(\kappa)\Big]. \tag{4.4.3}$$

where

$$\kappa = \frac{2\sinh(2\beta\epsilon)}{\cosh^2(2\beta\epsilon)} \qquad \textit{and} \qquad \kappa' = \kappa'(\epsilon\beta) := 2\tanh^2(2\beta\epsilon) - 1.$$

The functions κ and κ' are related by

$$\kappa^2 + {\kappa'}^2 = 1. \tag{4.4.4}$$

Moreover, $u_I(0,T)$ considered as a function of κ does not extend analytically around $\kappa = 1$.

Proof. (4.4.4) follows by a direct calculation. We show that

$$F(\kappa) := \kappa' K_1(\kappa) = \sqrt{1-\kappa^2}K_1(\kappa)$$

is not analytic in $\kappa = 1$. According to the DGL (4.4.1) we have

$$K_1'(\kappa)\sqrt{1-\kappa^2} = \frac{E_1(\kappa)}{\kappa\sqrt{1-\kappa^2}} - \frac{\sqrt{1-\kappa^2}K_1(\kappa)}{\kappa}.$$

Therefore

$$\begin{aligned} F'(\kappa) &= -\frac{\kappa}{\sqrt{1-\kappa^2}}K_1(\kappa) + K_1'(\kappa)\sqrt{1-\kappa^2} \\ &= -\frac{\kappa}{\sqrt{1-\kappa^2}}K_1(\kappa) + \frac{E_1(\kappa)}{\kappa\sqrt{1-\kappa^2}} - \frac{\sqrt{1-\kappa^2}K_1(\kappa)}{\kappa}. \end{aligned}$$

Now it follows from the asymptotic behaviour of $K_1(\kappa) \sim \log(4/\sqrt{1-\kappa^2})$ and $E_1(\kappa) \sim 1$ as $\kappa \to 1$ in Lemma 4.4.2 that $|F'(\kappa)| \to \infty$ as $\kappa \to 1$. □

We call the temperature T_c corresponding to $\kappa(\beta_c\epsilon) = 1$ where $\beta_c = 1/(kT_c)$ the *critical temperature*. This means that $\kappa'(\beta_c\epsilon) = 0$, or equivalently

$$\tanh(2\beta_c\epsilon) = \tanh\frac{2\epsilon}{kT_c} = \frac{1}{\sqrt{2}}, \qquad \text{and} \qquad \frac{\epsilon}{kT_c} = 0,4406868\cdots. \tag{4.4.5}$$

In particular, it holds

$$\cosh^2(2\beta_c\epsilon) = \frac{\sinh^2(2\beta_c\epsilon)}{\tanh^2(2\beta_c\epsilon)} = 2\sqrt{\cosh^2(2\beta_c\epsilon) - 1}$$

which gives

$$\begin{aligned} \cosh(2\beta_c\epsilon) &= \sqrt{2}, \\ \sinh(2\beta_c\epsilon) &= \tanh(2\beta\epsilon)\cosh(2\beta\epsilon) = 1. \end{aligned}$$

Heat capacity per spin: By using (4.4.3) and (4.4.1) one obtains

$$\begin{aligned} c_I(0,T) &= \frac{\partial u_I}{\partial T}(0,T) \\ &= \frac{2\kappa}{\pi}\big(\beta\epsilon \coth^2(2\beta\epsilon)\big)^2 \Big[2K_1(\kappa) - 2E_1(\kappa) - (1-\kappa')\Big(\frac{\pi}{2} + \kappa' K_1(\kappa)\Big)\Big]. \end{aligned}$$

The "heat capacity per spin" has a logarithmic singularity as $|T - T_c| \to 0$:

$$c_I(0,T) \sim C(\epsilon) \log\left|\frac{T-T_c}{T_c}\right| \qquad \text{as} \qquad T \to T_c.$$

Magnetization per spin: In order to calculate $m_I(0,T)$ we need an expression for the inner energy $a_I(B,T)$ for $B \neq 0$. Since we have assumed $B = 0$ in our calculations we cannot use the above formulas and present an expression of the magnetization/spin without a proof (for details see [9]):

$$m_I(B,T) = -\frac{\partial}{\partial B}\Big(\beta a_I(B,T)\Big)_{|B=0} = \begin{cases} 0 & \text{if } T > T_c, \\ \frac{(1+z^2)^{\frac{1}{4}}(1-6z^2+z^4)^{\frac{1}{8}}}{\sqrt{1-z^2}}, & \text{if } T < T_c. \end{cases}$$

Here we put $z = e^{-2\beta\epsilon}$.

Chapter 5

The renormalization group

(Robert Helling)

Chapter 6

Ideal gases

Within the mathematical framework of the CCR and CAR algebras we study thermodynamical models describing non-interacting particles in some bounded set $\Lambda \subset \mathbb{R}^n$. These are the so-called *free gases*. The simplifying assumption of non-interacting particles is a good approximation for a gas at high temperature and low pressure where the intermolecular forces become negligible.

6.1 The ideal Fermi gas

Let $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ be a "one-particle-Hilbert-space" over $\mathbb{C}$ and recall that the *Fermi-Fock space* was defined by

$$\mathfrak{F}_-(\mathfrak{h}) := P_- \mathfrak{F}(\mathfrak{h}).$$

Here we have:

- $\mathfrak{F}(\mathfrak{h}) = \bigoplus_{n \geq 0} \mathfrak{h}^n$ where $\mathfrak{h}^n = \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}$ with $n \in \mathbb{N}$ and $\mathfrak{h}^0 = \mathbb{C}$. (Fock space over $\mathfrak{h}$).
- $P_-=$ *projection onto the "anti-symmetric part" of* $\mathfrak{F}(\mathfrak{h})$.

Let H be a self-adjoint Hamiltonian operator on $\mathfrak{h}$ with *second quantization* $d\Gamma(H)$ on $\mathfrak{F}(\mathfrak{h}_-)$

$$d\Gamma(H) := \overline{\bigoplus_{n \geq 0} H_n} = \textit{self-adjoint closure}, \qquad H_0 = 0,$$

$$H_n\big(P_-(f_1 \otimes \cdots \otimes f_n)\big) : = P_-\Big(\sum_{i=1}^n f_1 \otimes f_2 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n\Big).$$

Put $\hbar = 1$ such that the Schrödinger equation for an arbitrary number of fermions moving independently is given by

$$i\frac{\Psi_t}{dt} = d\Gamma(H)\Psi_t$$

[1]

We consider the Gibbs grand canonical ensemble. Let $\mu \in \mathbb{R}$ (chemical potential) and $\beta \in \mathbb{R}$ (inverse temperature) and consider the modified Hamiltonian

$$K_\mu := d\Gamma(H - \mu I).$$

[1] With solution $\Psi_t = e^{-itd\Gamma(H)} = \Gamma(e^{-itH})\Psi$ and the evolution $\tau_t(A) = \Gamma(e^{itH})A\Gamma(e^{itH})$.

The *Gibbs equilibrium state* on the *CAR-algebra* $\mathcal{A}_{\text{CAR}}(\mathfrak{h})$ over $\mathfrak{h}$ takes the form

$$\omega(A) := \frac{\text{trace}(e^{-\beta K_\mu} A)}{\text{trace}(e^{-\beta K_\mu})}, \qquad \text{where} \qquad A \in \mathcal{A}(\mathfrak{h}).$$

Recall that $\mathcal{A}_{\text{CAR}}(\mathfrak{h})$ is the algebra generated by the identity I and $a(f)$ with $f \in \mathfrak{h}$ such that

(1) $\mathfrak{h} \ni f \mapsto a(f)$ is anti-linear,

(2) $\{a(f), a(g)\} = 0$

(3) $\{a(f), a(g)^*\} = \langle f, g \rangle I$.

Question: *Is the Gibbs state ω well-defined? More precisely: when is $e^{-\beta K_\mu}$ trace class?*

Lemma 6.1.1. *Let $\beta \in \mathbb{R}$, then (a) and (b) are equivalent:*

(a) $e^{-\beta H}$ *is trace class on* $\mathfrak{h}$,

(b) $e^{-\beta d\Gamma(H-\mu I)}$ *is trace class on* $\mathfrak{F}_-(\mathfrak{h})$ *for all* $\mu \in \mathbb{R}$.

Proof. Proposition 5.2.22 in Bratteli/Robinson. □

Remark 6.1.2. If the Gibbs state is not defined for all or some β (e.g. β negative) we can replace it by a τ-KMS state $\tilde{\omega}$ with respect to the following evolution

$$\mathcal{A}_{\text{CAR}}(\mathfrak{h}) \ni A \mapsto \tau_t(A) = e^{itK_\mu} A e^{-itK_\mu} \in \mathcal{A}_{\text{CAR}}(\mathfrak{h}). \tag{6.1.1}$$

Recall that the KMS-condition (which would be used in the following arguments) has the form:

$$\tilde{\omega}\big(A\tau_t(B)\big)_{|t=i\beta} = \tilde{\omega}(BA).$$

If the Gibbs state exists, then it is the unique τ-KMS state.

We consider the evolution (6.1.1) on generators $a^*(f)$ of $\mathcal{A}_{\text{CAR}}(\mathfrak{h})$.

Lemma 6.1.3. *Let $a(f) \in \mathcal{A}_{\text{CAR}}(\mathfrak{h})$ with $f \in \mathfrak{h}$, then we have for all t*

(i) $e^{itd\Gamma(H)} a^*(f) e^{-itd\Gamma(H)} = a^*\big(e^{itH} f\big)$,

(ii) $e^{itd\Gamma(H)} a(f) e^{-itd\Gamma(H)} = a\big(e^{itH} f\big)$.

Proof. We only show (i). Put $U_t := e^{itH}$ and recall that the second quantization relates the unitary one-parameter groups U_t corresponding to H and $d\Gamma(H)$ in the following way

$$e^{itd\Gamma(H)} = \Gamma(U_t) := \bigoplus_{n \geq 0} U_{n,t},$$

where $U_{0,t} = I$ and with $n \in \mathbb{N}$:

$$U_{n,t}\Big(P_-(f_1 \otimes f_2 \otimes \cdots \otimes f_n)\Big) := P_-\Big[U_t f_1 \otimes U_t f_2 \otimes \cdots \otimes U_t f_n\Big].$$

From this it follows:

$$\begin{aligned}
e^{itd\Gamma(H)}a^*(f)e^{-itd\Gamma(H)}P_-\big(f_1\otimes\cdots\otimes f_n\big) &= \Gamma(U_t)a^*(f)P_-\Big(U_{-t}f_1\otimes\cdots\otimes U_{-t}f_n\Big)\\
&= \frac{\sqrt{n+1}}{n!}\Gamma(U_t)P_-\Big(\sum_\pi \epsilon_\pi f\otimes U_{-t}f_{\pi_1}\otimes\cdots\otimes U_{-t}f_{\pi_n}\Big)\\
&= \frac{\sqrt{n+1}}{n!}P_-\Big(\sum_\pi \epsilon_\pi U_t f\otimes f_{\pi_1}\otimes\cdots\otimes f_{\pi_n}\Big)\\
&= P_-a^*(U_tf)P_-(f_1\otimes\cdots\otimes f_n).
\end{aligned}$$

Since $= P_-a^*(U_tf)P_- = a^*(U_tf)$ this finishes the proof. □

Write $z = e^{\beta\mu} > 0$ for the *activity*. Using the previous lemma we can calculate the so-called *two-point functions* of the Gibbs state ω.

Corollary 6.1.4. *Let $f, g \in \mathfrak{h}$, then we have*

$$\omega\big(a^*(f)a(g)\big) = \Big\langle g, ze^{-\beta H}(I+ze^{-\beta H})^{-1}f\Big\rangle. \tag{6.1.2}$$

Proof. In Lemma 6.1.3 we replace t by $i\beta$ and H by $H-\mu I$. Then

$$\begin{aligned}
\text{trace}\Big\{e^{-\beta K_\mu}a^*(f)a(g)\Big\} &= \text{trace}\Big\{e^{-\beta K_\mu}a^*(f)e^{\beta K_\mu}\, e^{-\beta K_\mu}a(g)\Big\}\\
&= \text{trace}\Big\{a^*\big(e^{-\beta(H-\mu I)}f\big)e^{-\beta K_\mu}a(g)\Big\}\\
&= z\ \text{trace}\Big\{e^{-\beta K_\mu}a(g)a^*\big(e^{-\beta H}f\big)\Big\} = (*).
\end{aligned}$$

Now we use the anti-commutation relations to switch $a^*(e^{-\beta H}f)$ back to the left:

$$(*) = -z\ \text{trace}\Big\{e^{-\beta K_\mu}a^*\big(e^{-\beta H}f\big)a(g)\Big\} + z\big\langle g, e^{-\beta H}f\big\rangle\text{trace}(e^{-\beta K_\mu}).$$

Dividing both sides by $\text{trace}(e^{-\beta K_\mu})$ gives

$$\omega\big(a^*(f)a(g)\big) = -z\omega\big(a^*(e^{-\beta H}f)a(g)\big) + z\big\langle g, e^{-\beta H}f\big\rangle$$

or equivalently

$$\omega\big(a^*\big([I+ze^{-\beta H}]f\big)a(g)\big) = z\big\langle g, e^{-\beta H}f\big\rangle.$$

Finally, (6.1.2) follows by replacing f with $(I+ze^{-\beta H})^{-1}f$. □

Definition 6.1.5. Consider the group of Bogoliubov transformations of $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ induced by

$$\tau_\theta\big[a(f)\big] := a\big(e^{i\theta}f\big), \qquad \text{where} \qquad \theta\in[0,2\pi).$$

These are the so-called *gauge transformations*. A state on $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ is called *gauge-invariant* if it is invariant under gauge transformations.

Remark 6.1.6. By a very similar argument one checks that the formula (6.1.2) generalizes to

$$\omega\left(\prod_{i=1}^{n} a^*(f_i)\prod_{j=1}^{m} a(g_j)\right) =$$

$$= \begin{cases} 0 & \text{if } n \neq m, \\ \sum_{\ell=1}^{n}(-1)^{n-\ell}\omega\big(a^*(f_1)a(g_\ell)\big)\omega\left(\prod_{i=2}^{n} a^*(f_i)\prod_{\substack{j=1\\ j\neq\ell}}^{m} a(g_j)\right), & \text{else.}\end{cases}$$

In particular,

(I) By iteration of this process it follows that the Gibbs state ω only depends on the values of all the *two-point functions*

$$\omega\big(a^*(f)a(g)\big) = \Big\langle g, ze^{-\beta H}(I + ze^{-\beta H})^{-1}f\Big\rangle.$$

The state ω is called *quasi-free.* [2]

(II) Remark (I) implies that the Gibbs state ω on $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ is gauge-invariant and quasi free.

Now we specify the discussion to the following case. Let $\Lambda \subset \mathbb{R}^n$ be a bounded and open subset and put

$$\mathfrak{h}_\Lambda := L^2(\Lambda), \qquad \text{and} \qquad \mathfrak{h} := L^2(\mathbb{R}^n),$$
$$C_0^\infty(\Omega) = \Big\{f \in C^\infty(\Omega) \;:\; \text{supp }(f) \subset \Omega \text{ is compact}\Big\}, \qquad \Omega \in \{\Lambda, \mathbb{R}^n\}.$$

Consider the (positive) Laplacian $-\Delta$ on $C_0^\infty(\Lambda)$. With respect to suitable units we define the Hamiltonians

$$H_\Lambda = \textit{some self-adjoint extension of } -\Delta \textit{ on } C_0^\infty(\Lambda),$$
$$H = \textit{self-adjoint extension of } -\Delta \textit{ on } C_0^\infty(\mathbb{R}^n).$$

There are various self-adjoint extensions H_Λ of $-\Delta$ on $L^2(\Lambda)$ according to the choice of boundary conditions. However, the Laplacian on $\mathbb{R}^n$ has a unique self-adjoint extension. The operators H_Λ typically have *discrete spectrum* with eigenvalue asymptotic (Weyl-asymptotic)

$$\lambda_\ell \sim \ell^{\frac{\dim\Lambda}{2}}, \qquad \text{as} \qquad \ell \to \infty$$

and therefore $e^{-\beta H_\Lambda}$ is trace class if $\beta > 0$. However, H has no discrete spectrum and $e^{-\beta H}$ is not of trace class for any $\beta \in \mathbb{R}$.

Remark 6.1.7 (*classical boundary conditions)***.** Let $\Lambda \subset \mathbb{R}^n$ be bounded and open with piecewise differentiable boundary $\partial\Lambda$. Recall Green's formula

$$\big\langle \Delta\psi, \varphi\big\rangle - \big\langle \psi, \Delta\varphi\big\rangle = \int_{\partial\Lambda}\left\{\overline{\psi}\,\frac{\partial\varphi}{\partial n} - \frac{\partial\overline{\psi}}{\partial n}\,\varphi\right\} d\sigma.$$

In order to make Δ symmetric on its domain of definition we must make sure that the integrand vanishes for all $\varphi, \psi \in \mathcal{D}(\Delta)$. We may choose

[2] We do not give the exact definition of a quasi-free state here which requires the notion of truncation functions. As for details see Bratteli/Robinson II, page 43.

(i) $\frac{\partial\varphi}{\partial n} = 0$ on $\partial\Lambda$, (*Neumann boundary conditions*),

(ii) $\varphi = 0$ on $\partial\Lambda$, (*Dirichlet boundary conditions*),

(iii) $\frac{\partial\varphi}{\partial n} = h\varphi$ where $h \in C^1(\partial\Lambda)$ is real-valued.

First we comment on the thermodynamical limit. In the following the limit $\Lambda \to \infty$ means that Λ is a sequence of open bounded sets that eventually contains all bounded $\widetilde{\Lambda} \subset \mathbb{R}^n$.

We write ω_Λ for the Gibbs equilibrium state over $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h}_\Lambda)$. Let ω be the gauge-invariant quasi-free state over $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ with two point functions (c.f. Corollary 6.1.4):

$$\omega\big(a^*(f)a(g)\big) = \Big\langle g, ze^{-\beta H}(I + ze^{-\beta H})^{-1} f\Big\rangle_{\mathfrak{h}}.$$

Proposition 6.1.8. *For all $A \in \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h}_\Lambda)$ it holds $\lim_{\Lambda\to\infty} \omega_\Lambda(A) = \omega(A)$.*

Proof. Bratteli/Robinson II. □

In particular, the thermodynamical limit of the "finite-volume equilibrium states" is uniquely defined and independent of the particular boundary conditions (*unique thermodynamic phase*).

6.2 Equilibrium phenomena

The explicit expression of the two point functions for the infinite idealized Fermi gas allows us to study some equilibrium phenomena.

Definition 6.2.1. Consider the *number functional* $\widehat{N}$ which measures the number of particles in a given state:

$$\widehat{N} : E_{\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})} \longrightarrow [0, \infty] : \widehat{N}(\tilde{\omega}) := \sup_F \sum_{\{f_i\}\subset F} \tilde{\omega}\big(a^*(f_i)a(f_i)\big). \tag{6.2.1}$$

F runs through finite dimensional subspaces of $\mathfrak{h}$ and $\{f_i\}$ through the ONBs of F.

Exercise 6.2.2. *Let $[e_i \,:\, i \in \mathbb{N}_0]$ and $[f_j \,:\, j \in \mathbb{N}_0]$ be orthonormal bases of $\mathfrak{h}$ and put*

$$\psi^{(m)} = P_-\big[e_{j_1} \otimes \cdots \otimes e_{j_m}\big],$$

where $m \in \mathbb{N}_0$ and the entries of $(j_1, \cdots, j_m) \in \mathbb{N}_0^n$ are pairwise distinct (otherwise $\Psi^{(m)} = 0$). With the number operator N on $\mathcal{F}_-(\mathfrak{h})$ show that

$$m = \big\langle \psi^{(m)}, N\psi^{(m)}\big\rangle = \sum_{n\geq 0} \Big\langle \psi^{(m)}, a^*(f_n)a(f_n)\psi^{(m)}\Big\rangle.$$

Consider the quasi-local CAR algebras

$$\mathcal{A}_\Lambda = \mathcal{A}_{\mathrm{CAR}}\big(\mathfrak{h}_\Lambda\big), \qquad \text{such that} \qquad \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h}) = \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}.$$

By choosing the sub-spaces F in (6.2.1) only in $\mathfrak{h}_\Lambda$ we obtain local number functionals

$$\widehat{N}_\Lambda : E_{\mathcal{A}_\Lambda} \to [0,\infty].$$

We calculate the following *density* for the Gibbs equilibrium state ω (=*number of particles per unit volume in* Λ):

Let $\{f_n\}_n$ be an orthonormal basis of $L^2(\Lambda)$, then we find from Corollary 6.1.4:

$$\begin{aligned}\rho(\beta,z) :&= \frac{\widehat{N}_\Lambda(\omega)}{|\Lambda|^{-1}}, \qquad \text{with} \quad |\Lambda| := \textit{volume of } \Lambda \\ &= |\Lambda|^{-1}\sum_{n\geq 0}\omega\big(a^*(f_n)a(f_n)\big) \\ &= |\Lambda|^{-1}\sum_{n\geq 0}\big\langle f_n, ze^{\beta\Delta}(I+ze^{\beta\Delta})^{-1}f_n\big\rangle_{L^2(\Lambda)} = (*).\end{aligned}$$

Via continuation by zero we can embed $L^2(\Lambda)$ into $L^2(\mathbb{R}^n)$. Let $\widehat{f}$ be the Fourier transform of $f\in L^2(\mathbb{R}^n)$, then:

$$\begin{aligned}\Big\langle f_n, ze^{\beta\Delta}(I+ze^{\beta\Delta})^{-1}f_n\Big\rangle_{L^2(\Lambda)} &= \Big\langle \widehat{f}_n, ze^{-\beta p^2}(1+ze^{-\beta p^2})^{-1}\widehat{f}_n\Big\rangle_{L^2(\mathbb{R}^n)} \\ &= \Big\langle |\widehat{f}_n|^2, ze^{-\beta p^2}(1+ze^{-\beta p^2})^{-1}\Big\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

With $p,x\in\mathbb{R}^n$ put $e_p(x) := (2\pi)^{-\frac{n}{2}}e^{ixp}$, then we have

$$\sum_{n\geq 0}|\widehat{f}_n|^2(p) = \sum_{n\geq 0}|\langle f_n, e_p\rangle|^2 = \|e_p\|^2_{L^2(\Lambda)} = \frac{|\Lambda|}{(2\pi)^n}.$$

Inserting this above gives

Lemma 6.2.3. *For each bounded open set $\Lambda\subset\mathbb{R}^n$ the density function $\rho(\beta,z)$ has the form*

$$\rho(\beta,z) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} ze^{-\beta p^2}\big(1+ze^{-\beta p^2}\big)^{-1}dp = \lambda^{-n}I(z) < \infty,$$

where $\lambda := \sqrt{4\pi\beta}$ ("thermal wave lenght of the individual particle") and the function $I(z)$ is given by

$$I(z) := \pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} ze^{-x^2}(1+ze^{-x^2})^{-1}dx.$$

In particular, $\rho(\beta,z)$ is independent of Λ (which is expected since the equilibrium state is invariant under space translations).

Next: *Calculate the local energy per unit volume.*

Let $\{f_n\}\subset C^1(\Lambda)$ be an orthonormal basis of $L^2(\Lambda)$. The *local energy per unit volume* of the state ω is given by

$$\begin{aligned}\varepsilon(\beta,z) &= |\Lambda|^{-1}\sum_{n\geq 0}\omega\big(a^*(\sqrt{-\Delta}f_n)a(\sqrt{-\Delta}f_n)\big) \\ &= |\Lambda|^{-1}\sum_{n\geq 0}\Big\langle f_n, ze^{\beta\Delta}(I+ze^{\beta\Delta})^{-1}(-\Delta)f_n\Big\rangle_{L^2(\Lambda)}.\end{aligned}$$

Exercise 6.2.4. *With the notation of Exercise 6.2.2 it holds*

$$\sum_{n\geq 0}\left\langle \psi^{(m)}, a^*(\sqrt{-\Delta}f_n)a(\sqrt{-\Delta}f_n)\psi^{(m)}\right\rangle_{\mathfrak{h}_\Lambda} = \left\langle \psi^{(m)}, T_\Lambda \psi^{(m)}\right\rangle_{\mathfrak{h}_\Lambda},$$

where $\{f_n\}_n$ *is a (suitable) orthonormal basis of* $\mathfrak{h}_\Lambda$ *and* T_Λ *is a self-adjoint extension of the second quantization* $\Gamma(-\Delta)$ *of* $-\Delta$ *w.r.t Neumann boundary conditions.* [3].

By a similar argument like the one we used for the density function $\rho(z,\beta)$ we obtain

$$\varepsilon(\beta,z) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} p^2 z e^{-\beta p^2}(1+ze^{-\beta p^2})^{-1}dp = (*).$$

Note that

$$\frac{zp_j^2 e^{-\beta p^2}}{1+ze^{-\beta p^2}} = -\frac{p_j}{2\beta}\frac{\partial}{\partial p_j}\log\left(1+ze^{-\beta p^2}\right)$$

and therefore one obtains via partial integration

$$\begin{aligned}(*) &= -\frac{1}{2\beta}(2\pi)^{-n}\sum_{j=1}^n\int_{\mathbb{R}^n} p_j\frac{\partial}{\partial p_j}\log\left(1+ze^{-\beta p^2}\right)dp\\ &= \frac{1}{2\beta}(2\pi)^{-n}\sum_{j=1}^n\int_{\mathbb{R}^n}\log\left(1+ze^{-\beta p^2}\right)dp\\ &= \frac{n}{2\beta}(2\pi)^{-n}\int_{\mathbb{R}^n}\log\left(1+ze^{-\beta p^2}\right)dp.\end{aligned}$$

Lemma 6.2.5. *For each bounded and open* $\Lambda\subset\mathbb{R}^n$ *the local energy per unit volume fulfills*

$$\varepsilon(\beta,z) = \frac{n}{2\beta}(2\pi)^{-n}\int_{\mathbb{R}^n}\log\left(1+ze^{-\beta p^2}\right)dp = \beta^{-1}\lambda^{-n}J(z) < \infty,$$

where $\lambda := \sqrt{4\pi\beta}$ *and the function* $J(z)$ *is given by*

$$J(z) := \pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} zx^2 e^{-x^2}(1+ze^{-x^2})^{-1}dx.$$

In particular, $\varepsilon(\beta,z)$ *is independent of* Λ.

Fermi sea: Consider the idealization of zero temperature: if we take $\beta\to\infty$, then the integrand in the expression of $\rho(\beta,z)$ behaves as follows (recall that $z=e^{\beta\mu}$):

$$\lim_{\beta\to\infty} ze^{-\beta p^2}\left(1+ze^{-\beta p^2}\right)^{-1} = \lim_{\beta\to\infty} e^{-\beta(p^2-\mu)}\left(1+e^{-\beta(p^2-\mu)}\right)^{-1} = \begin{cases}1, & \text{if } p^2<\mu\\ 0, & \text{if } p^2>\mu.\end{cases}$$

All states with energy $<\mu$ are occupied and states with energy greater than μ are empty. The critical value $\mu=p^2$ is called *Fermi surface.*

[3] If $\psi\in C_0^\infty(\Lambda)\subset\mathfrak{h}_\Lambda\subset\mathfrak{F}_-(\mathfrak{h}_\Lambda)$, then we have $T_\Lambda(\psi) = -\langle\psi,\Delta\psi\rangle_{\mathfrak{h}_\Lambda}$

6.3 The ideal Bose gas

Consider the *Bose Fock space* $\mathcal{F}_+(\mathfrak{h})$ over the one-particle Hilbert space $\mathfrak{h}$ with one-particle Hamiltonian H, i.e.

$$\mathcal{F}_+(\mathfrak{h}) := P_+\mathcal{F}(\mathfrak{h}),$$

where P_+ is the projection onto the *symmetric part* of $\mathcal{F}(\mathfrak{h})$. The Hamiltonian for a non-interacting system of Bosons is given as the second quantization $d\Gamma(H)$ of H.

The corresponding time evolution of observables $A \in \mathcal{L}\big(\mathcal{F}_+(\mathfrak{h})\big)$ has the form

$$A \mapsto \tau_t(A) := e^{itd\Gamma(H)}Ae^{-itd\Gamma(H)} = \Gamma\big(e^{itH}\big)A\Gamma\big(e^{-itH}\big). \tag{6.3.1}$$

Let $a_+(f)$ and $a_+^*(f)$ be the *annihilation* and *creation* operator on $\mathcal{F}_+(\mathfrak{h})$, respectively, which fulfill the *canonical commutation relations* (CCR) for all $f, g \in \mathfrak{h}$

(a) $\big[a_+(f), a_+(g)\big] = 0 = \big[a_+^*(f), a_+^*(g)\big] = 0,$

(b) $\big[a_+(f), a_+^*(g)\big] = \langle f, g\rangle I.$

The operators $a_+(f)$ and $a_+^*(f)$ with $f \in \mathfrak{h}$ are densely defined and unbounded in general. We pass to the family of *Weyl operators* $\mathcal{W} := \{W(f) \,:\, f \in \mathfrak{h}\}$ which are unitary

$$W(f) := e^{\frac{i}{\sqrt{2}}\overline{[a_+(f)+a_+^*(f)]}} \in \mathcal{L}\big(\mathcal{F}_+(\mathfrak{h})\big)$$

and satisfy

(a) $W(-f) = W(f)^*$ for all $f \in \mathfrak{h}$,

(b) $W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\,\langle f,g\rangle}W(f+g)$ for all $f, g \in \mathfrak{h}$.

Definition 6.3.1. The C^*-algebra $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ in $\mathcal{L}(\mathcal{F}_+(\mathfrak{h}))$ generated by $\mathcal{W}$ is called *CCR-algebra*.

We consider the action of τ_t on generators of the CCR algebra:

Lemma 6.3.2. *For all $t \in \mathbb{R}$ and $f \in \mathfrak{h}$ the $*$-automorphism τ_t acts on Weyl-operators as*

$$\tau_t\big(W(f)\big) = W\big(e^{itH}f\big). \tag{6.3.2}$$

In particular, $\{\tau_t\}_t$ defines a group of automorphisms on $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$.

Proof. Homework □

Remark 6.3.3. Recall that the one-parameter group of operators (6.3.2) is not strongly continuous.

With $\mu \in \mathbb{R}$ consider the generalize Hamiltonian $K_\mu := d\Gamma(H - \mu I)$ and assume that $e^{-\beta K_\mu}$ with $\beta \in \mathbb{R}$ is of trace class

Definition 6.3.4. The *Gibbs equilibrium state* on the *CCR-algebra* $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ takes the form

$$\omega(A) := \frac{\mathrm{trace}(e^{-\beta K_\mu}A)}{\mathrm{trace}(e^{-\beta K_\mu})}, \quad \text{where} \quad A \in \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h}).$$

Next step: *We extend the Gibbs state from $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ to polynomials in $a_+(f)$ and $a_+(g)$.*

Now, fix $n \in \mathbb{N}_0$ and put $f := (f_1, \cdots, f_n)$ with $f_j \in \mathfrak{h}$. Consider the operator

$$A_f := a(f_1)a(f_2)\cdots a(f_n)e^{-\frac{\beta}{2}K_\mu}. \tag{6.3.3}$$

The following result essentially distinguishes the existence of traces in case of the ideal Fermi and the ideal Bose gas, respectively, (c.f. Lemma 6.1.1).

Proposition 6.3.5. *Let $\mu, \beta \in \mathbb{R}$ and assume that $e^{-\beta H}$ is a trace class operator on $\mathfrak{h}$. Let $z := e^{\beta\mu}$ denote the "activity". Assume that $\beta(H - \mu I) > 0$, then*

(a) *The operator $e^{-\beta K_\mu}$ is of trace class.*

(b) *The operator $A_f^* A_f$ is of trace class.*

(c) *The two point functions $\omega(a^*(f)a(g))$ with $f, g \in \mathfrak{h}$ are well-defined and there is a constant $C(z, \beta)$ depending on z and β such that*

$$\big|\omega\big(a^*(f)a(g)\big)\big| \le C(z, \beta)\|f\| \cdot \|g\|. \tag{6.3.4}$$

Proof. (a): Let $\{\lambda_n\}_{n\ge 0}$ be the sequence of eigenvalues of H repeated according to the multiplicity and increasing (decreasing) if $\beta > 0$ (if $\beta < 0$). Let $\{e_n\} \subset \mathfrak{h}$ be an orthonormal basis of eigenvectors of H, i.e. $He_n = \lambda_n e_n$. With

$$0 \le j_1 < j_2 < \cdots < j_m,$$

where $m \in \mathbb{N}$ and *occupation numbers* $(n_{j_1}, \cdots, n_{j_m}) \in \mathbb{N}^m$ consider $E_{n_{j_1}\cdots, n_{j_m}} \in \mathcal{F}_+(\mathfrak{h})$ defined by:

$$E_{n_{j_1}\cdots, n_{j_m}} := P_+\big(\underbrace{e_{j_1} \otimes \cdots \otimes e_{j_1}}_{n_{j_1} \text{ times}} \otimes \underbrace{e_{j_2} \otimes \cdots \otimes e_{j_2}}_{n_{j_2} \text{ times}} \otimes \cdots \otimes \underbrace{e_{j_m} \otimes \cdots \otimes e_{j_m}}_{n_{j_m} \text{ times}}\big).$$

Note that $E_{n_1,\cdots,n_m}$ is an eigenvector of $e^{-\beta K_\mu}$. Put $N := n_{j_1} + n_{j_2} + \cdots + n_{j_m}$, then

$$\begin{aligned} e^{-\beta K_\mu} E_{n_1,\cdots,n_m} &= \Gamma(e^{-\beta(H-\mu I)})P_+\big(e_{j_1} \otimes \cdots \otimes e_{j_1} \otimes \cdots \otimes e_{j_m} \otimes \cdots \otimes e_{j_m}\big) \\ &= z^N P_+\left(e^{-\beta H}e_{j_1} \otimes \cdots \otimes e^{-\beta H}e_{j_1} \otimes \cdots \otimes e^{-\beta H}e_{j_m} \otimes \cdots \otimes e^{-\beta H}e_{j_m}\right) \\ &= z^N e^{-\beta(n_{j_1}\lambda_{j_1} + \cdots + n_{j_m}\lambda_{j_m})} E_{n_1,\cdots,n_m}. \end{aligned}$$

According to our assumption $\beta(H - \mu I) > 0$ we have $ze^{-\beta\lambda_j} = e^{-\beta(\lambda_j - \mu)} < 1$. Hence, we can estimate the trace of $e^{-\beta K_\mu}$ as follows:

$$\begin{aligned} \operatorname{trace}\left(e^{-\beta K_\mu}\right) &\le \prod_{j=0}^{\infty}\left(1 + ze^{-\beta\lambda_j} + z^2e^{-2\beta\lambda_j} + z^3e^{-3\beta\lambda_j} + \cdots\right) \\ &= \prod_{j=0}^{\infty}\left(1 - ze^{-\beta\lambda_j}\right)^{-1} \\ &= \exp\circ\log\Big\{\prod_{j=0}^{\infty}\big(1 + ze^{-\beta\lambda_j}(1 - ze^{-\beta\lambda_j})^{-1}\big)\Big\} \\ &= \exp\Big\{\sum_{j=0}^{\infty}\log\big(1 + ze^{-\beta\lambda_j}(1 - ze^{-\beta\lambda_j})^{-1}\big)\Big\} = (*). \end{aligned}$$

On the right hand side we apply the estimate $\log(1+x) \leq x$ whenever $x > 0$ and find

$$(*) \leq \exp\left\{\sum_{j=0}^{\infty} ze^{-\beta\lambda_j}(1 - ze^{-\beta\lambda_j})^{-1}\right\} = (**).$$

We only consider the case $\beta > 0$ in which we chose the eigenvalue sequence $\{\lambda_j\}_j$ to be increasing. Then we can estimate

$$\sup_{j \in \mathbb{N}_0}(1 - ze^{-\beta\lambda_j})^{-1} \leq (1 - ze^{-\beta\lambda_0})^{-1}$$

and therefore

$$(**) \leq \exp\left\{z(1 - ze^{-\beta\lambda_0})\sum_{j=0}^{\infty} e^{-\beta\lambda_j}\right\} = \exp\left\{z(1 - ze^{-\beta\lambda_0})\mathrm{trace}\left(e^{-\beta H}\right)\right\} < \infty.$$

(b): The trace of $A_f^* A_f$ can be estimated in a similar way

$$\begin{aligned} \mathrm{trace}\left(A_f^* A_f\right) &\leq \sum_{m=0}^{\infty} \sum_{(n_{j_1},\cdots,n_{j_m}) \in \mathbb{N}^m} \left\| a(f_1)\cdots a(f_n) e^{-\frac{\beta}{2}K_\mu} E_{n_1,\cdots,n_m}\right\|^2 \qquad (6.3.5) \\ &= \sum_{m=0}^{\infty} \sum_{(n_{j_1},\cdots,n_{j_m}) \in \mathbb{N}^m} z^N e^{-\beta(n_{j_1}\lambda_{j_1} + \cdots + n_{j_m}\lambda_{j_m})} \left\| a(f_1)\cdots a(f_n) E_{n_1,\cdots,n_m}\right\|^2. \end{aligned}$$

Now, we use the estimate

$$\|a(f_1)\cdots a(f_n)E_{n_1,\cdots,n_m}\| \leq N^{\frac{n}{2}}\|f_1\|\cdots\|f_n\| \cdot \underbrace{\|E_{n_1,\cdots,n_m}\|}_{=1},$$

which together with (6.3.5) gives

$$\begin{aligned} \mathrm{trace}\left(A_f^* A_f\right) &\leq \|f_1\|^2\cdots\|f_n\|^2 \sum_{m=0}^{\infty} \sum_{(n_{j_1},\cdots,n_{j_m}) \in \mathbb{N}^m} N^n z^N e^{-\beta(n_{j_1}\lambda_{j_1} + \cdots + n_{j_m}\lambda_{j_m})} \\ &= \|f_1\|^2\cdots\|f_n\|^2 \left(z\frac{d}{dz}\right)^n \sum_{m=0}^{\infty} \sum_{(n_{j_1},\cdots,n_{j_m}) \in \mathbb{N}^m} z^N e^{-\beta(n_{j_1}\lambda_{j_1} + \cdots + n_{j_m}\lambda_{j_m})} \\ &= \|f_1\|^2\cdots\|f_n\|^2 \left(z\frac{d}{dz}\right)^n \prod_{j=0}^{\infty}(1 - ze^{-\beta\lambda_j})^{-1} = (***). \qquad (6.3.6) \end{aligned}$$

We have seen in (a) that the infinite product on the right hand side converges under the condition $\beta(H - \mu I) > 0$ and it defines an analytic function in z. Therefore $(***)$ is finite which proves (b).

(c): Follows from the estimate (6.3.6) together with the Cauchy-Schwarz inequality:

$$\left|\omega\big(a^*(f)a(g)\big)\right|^2 \leq \left|\omega\big(a^*(f)a(f)\big)\right| \cdot \left|\omega\big(a^*(g)a(g)\big)\right| = \frac{\mathrm{trace}(A_f^* A_f)\mathrm{trace}(A_g^* A_g)}{\mathrm{trace}(e^{-\beta K_\mu})^2},$$

where $A_f = a(f)e^{-\frac{\beta K_\mu}{2}}$ and $A_g = a(g)e^{-\frac{\beta K_\mu}{2}}$. □

Under the condition of the previous lemma it follows that the two-point functions $\omega\big(a^*(f)a(g)\big)$ are well-defined. We calculate their value

$$\begin{aligned}\operatorname{trace}\left\{e^{-\beta K_\mu}a^*(f)a(g)\right\} &= \operatorname{trace}\left\{e^{-\frac{\beta}{2}K_\mu}a^*(f)e^{\frac{\beta}{2}K_\mu}\, e^{-\beta K_\mu}\, e^{\frac{\beta}{2}K_\mu}a(g)e^{-\frac{\beta}{2}K_\mu}\right\}\\ &= \operatorname{trace}\left\{a^*\big(e^{-\frac{\beta}{2}(H-\mu I)}f\big)e^{-\beta K_\mu}a\big(e^{-\frac{\beta}{2}(H-\mu I)}g\big)\right\}\\ &= \operatorname{trace}\left\{e^{-\beta K_\mu}a\big(e^{-\frac{\beta}{2}(H-\mu I)}g\big)a^*\big(e^{-\frac{\beta}{2}(H-\mu I)}f\big)\right\} = (*).\end{aligned}$$

Now, we use the CCR-relations to switch $a^*(\cdots)$ back to the left:

$$(*) = \operatorname{trace}\left\{e^{-\beta K_\mu}a^*\big(e^{-\frac{\beta}{2}(H-\mu I)}f\big)a\big(e^{-\frac{\beta}{2}(H-\mu I)}g\big)\right\} + \left\langle g, e^{-\beta(H-\mu I)}f\right\rangle \operatorname{trace}\left(e^{-\beta K_\mu}\right).$$

Dividing by $\operatorname{trace}(e^{-\beta K_\mu})$ gives

$$\omega\big(a^*(f)a(g)\big) = \omega\left(a^*\big(e^{-\frac{\beta}{2}(H-\mu I)}f\big)a\big(e^{-\frac{\beta}{2}(H-\mu I)}g\big)\right) + \left\langle g, e^{-\beta(H-\mu I)}f\right\rangle.$$

If we iterate this algorithm N times we obtain:

$$\omega\big(a^*(f)a(g)\big) = \omega\left(a^*\big(e^{-\frac{N\beta}{2}(H-\mu I)}f\big)a\big(e^{-\frac{N\beta}{2}(H-\mu I)}g\big)\right) + \sum_{m=1}^{N}\left\langle g, e^{-\beta m(H-\mu I)}f\right\rangle. \tag{6.3.7}$$

Under the assumptions of Proposition 6.3.5 we have $\beta(H-\mu I) > 0$ and therefore

$$\lim_{N\to\infty}\left\|e^{-\frac{N\beta}{2}(H-\mu I)}f\right\| = 0.$$

Taking the limit $N \to \infty$ on the right of (6.3.7) and using the estimate in Proposition 6.3.5, (c)

$$\left|\omega\big(a^*(f)a(g)\big)\right| \le C(z,\beta)\|f\|\cdot\|g\|$$

we obtain:

$$\lim_{N\to\infty}\omega\left(a^*\big(e^{-\frac{N\beta}{2}(H-\mu I)}f\big)a\big(e^{-\frac{N\beta}{2}(H-\mu I)}g\big)\right) = 0. \tag{6.3.8}$$

Hence we end up with the following two-point functions for the Bose gas.

Proposition 6.3.6. *Let $\mu, \beta \in \mathbb{R}$ and assume that $e^{-\beta H}$ is a trace class operator on $\mathfrak{h}$. If $\beta(H-\mu I) > 0$, then the two-point functions of the Gibbs-state ω are given by*

$$\omega\big(a^*(f)a(g)\big) = \left\langle g, ze^{-\beta H}(I - ze^{-\beta H})^{-1}f\right\rangle. \tag{6.3.9}$$

Moreover, on the Weyl operator ω acts as

$$\omega(W(f)) = \exp\left\{-\frac{1}{4}\left\langle f, (I + ze^{-\beta H})(I - ze^{-\beta H})^{-1}f\right\rangle\right\}.$$

Proof. We only show the first statement: note that

$$\sum_{m=1}^{\infty}e^{-\beta m(H-\mu I)} = \sum_{m=1}^{\infty}\left(ze^{-\beta H}\right)^m = \left(I - ze^{-\beta H}\right)^{-1} - I = ze^{-\beta H}(I - ze^{-\beta H})^{-1}.$$

Hence the assertion follows from (6.3.8) and (6.3.7). □

6.4 Equilibrium phenomena

Assume that the operator $ze^{-\beta H}(I - ze^{-\beta H})^{-1}$ is positive self-adjoint (not necessarily bounded or with discrete spectrum). Then the associated sesquilinear form on the right of (6.3.9) determines a quasi-free state.

Let ω be the gauge-invariant quasi-free state over $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ with $\mathfrak{h} := L^2(\mathbb{R}^n)$ and two point functions

$$\omega\big(a^*(f)a(g)\big) = \Big\langle g, ze^{-\beta H}(I - ze^{-\beta H})^{-1} f\Big\rangle_{\mathfrak{h}},$$

where H is the self-adjoint extension of $-\Delta$ on $L^2(\mathbb{R}^n)$. Put $\mathfrak{h}_\Lambda := L^2(\Lambda)$

$$\mathcal{A}_\Lambda := \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h}_\Lambda) \quad \text{and} \quad \mathcal{A} := \mathcal{A}_{\mathrm{CCR}}(\tilde{\mathfrak{h}}) \quad \text{where} \quad \tilde{\mathfrak{h}} := \bigcup_{\Lambda \subset \mathbb{R}^n} L^2(\Lambda).$$

If ω_Λ denotes the Gibbs state on $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h}_\Lambda)$ with respect to a self-adjoint extension H_Λ of the Laplacian $-\Delta$ on $L^2(\Lambda)$ ($\Lambda \subset \mathbb{R}^n$ bounded and open) and parameters β and μ, then we have the following result on the thermodynamical limit:

Proposition 6.4.1. *If there is $c > 0$ with $H_\Lambda - \mu I \geq cI$ for all Λ, then it follows*

$$\lim_{\tilde{\Lambda} \to \infty} \omega_{\tilde{\Lambda}}(A) = \omega(A), \qquad A \in \mathcal{A}_\Lambda.$$

Proof. Bratteli/Robinson II. □

Now we specify the discussion to an open square box Λ_L with edges of length $L > 0$

$$\Lambda_L := \Big(-\frac{L}{2}, \frac{L}{2}\Big) \times \cdots \times \Big(-\frac{L}{2}, \frac{L}{2}\Big) \subset \mathbb{R}^n$$

and we assume Dirichlet boundary conditions for the Laplacian $-\Delta$ on Λ_L. Consider the local density

$$\rho_{\Lambda_L}(\beta, z) := \frac{1}{|\Lambda_L|} \sum_{n \geq 0} \omega_{\Lambda_L}\big(a^*(f_n)a(f_n)\big) = (*),$$

where $\{f_n\}$ is an orthonormal basis of eigenfunctions $-\Delta$ in $\mathcal{D}(-\Delta)$. Assuming that $\beta(H - \mu I) > 0$ we find from the definition of the two point functions of ω_{Λ_L} in Proposition 6.3.6 that

$$\begin{aligned} (*) &= L^{-n} \sum_{n \geq 0} \Big\langle f_n, ze^{\beta\Delta}(I - ze^{\beta\Delta})^{-1} f_n\Big\rangle_{L^2(\Lambda_L)} \\ &= L^{-n} \sum_{\alpha \in \mathbb{N}^n} ze^{-\beta\gamma_\alpha(L)}(1 - ze^{-\beta\gamma_\alpha(L)})^{-1}. \end{aligned}$$

Note that the eigenvalues of $-\Delta$ on Λ_L are given by the numbers

$$E_\Delta(L) := \Big\{\gamma_\alpha(L) := \frac{\pi^2}{L^2}(\alpha_1^2 + \cdots + \alpha_n^2) \;:\; \alpha \in \mathbb{N}^n\Big\},$$

with corresponding eigenfunctions

$$F_\alpha^L(x_1, \cdots, x_n) := \prod_{j=1}^n \sin\left(\frac{\pi\alpha_j}{L}\left[x_j - \frac{L}{2}\right]\right).$$

Since $H_{\Lambda_L} \geq \gamma_{(1,1\cdots,1)}(L)I$ it follows that the condition $H_{\Lambda_L} - \mu I \geq cI$ for all $L > 0$ which appears in Proposition 6.4.1 can be fulfilled if

$$0 < c \leq \big(\gamma_{(1,1\cdots,1)}(L) - \mu\big) = \frac{n\pi^2}{L^2} - \mu, \qquad \text{for all} \quad L > 0$$

and therefore we need $\mu < 0$. Since $\beta > 0$ we have

$$0 < z = e^{\mu\beta} < 1.$$

In this region (*single phase region*) we have the thermodynamical limit in Proposition 6.4.1 and a unique thermodynamical phase of the infinitely extended Bose gas. However, note that $\rho_{\Lambda_L}(\beta, z)$ has a pole with respect to the activity z as z approaches

$$e^{\beta\gamma_{(1,1,\cdots,1)}(L)} = e^{\beta\frac{n\pi^2}{L^2}} \longrightarrow 1 \qquad \text{as} \quad L \to \infty.$$

If we choose *von Neumann boundary conditions*, then the Laplacian in Λ has a zero-eigenvalue and the same unboundedness of the local density happens for $z \to 1$ independently of the choice of box size L. This phenomenon is called *Bose-Einstein-condensation.*

Remark 6.4.2. We may also look at the local density with respect to the equilibrium state ω of the infinite extended Bose gas in Proposition 6.4.1. Let $\emptyset \neq \Lambda \subset \mathbb{R}^n$ be bounded and open and $\{f_n\}_{n\geq 0}$ and orthonormal basis of $L^2(\Lambda)$. Then

$$\begin{aligned}
\rho(z,\beta) &= \frac{1}{|\Lambda|}\sum_{n\geq 0} \omega\big(a^*(f_n)a(f_n)\big) \\
&= \frac{1}{|\Lambda|}\sum_{n\geq 0} \Big\langle \widehat{f}_n, ze^{-\beta p^2}\big(1 - z^{-\beta p^2}\big)^{-1}\widehat{f}_n\Big\rangle_{L^2(\mathbb{R}^n)} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} ze^{-\beta p^2}\big(1 - ze^{-\beta p^2}\big)^{-1} dp \\
&= \lambda^{-n}\pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} ze^{-x^2}\big(1 - ze^{-x^2}\big)^{-1} dx,
\end{aligned}$$

where $\lambda := \sqrt{4\pi\beta}$. Note that for all $x \in \mathbb{R}^n$ the map

$$[0,1] \ni z \mapsto ze^{-x^2}\big(1 - ze^{-x^2}\big)^{-1}$$

is monotonely increasing. Therefore, $z \mapsto \rho(\beta, z)$ is strictly increasing and we see that

$$\rho(z,\beta) \leq \lambda^{-n}\pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-x^2}\big(1 - e^{-x^2}\big)^{-1} dx.$$

Moreover, we have for the integral

$$\int_{\mathbb{R}^n} e^{-x^2}\big(1 - e^{-x^2}\big)^{-1} dx \begin{cases} = \infty, & \text{if } n = 1, 2 \\ < \infty, & \text{if } n \geq 3. \end{cases}$$

Thus, we see that $\rho(z, \beta)$ remains bounded for $z \in [0, 1]$ in dimensions $n \geq 3$. This does not reflect the unboundedness effect that arises for a finite box as was discussed above. However, for all $0 < z \leq 1$ one has

$$\lim_{L\to\infty} \rho_{\Lambda_L}(\beta, z) = \rho(\beta, z).$$

Next: *Analyse the "Bose-Einstein-condensation" appearing when $z = 1$. We look at the thermodynamical limit as $L \to \infty$ for fixed densities $\rho_{\Lambda_L}(\beta, z)$.*

Let $n \geq 3$ and fix $\beta, \tilde{\rho} > 0$. Since $\rho_{\Lambda_L}(\beta, \cdot)$ is monotonely increasing to $+\infty$ as $z \uparrow e^{\beta \frac{n\pi^2}{L^2}}$ we can uniquely solve

$$\rho_{\Lambda_L}(\beta, z_L) = \tilde{\rho} \quad \text{where} \quad 0 < z_L < e^{\beta \frac{n\pi^2}{L^2}}. \tag{6.4.1}$$

One always has $\rho_{\Lambda_L}(\beta, z) \leq \rho(\beta, z)$ whenever $0 < z \leq 1$ and $L > 0$. Moreover, both functions are monotonely increasing in z. Two cases are possible

I. Assume that $0 < \tilde{\rho} \leq \rho(\beta, 1)$. Then we can also uniquely solve the equation $\rho(\beta, \tilde{z}) = \tilde{\rho}$ where $\tilde{z} \in (0, 1]$ and from

$$\rho_{\Lambda_L}(\beta, \tilde{z}) \leq \rho(\beta, \tilde{z}) = \tilde{\rho} = \rho_{\Lambda_L}(\beta, z_L)$$

we find that $0 < \tilde{z} \leq z_L$. It can be shown that

$$\lim_{L\to\infty} z_L = \tilde{z}. \tag{6.4.2}$$

II. Assume that $\rho(\beta, 1) < \tilde{\rho}$. We have $z_L > 1$ since otherwise we would arrive at the contradiction

$$\rho(\beta, 1) < \tilde{\rho} = \rho_{\Lambda_L}(\beta, z_L) \leq \rho(\beta, z_L) \leq \rho(\beta, 1).$$

In this case it can be shown that $\lim_{L\to\infty} z_L = 1$ and

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} z_L e^{-\beta\gamma_{(1,\cdots,1)}(L)} \big(1 - z_L e^{-\beta\gamma_{(1,\cdots,1)}(L)}\big)^{-1} = \tilde{\rho} - \rho(\beta, 1) > 0. \tag{6.4.3}$$

Recall that

$$\gamma_{(1,\cdots,1)}(L) = \frac{n\pi^2}{L^2}$$

is the smallest eigenvalue of the Laplacian $-\Delta$ on Λ_L with respect to *Dirichlet boundary conditions* and $|\Lambda_L| = L^n$ is the volume of the box.

Moreover, if $\alpha \in \mathbb{N}^n$ with $\alpha \neq (1, \cdots, 1)$, then we have

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} z_L e^{-\beta\gamma_\alpha(L)} \big(1 - z_L e^{-\beta\gamma_\alpha(L)}\big)^{-1} = 0. \tag{6.4.4}$$

Now we state the main result:

Theorem 6.4.3. *Let $n \geq 3$ and fix $\tilde{\rho}, \beta > 0$. With $L > 0$ consider the "square boxes" Λ_L having side-length L as above. Moreover, put*

(a) *H_{Λ_L}:=self-adjoint extension of $-\Delta$ on Λ_L w.r.t. Dirichlet boundary conditions and H the selfadjoint extension of $-\Delta$ on $\mathbb{R}^n$.*

(b) *ω_{Λ_L} the Gibbs state on $\mathcal{A}_{\mathrm{CCR}}(L^2(\Lambda_L))$ with respect to β and the activity z_L which is chosen as the unique solution of*

$$\rho_{\Lambda_L}(\beta, z_L) = \tilde{\rho}, \quad \text{where} \quad \tilde{\rho} > 0.$$

Here, $\rho_{\Lambda_L}(\beta, z)$ means the local density with respect to ω_{Λ_L}.

(c) *$\rho(\beta, z)$, the local density of the infinite extended Bose gas, i.e.*

$$\rho(\beta, z) = \lim_{L\to\infty} \rho_{\Lambda_L}(\beta, z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} z e^{-\beta p^2} \left(1 - z e^{-\beta p^2}\right)^{-1} dp, \qquad 0 < z \leq 1.$$

Then the following limit exists

$$\omega_{\tilde{\rho}}(A) = \lim_{L\to\infty} \omega_{\Lambda_L}(A), \quad \text{where} \quad A \in \overline{\bigcup_{\Lambda} \mathcal{A}_{\mathrm{CCR}}(L^2(\Lambda))}.$$

Moreover, $\omega_{\tilde{\rho}}$ acts as follows on generators of the CCR-algebra

(A) *If $\tilde{\rho} \leq \rho(\beta, 1)$ and $\tilde{z} = \lim_{L\to\infty} z_L$ is the unique solution to $\tilde{\rho} = \rho(\beta, \tilde{z})$, then*

$$\omega_{\tilde{\rho}}\big(W(f)\big) = \exp\left\{-\frac{1}{4}\left\langle f, (I + \tilde{z}e^{-\beta H})(I - \tilde{z}e^{-\beta H})^{-1} f\right\rangle_{L^2(\mathbb{R}^n)}\right\}.$$

(B) *If $\tilde{\rho} > \rho(\beta, 1)$, then $\lim_{L\to\infty} z_L = 1$ and*

$$\omega_{\tilde{\rho}}\big(W(f)\big) = \exp\Big\{ - 2^{n-1}\big(\tilde{\rho} - \rho(\beta, 1)\big)\Big| \int_{\mathbb{R}^n} f(x)dx\Big|^2 - \\ - \frac{1}{4}\left\langle f, (I + e^{-\beta H})(I - e^{-\beta H})^{-1} f\right\rangle_{L^2(\mathbb{R}^n)}\Big\}.$$

Proof. We only comment on (B): Let $f \in L^2(\Lambda_L)$ and recall from Proposition 6.3.6 that

$$\omega_{\Lambda_L}(W(f)) = \exp\Big\{ - \frac{1}{4} \underbrace{\left\langle f, (I + z_L e^{-\beta H_{\Lambda_L}})(I - z e^{-\beta H_{\Lambda_L}})^{-1} f\right\rangle}_{=:I_L(f)} \Big\}.$$

With the eigenvalues $\gamma_\alpha(L)$, $\alpha \in \mathbb{N}^n$ of H_Λ and the orthogonal projections $P_{k(\alpha)}(L)$ where $k(\alpha) = \alpha_1^2 + \cdots + \alpha_n^2$ onto the corresponding eigenspace we can write

$$I_L(f) = \sum_{\alpha\in\mathbb{N}^n} \frac{1 + z_L e^{-\beta\gamma_\alpha(L)}}{1 - z_L e^{-\beta\gamma_\alpha(L)}} \left\langle f, P_{k(\alpha)}(L) f\right\rangle.$$

Recall that the family of normalized eigenfunction of H_{Λ_L} with respect to Dirichlet boundary conditions and corresponding eigenvalues $\gamma_\alpha(L)$ was given by

$$\Psi_\alpha^L(x_1, \cdots, x_n) = \frac{F_\alpha^L(x_1, \cdots, x_n)}{\|F_\alpha^L\|} = \|F_\alpha^L\|^{-1} \prod_{j=1}^n \sin\left(\frac{\pi\alpha_j}{L}\left[x_j - \frac{L}{2}\right]\right), \qquad \text{where} \qquad \alpha \in \mathbb{N}^n.$$

Note that $\|F_\alpha^L\|^{-1} = \sqrt{\frac{2^n}{L^n}}$ is independent of α. In particular, if $\alpha = (1, \cdots, 1)$, then the above expression simplifies to

$$\Psi_{(1,\cdots,1)}^L(x_1, \cdots, x_n) = \frac{(-1)^n 2^{\frac{n}{2}}}{L^{\frac{n}{2}}} \prod_{j=1}^n \cos\left(\frac{\pi x_j}{L}\right).$$

Then we have

$$\begin{aligned}\big\langle f, P_{k(1,\cdots,1)}(L)f\big\rangle &= \|P_{k(1,\cdots,1)}(L)f\|^2 \\ &= \big|\big\langle f, \Psi^L_{(1,\cdots,1)}\big\rangle\big|^2 = \frac{2^n}{L^n}\left|\int_{\Lambda_L} f(x)\prod_{j=1}^n \cos\Big(\frac{\pi x_j}{L}\Big)\,dx\right|^2\end{aligned}$$

and therefore

$$\lim_{L\to\infty} L^n\big\langle f, P_{k(1,\cdots,1)}(L)f\big\rangle = 2^n\Big|\int_{\mathbb{R}^n} f(x)dx\Big|^2. \tag{6.4.5}$$

Moreover, we find from (6.4.3) that

$$\lim_{L\to\infty} L^{-n}\,\frac{1+z_Le^{-\beta\gamma_{(1,\cdots,1)}(L)}}{1-z_Le^{-\beta\gamma_{(1,\cdots,1)}(L)}} = 2\big(\tilde\rho-\rho(\beta,1)\big). \tag{6.4.6}$$

Combining (6.4.5) and (6.4.6) gives

$$\lim_{L\to\infty}\frac{1+z_Le^{-\beta\gamma_{(1,\cdots,1)}(L)}}{1-z_Le^{-\beta\gamma_{(1,\cdots,1)}(L)}}\,\big\langle f, P_{k(1,\cdots,1)}(L)f\big\rangle = 2^{n+1}\Big|\int_{\mathbb{R}^n} f(x)dx\Big|^2\big(\tilde\rho-\rho(\beta,1)\big).$$

The higher energy states give no contribution to the density. Indeed, if we choose $\alpha\in\mathbb{N}^n$ with $\alpha\neq(1,\cdots,1)$, then

$$\begin{aligned}\big|\big\langle f, P_{k(\alpha)}f\big\rangle\big|^2 &= \sum_{k(\beta)=k(\alpha)}\big|\big\langle f,\Psi^L_\beta\big\rangle\big|^2 \\ &= \frac{1}{\|F^L_\alpha\|^2}\sum_{k(\beta)=k(\alpha)}\Big|\int_{\Lambda_L} f(x)F^L_\beta(x)dx\Big|^2 \\ &\le \frac{2^n}{L^n}\sum_{k(\beta)=k(\alpha)}\Big\{\int_{\mathbb{R}^n}|f(x)|dx\Big\}^2.\end{aligned}$$

Therefore, we conclude that there is a constant $C_\alpha>0$ independent of L such that

$$L^n\big|\big\langle f, P_{k(\alpha)}f\big\rangle\big|^2 \le C_\alpha\Big\{\int_{\mathbb{R}^n}|f(x)|dx\Big\}^2. \tag{6.4.7}$$

From (6.4.4) recall that

$$\lim_{L\to\infty} L^{-n}z_Le^{-\beta\gamma_\alpha(L)}\big(1-z_Le^{-\beta\gamma_\alpha(L)}\big)^{-1} = 0. \tag{6.4.8}$$

By combining (6.4.7) and (6.4.8) one finds for all $m\in\mathbb{N}$ with $m\geq n$ that

$$\lim_{L\to\infty}\sum_{\substack{\alpha\neq(1,\cdots,1)\\ k(\alpha)\le m}}\frac{1+z_Le^{-\beta\gamma_\alpha(L)}}{1-z_Le^{-\beta\gamma_\alpha(L)}}\big\langle f, P_{k(\alpha)}(L)f\big\rangle = 0$$

and therefore

$$\lim_{L\to\infty}\Big\{I_L(f)-\underbrace{\sum_{k(\alpha)>m}\frac{1+z_Le^{-\beta\gamma_\alpha(L)}}{1-z_Le^{-\beta\gamma_\alpha(L)}}\big\langle f, P_{k(\alpha)}(L)f\big\rangle}_{=:I^m_L(f)}\Big\} = 2^{n+1}\Big|\int_{\mathbb{R}^n} f(x)dx\Big|^2\big(\tilde\rho-\rho(\beta,1)\big). \tag{6.4.9}$$

Finally, one shows that

$$\lim_{m\to\infty}\lim_{L\to\infty}\Big\{I_L^m(f)-\big\langle f,(I+e^{-\beta H})(I-e^{-\beta H})^{-1}f\big\rangle\Big\}=0.$$

Let $\varepsilon>0$ and choose $m>0$ such that

$$\Big|\lim_{L\to\infty}I_L(f)-\lim_{L\to\infty}\Big\{I_L(f)-I_L^m(f)\Big\}-\big\langle f,(I+e^{-\beta H})(I-e^{-\beta H})^{-1}f\big\rangle\Big|=$$

$$=\Big|\lim_{L\to\infty}\Big\{I_L^m(f)-\big\langle f,(I+e^{-\beta H})(I-e^{-\beta H})^{-1}f\big\rangle\Big\}\Big|<\varepsilon.$$

Since $\varepsilon>0$ was chosen arbitrarily and the left hand side does not depend on m we find from (6.4.9) that

$$\lim_{L\to\infty}I_L(f)=2^{n+1}\Big|\int_{\mathbb{R}^n}f(x)dx\Big|^2\big(\tilde\rho-\rho(\beta,1)\big)+\big\langle f,(I+e^{-\beta H})(I-e^{-\beta H})^{-1}f\big\rangle,$$

which finishes the proof of (B). □

Remark 6.4.4. We give some comments on the phenomenon of Bose-Einstein-condensation.

(a) In the high density region we have $z=1$ and *Bose-Einstein condensation* takes place, i.e. a finite proportion of particles are in the lowest energy state. This effect corresponds to a *phase transition* of the system of non-interacting Bosons.

(b) In the region $z=1$ there is a family of equilibrium states at the same temperature and parametrized by their particle densities $\tilde\rho\in[\rho(\beta,1),\infty)$.

(c) The equilibrium states corresponding to $z=1$ have less ergodic properties than the states in the single phase region.

(d) Consider the equilibrium state $\omega_{\tilde\rho}$ corresponding to $\tilde\rho\in[\rho(\beta,1),\infty)$. The calculation in the proof of Theorem 6.4.3 shows that the two-point-functions of $\omega_{\tilde\rho}$ are given by

$$\omega_{\tilde\rho}\big(a^*(f)a(g)\big)=2^n\big[\tilde\rho-\rho(\beta,1)\big]\int_{\mathbb{R}^n}\overline{g(x)}dx\int_{\mathbb{R}^n}f(x)dx+$$

$$+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}(p)\overline{\widehat{g}(p)}e^{-\beta p^2}\big(1-e^{-\beta p^2}\big)^{-1}dp.$$

The local densities take the form

$$\tilde\rho(\beta,1)=|\Lambda_L|^{-1}\sum_{\{f_n\}}\omega_{\tilde\rho}\big(a^*(f_n)a(f_n)\big)=2^n\big[\tilde\rho-\rho(\beta,1)\big]+\rho(\beta,1).$$

Recall that the factor "2^n" on the right appeared in the proof of Theorem 6.4.3 when we took the limit

$$\lim_{L\to\infty}L^n\big|\big\langle f,\Psi^L_{(1,\cdots,1)}\big\rangle\big|^2=\lim_{L\to\infty}L^n\big|\Psi^L_{(1,\cdots,1)}(0)\big|^2\Big|\int_{\mathbb{R}^n}f(x)dx\Big|^2.$$

More precisely, in the case of Dirichlet boundary conditions and with the lowest energy eigenfunction $\Psi^L_{(1,\cdots,1)}$ of the Dirichlet Laplacian H_{Λ_L} we had

$$2^n=\lim_{L\to\infty}L^n\big|\Psi^L_{(1,\cdots,1)}(0)\big|^2.$$

Note that this value, which is interpreted as the relative proportion of the condensate at the origin, is sensitive under the particular choice of boundary conditions.

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