~~Lin~~ $A \in M(2\times 2, \mathbb{C})$

linear $\Rightarrow$ $\omega(A) = \cancel{\mathrm{tr}\, \rho A}$ $\rho_{11} a_{11} + \rho_{21} a_{12} + \rho_{12} a_{21} + \rho_{22} a_{22}$

$= \mathrm{tr}\, \rho A$ for some $\rho \in M(2\times 2, \mathbb{C})$.

$A \geq 0 \Rightarrow$ diagonalize $A = \begin{pmatrix} a_{11} & \\ & a_{22} \end{pmatrix}$ $a_{11} \geq 0$ $a_{22} \geq 0$

$\Rightarrow$ $\rho_{11} \geq 0$ $\rho_{22} \geq 0$.

$\langle v, \rho v \rangle = (\bar{v}_1, \bar{v}_2) \begin{pmatrix} \rho_{11} v_1 + \rho_{12} v_2 \\ \rho_{21} v_1 + \rho_{22} v_2 \end{pmatrix}$

$= \rho_{11} |v_1|^2 + \rho_{22} |v_2|^2 + (\cancel{\rho_{12} + \rho_{21}}) (\bar{v}_1 v_2)$

$\rho_{12} \bar{v}_1 v_2 + \rho_{21} \bar{v}_2 v_1$

$A = B B^*$

$\mathrm{tr}\, B^+ \rho B$ $B = (\vec{b}_1\ \vec{b}_2)$

$\rho B = (\rho \vec{b}_1, \rho \vec{b}_2)$

$\mathrm{tr}\, B^+ \rho B = \mathrm{tr} \begin{pmatrix} \langle \vec{b}_1, \rho \vec{b}_1 \rangle & \\ 0 & \langle \vec{b}_2, \rho \vec{b}_2 \rangle \end{pmatrix}$

$= \langle \vec{b}_1, \rho \vec{b}_1 \rangle + \langle \vec{b}_2, \rho \vec{b}_2 \rangle$

$\Rightarrow \rho \geq 0$.

1

$\Rightarrow$ diagonalize $\rho = \begin{pmatrix} \rho_1 & \\ & \rho_2 \end{pmatrix}$

$1 = tr(\rho \mathbb{1}) = \rho_1 + \rho_2 = tr\,\rho.$

$\Rightarrow$ density matrix.

GNS construction: Two cases:

I) pure state: OBdA $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathcal{J} = \{A \in M_{2\times 2} \mid tr(\rho A^* A) = 0\}$

$0 = tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(A^* A \right)\right) = |a_{11}|^2 + |a_{12}|^2$

$\mathcal{J} = \left\{\begin{pmatrix} 0 & a_{21} \\ 0 & a_{22} \end{pmatrix}\right\}$

$\mathcal{H} = \mathfrak{A} / \mathcal{J} = \left\{\left[\begin{pmatrix} b_{11} & 0 \\ b_{12} & 0 \end{pmatrix}\right]\right\}$

$\pi\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \left[\begin{pmatrix} b_{11} & 0 \\ b_{12} & 0 \end{pmatrix}\right] =$

$\left[\begin{pmatrix} a_{11} b_{11} + a_{21} b_{12} & 0 \\ a_{12} b_{11} + a_{22} b_{12} & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} a\vec{b} & 0 \end{pmatrix}\right].$

$\cong \mathbb{C}^2$ representing $M(2,2)$.

~~III) mixed state~~

Why is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ pure?

other $\lambda = \begin{pmatrix} 1 & a \\ \bar{a} & 0 \end{pmatrix}$ $\quad \det \lambda = 1 + |a|^2$

$\Rightarrow a = 0.$

II mixed state:

$$g = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \qquad 0 < s_1, s_2 < 1$$

$$J = \left\{ A \in M_2(\mathbb{C}) \,\middle|\, 0 = \mathrm{tr}(g\, A^* A) \right\}$$

$$\Rightarrow \quad 0 = s_1 \left(|a_{11}|^2 + |a_{12}|^2 \right) + s_2 \left(|a_{21}|^2 + |a_{22}|^2 \right)$$

$$\Rightarrow J = \{0\}$$

i.e. repr. on $\left[\left(\vec{b}_1 \; \vec{b}_2 \right) \right] = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}$

$$\pi(A) \left[\left(\vec{b}_1 \; \vec{b}_2 \right) \right] = \left[\left(A\vec{b}_1 \quad A\vec{b}_2 \right) \right] = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array} \right) \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}$$

$\mathbb{C}^2 \oplus \mathbb{C}^2$ not irreducible

$$\left\langle \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}, \begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} \right\rangle = \omega\left(\left(\vec{b}_1, \vec{b}_2 \right)^+ \left(\vec{c}_1 \; \vec{c}_2 \right) \right) = s_1 \langle \vec{b}_1, \vec{c}_1 \rangle + s_2 \langle \vec{b}_2, \vec{c}_2 \rangle.$$

two $\mathcal{H}$'s no mixture.

Fock space, Canonical (anti)-commutation relations.

Two approaches to C.C.R.: HS or C*.

Recall "Harmonic oscillator": (1x dynamics)

$$(x, p) \longrightarrow a, a^+ = \frac{1}{\sqrt{2}}(x \pm ip)$$

ON-Basis of $\mathcal{H}$: $\{|n\rangle \mid n \in \mathbb{N}_0\}$

$$a^+ |n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$a\ |n\rangle = \sqrt{n}\ |n-1\rangle$$

makes $a^+ = a^*$.

"adding photons ~~kaza~~ (bosonic) excitations into system"

Simplest example of Fock space, excitation can only have one state, i.e. 1 particle $HS = \mathbb{C}$.

In general: "Particles" can have many different states in 1-particle Hilbert space h.

From this we build n-particle HS (distinguishable particles)

$$\mathfrak{h}^n = \underbrace{\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}}_{n\text{-factors}}$$

(recall: $\mathcal{G} \otimes \mathcal{H}$ has elements of the form

$$g \otimes h \quad \text{with} \quad g \in G, \; h \in \mathcal{H}$$

and (ℓ^2)-linear combinations thereof with

$$(\lambda g) \otimes h = g \otimes (\lambda h) \qquad \text{for } \lambda \in \mathbb{C}$$

and

$$\langle g_1 \otimes h_1 | g_2 \otimes h_2 \rangle = \langle g_1 | g_2 \rangle \langle h_1 | h_2 \rangle \quad)$$

Def <u>Fock space</u> $\mathcal{F}(\mathfrak{h}) = \overline{\bigoplus_{n \geq 0} \mathfrak{h}^n}$ w/ $\mathfrak{h}^0 \equiv \mathbb{C}$

Symmetrization (Bosons)

$$P_+ (f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in \sigma_n} f_{\pi_1} \otimes \cdots \otimes f_{\pi_n}$$

Antisymmetrization

$$P_- (f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in \sigma_n} (-1)^{\operatorname{sgn} \pi} f_{\pi_1} \otimes \cdots \otimes f_{\pi_n}$$

$$\mathcal{F}_\pm(\mathfrak{h}) := P_\pm (\mathcal{F}(\mathfrak{h}))$$

Number operator

$$N\psi = \{ n\psi^{(n)} \}_{n \geq 0}$$

$$D(N) = \{ \psi = (\psi^{(n)})_{n \geq 0} \in \mathcal{F}_{(\pm)}(h) \mid \sum_n n^2 \|\psi^{(n)}\|^2 < \infty \}$$

is self-adjoint.

e^{itN} leaves $\mathcal{F}_{\pm}$ invariant.

Second quantization

H s.a. on h. Define H_n on $h_{\pm}^{n}$ by

$$H_n(P_{\pm}(f_1 \otimes \cdots \otimes f_n)) := P_{\pm}\left(\sum_{i=1}^{n} f_1 \otimes f_2 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n\right)$$

$\sum_n H_n$ is essentially s.a. on

$$d\Gamma(H) = \bigoplus_{n \geq 0} H_n$$

U unitary on h.

$$U_n(P_{\pm}(f_1 \otimes \cdots \otimes f_n)) = P_{\pm}(Uf_1 \otimes \cdots \otimes Uf_n)$$

$$\Gamma(U) = \bigoplus_{n \geq 0} U_n$$

NB if $U = e^{itH}$ $\quad \Gamma(U) = e^{it\, d\Gamma(H)}$.

annihilation op: $f \in h$

$$a(f)\,\psi^{(0)} = 0$$

$$a(f)\,(f_1 \otimes \cdots \otimes f_n) = \sqrt{n}\,\langle f \mid f_1 \rangle\; f_2 \otimes \cdots \otimes f_n$$

anti-lin in f!

creation op: $f \in h$

$$a^*(f)\,\psi^{(0)} = f.$$

$$a^*(f)\,(f_1 \otimes \cdots \otimes f_n) := \sqrt{n+1}\;\; f \otimes f_1 \otimes \cdots \otimes f_n$$

linear in f.

NB: Physics literature: Use $e^{i\vec{k}\vec{x}}$ "$\in$" $L^2(\mathbb{R}^n)$

and write $a^{(*)}(\vec{k})$ for $a^{(*)}(e^{i\vec{k}\vec{x}})$

check $\|a(f)\,\psi^{(n)}\| \le \sqrt{n}\,\|f\|\,\|\psi^{(n)}\|$

$\|a^*(f)\,\psi^{(n)}\| \le \sqrt{n+1}\;\|f\|\,\|\psi^{(n)}\|$

$\Rightarrow$ a, a^* can be extended to $D(\sqrt{N})$

$$\|a^{\#}(f)\,\psi\| \le \|f\|\;\|\sqrt{N+1}\,\psi\|$$

and of course $\langle a^*(f)\,\psi \mid \varphi\rangle - \langle \psi \mid a(f)\,\varphi\rangle$

and bosonic / fermionic w...

$$a_\pm^\#(f) = P_\pm\, a^\#(f)\, P_\pm$$

<u>Prop</u>

$$[a_+(f), a_+(g)] = 0 = [a_+^*(f), a_+^*(g)]$$

CCR

$$[a_+(f), a_+^*(g)] = \langle f, g\rangle \,\mathrm{id}$$

$$\{a_-(f), a_-(g)\} = 0 = \{a_-^*(f), a_-^*(g)\}$$

CAR.

$$\{a_-(f), a_-^*(g)\} = \langle f, g\rangle \,\mathrm{id}.$$

<u>Remark:</u> Look very similar but have very different properties, partly due to

$$\|a_-^*(f)\| = \|f\| < \infty \qquad \text{while} \qquad \|a_+^\#(f)\| = \infty.$$

<u>Pf (CAR)</u>

$$(a^*(f)a(f))^2 = a^*(f)\,a(f)\,a^*(f)\,a(f)$$

$$\overset{a(f)a(f)=0}{=} a^*(f)\,\{a(f), a^*(f)\}\,a(f)$$

$$= \|f\|^2\, a^*(f)\,a(f)$$

$$\|a(f)\|^4 \underset{C^*}{=} \|(a^*(f)a(f))^2\| = \|f\|^2\,\|a^*(f)a(f)\| = \|f\|^2\,\|a(f)\|^2$$

ON-Basis: Define $\Omega = (1, 0, 0, \dots)$ and let $\{f_\alpha\}$ be ON-Basis of h.

Then

$$a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_n})\Omega$$

for subsets $\{f_{\alpha_1}, \dots, f_{\alpha_n}\}$ of $\{f_\alpha\}$.

form ON-Basis of $\mathcal{F}_-(h)$!

CCR-relations

Problem: Let $\psi^{(n)} = f \otimes \cdots \otimes f$. Then

$$\|a_+(f)\, \psi^{(n)}\| = \sqrt{n}\ \|\psi^{(n)}\|\ \|f\|$$

$\Rightarrow \|a_+(f)\| = \infty.$

This can cause a lot of pain. Avoid by instead considering bounded functions of $a^\#_+(f)$.

Field ~~relationship~~

$$\Phi(f) := \frac{1}{\sqrt{2}}\left(a(f) + a^*(f)\right)$$

" Momentum

$$\Pi(f) := \Phi(if) = -i\frac{1}{\sqrt{2}}\left(a(f) - a^*(f)\right)$$

NB: Recover $a^\#(f)$ from $\Phi(f)$!

Proposition. Let $\mathcal{F}(\mathcal{H}) = \{ \psi \in \mathcal{F}_+(\mathcal{H}) \mid$ all most all $\psi^{(n)} = 0 \}$

finite particle states

(i) $\Phi(f)$ is essentially s.a. on $\mathcal{F}(\mathcal{H})$

$$\|f_\alpha \to f\| \to 0 \Rightarrow \| \Phi(f_\alpha)\psi - \Phi(f)\psi\| \to 0$$
$$\psi \in D(\sqrt{N})$$

(ii) $\Omega := (1, 0, 0, \dots)$

$$\{ \Phi(f_1) \dots \Phi(f_n)\Omega \ f_k \mid n = 0, 1, \dots \}$$

is dense in $\mathcal{F}_+(\mathcal{H})$.

(iii) For $\psi \in D(N)$, $f, g \in \mathcal{H}$:

$$\left(\Phi(f)\Phi(g) - \Phi(g)\Phi(f) \right)\psi = i \operatorname{Im}(\langle f, g \rangle)\psi$$

pf (iii) by direct computation.

$\Phi(f)$ can be closed to a s.a. operator.

And we will now consider the corresponding 1-parameter grp

$$W(f) := e^{i\Phi(f)} \qquad \text{unitary.} \Rightarrow \|W(f)\| \le 1$$

"Weyl-operator"

"recover" $\Phi(f) = -i \frac{d}{dt} W(tf)\big|_{t=0}$ if it exists.

Proposition

(1) $f, g \in \mathfrak{h}$. We have $W(f)\, D(\overline{\Phi(g)}) = D(\Phi(g))$ and

$$W(f)\, \Phi(g)\, W(f)^* = \Phi(g) - \Im(\langle f, g\rangle)\, id$$

(2) $f, g \in \mathfrak{h}$

$$W(f)\, W(g) = e^{-i \Im(\langle f, g\rangle)}\, W(f+g)$$

(3) $\|f_\alpha - f\| \to 0 \Rightarrow \forall \psi \in \mathfrak{F}_+(\mathfrak{h})$

$$\|(W(f_\alpha) - W(f))\psi\| \to 0.$$

(4) $f \in \mathfrak{h} \setminus \{0\}$

$$\| W(f) - 1\!\!1 \| = 2. \qquad \text{(i.e. discont in } f!)$$

Pf (1) formally (Baker: power series)

$$[\phi, W] = i\, [a, \phi]\, W$$

(2) $\frac{d}{dt} \left(W(tf)\, W(tg)\, W(t(f+g))^* \right) \psi =$

$$W(tf)\, [\, i\,\Phi(f)\,, W(tg)\,]\, W(t(f+g))^* \psi =$$

$$i\, t\, \Im \langle f, g\rangle\, W(tf)\, W(tg)\, W(t(f+g))^* \quad \text{\& Integrate}$$

(c) (c) implies ~~that~~ the spectrum is the whole

$$W(itf)\,\Phi(f)\,W(itf)^* = \Phi(f) - t\|f\|^2\,\mathrm{id}$$

~~so any ... can be~~ so unitary transf. shifts spectrum by $t\|f\|^2$. i.e. $\sigma(\Phi(f)) = \mathbb{R}$.

$$\Rightarrow \quad \sigma(W(f)) = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

$$\Rightarrow \quad \sigma(W(f) - \mathrm{id}) = \{ z \in \mathbb{C} \mid |z+1| = 1 \}$$

$$\Rightarrow \quad \|W(f) - \mathrm{id}\| = 2$$

Examples of C^*-algebras

CAR:

Thm: Let h be pre-HS w/ closure $\bar{h}$. Then, up to isomorphism, there is a unique C^*-alg generated by id and $a(f)$, $f \in h$ satisfying

(1) $f \mapsto a(f)$ anti-linear

(2) $\{a(f), a(g)\} = 0$

(3) $\{a(f), a^*(g)\} = (f, g)\,\mathrm{id}$

We then have

(i) $\|a(f)\| = \|f\| \quad \forall f \in h$

(ii) If $\dim h = n < \infty$, $\mathfrak{A}(h) \cong M(2^n \times 2^n, \mathbb{C})$

(iii) $\mathfrak{A}(h)$ is separable iff h is.

(iv) ~~tde~~ For $U : h \to h$ bounded linear an
$V : h \to h$ bounded anti-linear w/

$$V^* U + U^* V = 0 = U^* V^* + V U^*$$

$$U^* U + V^* V = \mathrm{id} = U U^* + V V^*$$

there exists a unique $*$-automorphism of $\mathfrak{A}(h)$

$$\gamma(a(f)) = a(Uf) + a^*(Vf)$$

further

$$\gamma^{-1}(a(f)) = a(U^* f) + a^*(V^* f)$$

"Bogoliubov transformations

Alternatively, as for CCR define field op's

$$B(f) = \frac{1}{\sqrt{2}}\left(a(f) + a^*(f) \right)$$

(invert as $\quad a(f) = \frac{1}{\sqrt{2}}\left(B(f) + i B(if) \right)$

Then $\quad \{ B(f), B(g) \} = \mathrm{Re}(\langle f, g \rangle)$

This suggests a real approach: Starting from real HS H with s: real positive symmetric bilinear.

If J is an operator with

$$s(Jf, g) = -s(f, Jg) \quad \text{and} \quad J^2 = -id$$

(complex structure)

one can define

$$a_J(f) := \frac{1}{\sqrt{2}} \left(B(f) + i\, B(Jf) \right)$$

$$a_J^*(f) := \frac{1}{\sqrt{2}} \left(B(f) - i\, B(Jf) \right)$$

such that

$$\left\{ a_J(f), a_J^*(g) \right\} = s(f, g) + i\, s(f, Jg)$$

But this is not more general since real HS + cplx str is complex HS.

Finally if T is Bogoliubov, i.e. invertible ...

$$s(Tf, Tg) = s(f, g)$$

there is $*$-aut $\gamma(B(f)) := B(Tf)$

which is Bogoliubov w

$$U = \frac{1}{2}(T - JTJ) \qquad V = \frac{1}{2}(T + JTJ).$$

CCR

Use Weyl operators.

Start w/ <u>real</u> HS $\mathcal{H}$ w/ non-deg sympl fn

$$\sigma : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

w/ $\sigma(f, g) = -\sigma(g, f)$

and $\forall g \in \mathcal{H} : \sigma(f, g) = 0 \Rightarrow f = 0.$

eg. Complex HS with $\sigma(f, g) = \operatorname{Im}\langle f, g\rangle$

Difference is $\exists$ operator J w/

$$\sigma(Jf, g) = -\sigma(f, Jg) = \sigma(Jg, f)$$

and $J^2 = -id$ 'complex structure'

$$\langle f, g\rangle = \sigma(f, Jg) + i\sigma(f, g)$$

Thm $\mathcal{H}, \sigma$ as above. Then there is a unique C^*-alg $\mathfrak{A}(\mathcal{H})$ generated by $W(f)$ satisfying

(1) $W(-f) = W(f)^*$

(2) $W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)} W(f+g) \quad \forall f, g \in \mathcal{H}.$

We have

(i) $W(0) = id$, $W(f)$ is unitary for $f \neq 0$ and $\|W(f) - id\| = 2 \quad \forall f \in \mathcal{H}, f \neq 0.$

(ii) $\mathfrak{A}(\mathcal{H})$ is non-separable for $\mathcal{H} \neq 0$

(iii) T real invertible op w/ $\sigma(Tf, Tg) = \sigma(f,g)$ "symplectomorphism" induces $*$-automorphism γ

$$\gamma(W(f)) = W(Tf)$$

Pf ad ii) follows from (i).

Def Let I be directed ortho set w/ $\perp$. A <u>quasi-local alg</u> is a C^*-alg $\mathfrak{A}$ with σ as above and net $\{\mathfrak{A}_\alpha\}_{\alpha \in I}$ of C^*-sub-algs w/

(a) $\beta < \alpha \Rightarrow \mathfrak{A}_\beta \subset \mathfrak{A}_\alpha$

(b) all ~~algs~~ $\mathfrak{A}_\alpha$ have a common unit.

(c) $\bigcup_{\alpha \in I} \mathfrak{A}_\alpha$ is dense in $\mathfrak{A}$

(d) $\sigma(\mathfrak{A}_\alpha) = \mathfrak{A}_\alpha$

(e) $\alpha \perp \beta \Rightarrow [\mathfrak{A}^e_\alpha, \mathfrak{A}^e_\beta] = \{0\}$

$[\mathfrak{A}^e_\alpha, \mathfrak{A}^o_\beta] = \{0\}$

$\{\mathfrak{A}^o_\alpha, \mathfrak{A}^o_\beta\} = \{0\}$.

ex $I = \{\Lambda \subset \mathbb{Z}^n \mid \Lambda \text{ finite}\}$ directed by inclusion, $\perp$ is disjoint.
Let $\Lambda \subset I$ and to each $x \in \Lambda$ assign a finite dimensional HS $\mathcal{H}_x$. Consider $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$, $\mathfrak{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$, $\sigma = id$

As I is countable: "UHF" alg uniformly hyperfinite

ex $(H, \langle , \rangle)$ HS, $I \subset \{\Pi \subset I \mid \Pi \text{ closed}\}$, $\leq = \subset$
w $\bigcup_{\Pi \in I} \Pi \subset H$ dense. $\perp = \perp$. Let $\mathfrak{A}_{CAR}(H)$ the CAR alg of H. For $\Pi \in I$: $\mathfrak{A}_{CAR}(\Pi) = \{\text{gen. by } a(f) \mid f \in \Pi\}$
$\sigma(a(f)) = -a(f)$.

Ex H real HS w/ σ. $I = \{\Pi \subset H \mid \text{sub space}\}$, $\perp$ via σ
In particular $\bigcup_{\Pi \in I} \Pi = H$ (not only dense)
$\mathfrak{A}_{CCR}(H)$, $\mathfrak{A}_{CCR}(\Pi) = \overset{\text{generated}}{\{W(f) \mid f \in \Pi\}}$, $\sigma = id$.

NB: disj union is not enough!

4. States, representations, GNS construction

Let $\mathfrak{A}$ be unital C^*-alg. Recall s.a. function calculus (~~[illegible]~~ 8): for A w/ $A = A^*$ $\mathfrak{A}_A$ C^*-alg gen by A is commutative (fcts on spectrum).

Gelfand $\pi : \mathfrak{A}_A \to C(\sigma(A))$

$\pi \circ p(A) = p(z)$ $\forall$ polyn.

Given $f \in C(\sigma(A))$ define $f(A) := \pi^{-1}(f)$

"functional calculus"

Implies $\sigma(A) \subset \mathbb{R}$.

Def A positive ($A \geq 0$) iff $A = A^*$ and $\sigma(A) \subset \mathbb{R}_{\geq 0}$,

$A \geq B$ by $A - B \geq 0$.

Def $C \subset \mathfrak{A}$ is called a cone if it is invariant under multiplication by λ, $\lambda \geq 0$.

$\{A \in \mathfrak{A} \mid A \geq 0\}$ is a closed convex cone.

Elements of the form AA^* are all positive.

We write $\mathfrak{A}^*$ for topological dual ~~bounded~~ lin. functional

$\omega : \mathfrak{A} \to \mathbb{C}$ w/ op norm.

Def A linear functional $f \in \mathfrak{A}^*$ is called positive if $A \geq 0 \Rightarrow \varphi(A) \geq 0$. It is called a state if in addition $\varphi(e) = 1$. (continuity need not to be assumed.)

Remarks on GNS

traditional: $\mathcal{H} \rightsquigarrow \mathcal{B}(\mathcal{H}) \rightsquigarrow \mathcal{S}'(\mathcal{H})$

$\qquad\qquad \psi \qquad A \qquad S$

here: $\mathfrak{A} \rightsquigarrow E_{\mathfrak{A}} \rightsquigarrow \mathcal{H}$

$\qquad\qquad A \qquad \omega \qquad \psi$

What do we gain?

There could be inequivalent representations.

we saw already $M(2\times 2)$ on $\mathbb{C}^2$ and $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$

but that's somewhat trivial.

*

And in fact

$\dim \mathfrak{A} < \infty$: $M(n\times n)$ has $\mathbb{C}^n$ as its only irrep (up to unitary equivalence)

QM: $CCR(\mathbb{C}^n)$ (i.e. finitely many x's and p's)

Def: $\pi : CCR(\cancel{h}) \to \mathcal{B}(\mathcal{H})$ representation is called regular iff $f \mapsto \pi(W(f))$ is strongly continuous.

Def Strong topology on $\mathcal{B}(\mathcal{H})$: generated by semi-norms $\forall \psi \in \mathcal{H}$ $\quad \|A\|_\psi := \|A\psi\|$.

(*) GNS ~~theorem~~ takes besides $\mathfrak{A}$ an ω to construct $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Does it depend on ω?

<u>Uniqueness</u>: Let $(\mathcal{H}, \pi, \Omega)$ be a representation of $\mathfrak{A}$ with $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$. Then it is unitarily equivalent to GNS $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$.

<u>Pf</u> HW

but even: ~~Let $(\mathcal{H}_\omega, \pi, \Omega)$~~ Take $(\mathfrak{A}, \omega)$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Then take $\psi \in \mathcal{H}_\omega$ and build

$$\omega_\psi(A) := \langle \psi, \pi_\omega(A)\psi \rangle.$$

Then $(\mathcal{H}_{\omega_\psi}, \pi_{\omega_\psi}) \cong (\mathcal{H}_\omega, \pi_\omega)$ (of course with different Ω).

Even when $\rho \in \mathcal{S}^1(\mathcal{H}_\omega)$

$$\omega_\rho(A) := \mathrm{tr}_{\mathcal{H}_\omega}(\rho\, \pi_\omega(A))$$

leads to $\mathcal{H}_{\omega_\rho} \cong$ direct sum of copies of $\mathcal{H}_\omega$.

<u>Def</u> Given $(\mathfrak{A}, \omega)$. A state $\tilde{\omega}$ which can be written as $\tilde{\omega}(A) = \mathrm{tr}_{\mathcal{H}_\omega}(\rho\, \pi_\omega(A))$

for some $\rho \in \mathcal{S}^1(\mathcal{H}_\omega)$ is called normal to ω.
$\{\tilde{\omega} \mid \tilde{\omega}$ normal to $\omega\}$ is called the <u>folium</u> of ω.

In fact

__Thm__ (Fell) ~~The norm~~ given ω, the ~~norm~~ folium is weakly dense in $E_{\mathfrak{A}}$.

__Def__ __weak topology__: Generated by semi-norms from $\{A_1, \ldots, A_n\} \in \mathfrak{A}$ finite

$$\|\varphi\|_{A_1 \ldots A_n} = \sup \{ |\varphi(A_i)| \}.$$

__Rem__: ~~Ex~~ i.e. ~~up to an error~~ $\forall \varepsilon > 0$ and $\omega \in E_{\mathfrak{A}}$
Given n ~~measurements~~ observables, there is always a density matrix in $\mathcal{H}_\omega$ that gives measurements (of expectation values) of these observables up to an error ε.

__Thm__ (Stone, v Neumann): The Schrödinger representation π_S
$\mathcal{H} = L^2(\mathbb{R}^n)$ for $CCR(\mathbb{C}^n)$ which for $\vec{q} \in \mathbb{R}^n \subset \mathbb{C}^n$
ψ

$$\left(\pi_S(W(\vec{q}))\psi\right)(\vec{x}) = \psi(\vec{x} - \vec{q})$$

and $i\vec{p} \in i\mathbb{R}^n \subset \mathbb{C}^n$

$$\left(\pi_S(W(i\vec{p}))\right)(\vec{x}) = e^{i\vec{p}\cdot\vec{x}}\psi(\vec{x})$$

is the only regular irrep of $CCR(\mathbb{C}^n)$.

rem: In a hw we saw, "regular" is essential for this uniqueness.

BUT

for $\dim h = \infty$ (~~fat~~ QFT!)

$CCR(h)$ has many inequivalent regular reps

(charge behind moon to Coulomb Field)

not obvious.

easier:

e.g. Ising model:

local algebras

$$\Lambda \subset \mathbb{Z}^n \qquad \mathfrak{A}_\Lambda := \bigotimes_{x\in\Lambda} M(2\times 2, \mathbb{C})_x$$

$$\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}_\Lambda}^{\|\cdot\|}$$ quasi local algebra

has a representation ~~with~~ ~~cont~~ carry fn

$$\cancel{\mathcal{H} = \{ S : \mathbb{Z}^n \to \mathbb{C}^2}$$

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^2_x$$ where $A \in \mathfrak{A}_\Lambda$ acts

matrix product

$$AS = \bigotimes_{x\in\Lambda} A_x S_x$$

and $\omega_\Lambda^{\uparrow}(A) = \prod_{x \in \Lambda} \left\langle \binom{1}{0} \middle| A_x \binom{1}{0} \right\rangle$

and $\omega_\Lambda^{\downarrow} = \prod_{x \in \Lambda} \left\langle \binom{0}{1} \middle| A_x \binom{0}{1} \right\rangle$

It is then possible (we will see this) to extend $(\mathfrak{A}_\Lambda, \omega_\Lambda^{\uparrow\downarrow})$ and thus $(\mathcal{H}_\Lambda^{\uparrow\downarrow}, \pi_\Lambda^{\uparrow\downarrow}, \Omega_\Lambda^{\uparrow\downarrow})$

"thermodynamic limit"

to $(\mathfrak{A}, \omega^{\uparrow\downarrow})$. But $\omega^{\uparrow}$ is not normal

to $\omega^{\downarrow}$ (as "$\bigotimes_{x \in \mathbb{Z}^n} \sigma_x^{(x)}$" $\notin \overline{\bigcup_{\Lambda} \mathfrak{A}_\Lambda}^{\|\cdot\|}$).

infinite spin flip.

We will need those for ~~...~~ phase transitions

(as those are the two g.s. of ferromagnetic Ising).

So we should not fix $\mathcal{H}$ at the start!

---

existence of states

__Fact:__ It is a consequence of the Hahn-Banach-thm that each C*-algebra has a state (in fact: many).

So we can form $\mathcal{H}_u = \bigoplus_{\omega \in E_{\mathfrak{A}}} \mathcal{H}_\omega$ ("sep norm."), $\bigoplus_{\omega \in E_{\mathfrak{A}}} \pi_\omega$, This (huge, $\mathcal{H}_u$ is typically not separable) representation can be seen to be faithful.

$$\Rightarrow \quad \mathfrak{A} \cong \pi_u(\mathfrak{A}) \in \mathcal{B}(\mathcal{H}_u).$$

prop: So, every C*-algebra is unitary equivalent to a closed ~~+ sub~~ algebra of some $\mathcal{B}(\mathcal{H})$.

Thermodynamics

So far, we have been doing kinematics. We want ~~to~~ finally to talk about temperature or $\beta = \frac{1}{kT}$.

We have learned ~~that~~ to consider the "Gibbs state".

$$\omega_\beta(A) = tr(e^{-\beta H} A) \Big/ tr(e^{-\beta H})$$

which maximises the "entropy" as the thermal equilibrium state. We will have to refine this notion as

these problems are related!

- this seems to be unique (in conflict with what we expect for phase transitions)
- this lives on a Hilbert space
- $e^{-\beta H}$ is most often not trace class (e.g. cont. spectrum of H)!

But for $\dim \mathfrak{A} < \infty$ everything is fine:

$H = H^*$ has discrete spectrum and so that

$$\omega_{\mathfrak{A}, H, \beta}(A) = \operatorname{tr}\left(e^{-\beta H} A\right) / \operatorname{tr} e^{-\beta H}$$

makes sense.

~~N~~ Even though equilibrium states are time independent, dynamics in the form of a Hamiltonian is essential.

As before, (because in general the typical Hamiltonian is an unbounded operator), it is easier to treat the integrated version, the time evolution which for operators in the Heisenberg picture reads

$$e^{iHt} A e^{-iHt} = \alpha_t(A).$$

In general,

<u>Def</u> A C^*-dynamical system ~~is~~ is a pair $(\mathfrak{A}, \alpha)$ where $\alpha : \mathbb{R} \to *\text{-Aut}(\mathfrak{A})$ is a ~~weakly~~ strongly continuous group ~~homo~~ homomorphism.

~~Unitary~~ In the finite dimensional case we compute

$$\omega_q(AB) = \mathrm{tr}(e^{-\beta H}AB)$$

$$\overset{\text{cycl}}{=} \mathrm{tr}(B\, e^{-\beta H} A)$$

$$= \mathrm{tr}(e^{\beta H} e^{-\beta H} B e^{-\beta H} A)$$

$$\overset{\text{cycl}}{=} \mathrm{tr}(e^{-\beta H} B\, e^{-\beta H} A\, e^{\beta H})$$

$$= \mathrm{tr}(e^{-\beta H} B\, \alpha_{i\beta}(A))$$

$$= \omega_q(B\, \alpha_{i\beta}(A))$$

This relation will turn out to be crucial as it generalizes to the general situation that has problems ¡.

So we define

<u>Def</u> ! $(\mathfrak{A}, \alpha)$ be a C^*-dynamical system. The state ω over $\mathfrak{A}$ is called an <u>α-KMS state</u> ("Kubo-Martin-Schwinger") at <u>inverse temperature</u> $\beta \in \mathbb{R}$ if

$$\omega(A\, \alpha_{i\beta}(B)) = \omega(BA)$$

for all A, B in a norm-dense, α-invariant $*$-subalgebra of $\mathfrak{A}$.

In the following, we want to convince ourselves that indeed the KMS-states are the equilibrium states. We will see

- KMS $\Rightarrow$ stationary
- for dim $\mathfrak{A} < \infty$: KMS $\Leftrightarrow$ Gibbs
- ω passive $\Leftrightarrow$ ω combination of KMS states
- KMS is stable
- quasi-local KMS $\Rightarrow$ restricts to local Gibbs.

<u>Thm</u> KMS $\Rightarrow$ $\omega \circ \alpha_t = \omega$

<u>proof</u> define $F(z) = \omega(\alpha_z(B))$ for some B for which KMS hold

OBdA : $\beta = 1$ (rescale t)
this is analytic and bounded on the strip

$$\overline{D} = \{ z \in \mathbb{C} \mid -\beta \leq \Im z \leq 0 \}$$

by $M := \sup \{ \|\alpha_{i\gamma}(B)\| \mid \gamma \in [-1, 0] \}$,

since

$$|F(z)| \overset{MP}{\leq} \|\alpha_z(B)\| = \|\alpha_{\Re z} \circ \alpha_{i \Im z}(B)\|$$

$$= \|\alpha_{i\Im z}(B)\|$$

Now use KMS w/ $A = \mathbb{1}$

$$F(z-i) = \omega(\mathbb{1}\ \alpha_{-i} \circ \alpha_z(B)) \overset{KMS}{=} \omega(\alpha_z(B)\mathbb{1}) = F(z)$$

$\Rightarrow$ F periodic (Matsubara!) + analytic $\overset{\text{Liouville}}{\Longrightarrow}$ F constant.

** GNS with symmetries:

Let α be a $*$-automorphism of $\mathfrak{A}$ and ω an invariant state $\omega \circ \alpha = \omega$

Prop Then there is a $U \in \mathcal{U}(\mathcal{H}_\omega)$ with

$$\pi(\alpha(A)) = U \pi(A) U^* \quad \forall A \in \mathfrak{A} \quad (1)$$

a unitary implementer

Pf On $[A] \in \mathcal{H}_\omega$ define $U[A] := [\alpha(A)]$

This is well defined as $\alpha(N) \subset N$ and

Unitary as

$$\langle U[A], U[B] \rangle =$$
$$\langle [\alpha(A)], [\alpha(B)] \rangle =$$
$$\omega(\alpha(A)^* \alpha(B)) \overset{\text{aut}}{=}$$
$$\omega(\alpha(A^* B)) \overset{\text{inv}}{=}$$
$$\omega(A^* B) = \langle [A], [B] \rangle$$

Rem (1) defines U only up to a phase. (Cocycles)
The phase is essential for group law $\alpha_1, \alpha_2 \in *\text{-aut}(\mathfrak{A})$

$$U_{\alpha_2} U_{\alpha_1} \overset{?}{=} U_{\alpha_2 \circ \alpha_1}$$

Our construction based on invariant ω does this.
This is not the case in general $\rightsquigarrow$ anomalies.

Remark for $\dim \mathfrak{A} < \infty$:

In this case any $\omega(\cdot) = \mathrm{tr}(\rho\, \cdot)$ and

any α_t is $e^{-iHt} \cdot e^{iHt}$,

Choose A such that $[A, H] = 0$ (eg $A = f(H)$)

then

$$\mathrm{tr}\, \rho A B = \mathrm{tr}\, \rho B A = \mathrm{tr}\, A \rho B$$

$$\Rightarrow \quad 0 = \mathrm{tr}\,([\rho, A] B)$$

as this holds $\forall B$ we have

$$[\rho, A] = 0$$

double commutant (or in diagonal basis)

$\Rightarrow$ $\rho = g(H)$ (or ρ is diagonal in ~~basis~~ energy basis)

Now, take ψ_1, ψ_2 eigenstates of H w/ eigenvalues $E_{1/2}$

and evaluate KMS for $A = |\psi_1\rangle\langle\psi_2|$, $B = A^+ = |\psi_2\rangle\langle\psi_1|$

$$\mathrm{tr}\, \rho A B = \mathrm{tr}\, \rho B e^{-\beta H} A\, e^{\beta H}$$

$$\mathrm{tr}\, \rho |\psi_1\rangle\langle\psi_2|\psi_2\rangle\langle\psi_1| = \langle\psi_1|\rho|\psi_1\rangle$$

$$\mathrm{tr}\, \rho |\psi_2\rangle\langle\psi_1| e^{-\beta H} |\psi_1\rangle\langle\psi_2| e^{\beta H} = \mathrm{tr}\, \rho |\psi_2\rangle e^{-\beta E_1} e^{\beta E_2} \langle\psi_2| = e^{-\beta(E_1 - E_2)} \langle\psi_2|\rho|\psi_2\rangle$$

$$\Rightarrow \rho = c\, e^{-\beta H}$$

Alternatively, define

$$F^{(\beta)}_{A,B}(z) = \omega_\beta(B\,\alpha_z(A))$$

$$G^{(\beta)}_{AB}(z) = \omega_\beta(\alpha_z(A)B)$$

for ~~finite system~~ finite Box $\quad z = t + i\gamma$

$$F^{(\beta)}_{A,B}(z) = Z^{-1}\,\mathrm{tr}\left(B e^{iHt} e^{-\gamma H} A e^{-iHt} e^{-(\beta-\gamma)H}\right)$$

as $H e^{-H\alpha}$ is $\mathcal{S}'$ for $\alpha > 0$

The KMS condition then reads

$$G^{(\beta)}_{AB}(t) = F^{(\beta)}_{A,B}(t + i\beta)$$

$$\tilde{G}(\varepsilon) = e^{-\beta\varepsilon}\tilde{F}(\varepsilon)$$

A number of facts

Thm Let $(\mathfrak{A}, \alpha)$ be a C^*-dynamical system w/ $\mathbb{1}$

For $\beta \in \mathbb{R}$ denote by K_β the set of (β,τ)-KMS states. then

(1) K_β is convex and weak*-cpt

(2) K_β is a simplex. (i.e. has unique decomposition into extremal states $\hat{=}$ phases)

(3) $\omega \in K_\beta$ is extremal iff

$\Leftrightarrow$ $\pi(\mathfrak{A})''$ is factor

$\Leftrightarrow Z(\pi(\mathfrak{A})'') = \mathbb{C}\mathbb{1}$

$\llcorner$ global observables

Thm (Consequence of Tomita-Takesaki-Modular Theory)
Given $\mathfrak{A}$ and some state ω there is a dynamics $\alpha: \mathbb{R} \to \mathrm{Aut}(\mathfrak{A})$ s.th. ω is $(\alpha, 1)$-KMS.

Def Generators: given $\mathfrak{A}, \alpha$ there is a dense $D(\delta) \subset \mathfrak{A}$ where $D(\delta) \ni A$

$$\delta A = \frac{d}{dt} \alpha_t(A) \Big|_{t=0}$$

is defined. This in general is an unbounded derivation

In the GNS representation (ω static!) where

$$U(t)\pi(A)U(t)^{-1} = \pi(\alpha(t)(A))$$

it is given by the s.a. Hamiltonian

$$H = \frac{d}{dt} U(t)|_{t=0}$$

(NB: can depend of rep!)

<u>Prop</u> Let ω be a stationary state wrt α and KMS wrt τ. Then α and τ commute

<u>pf</u> only for finite dimensional case: $\tau(A) = \mathrm{tr}(e^{-\beta\bar{H}}A)$

and α com then $U(t) = e^{itH}$. $\forall A \in \mathfrak{A}$.

$$\mathrm{tr}(e^{-\beta\bar{H}}A) \overset{\text{inv}}{=} \mathrm{tr}(e^{-\beta\bar{H}} e^{itH} A e^{-itH})$$

$$= \mathrm{tr}(e^{itH} e^{-\beta\bar{H}} e^{-itH} A)$$

$$\Leftrightarrow \quad e^{-\beta\bar{H}} = e^{itH} e^{-\beta\bar{H}} e^{-itH}$$

$$\Leftrightarrow \quad [e^{-\beta\bar{H}}, e^{itH}] = 0$$

diverting $\Rightarrow [\bar{H}, H]$ □.

Stability

! $(\mathfrak{A}, \alpha)$ be C^*-dynamical system and ω be (α, β)-KMS. $\tilde{\mathfrak{A}}$ time analytic sub-algebra.

We want to change the dynamical law by a bit, i.e. perturb Hamiltonian by $h \in \mathfrak{A}$ (small)

This leads to a new time evolution "$e^{it(H+h)}$"

As $H \notin \mathfrak{A}$, lets define it by

$$\frac{d}{dt} \alpha_t^{-1} \alpha_t^{\lambda h} A \Big|_{t=0} = i\lambda [h, A]$$

$$\alpha_{t+t'}^{\lambda h} = \alpha_t^{\lambda h} \alpha_{t'}^{\lambda h},$$

$$\alpha_0^{\lambda h} = id$$

Haag.

BR gave a more concrete repr.

$$\alpha_t^{\lambda h} A = \alpha_t A + \sum_{n=1}^{\infty} i^n \int_{0 \le t_1 \le \dots \le t_n \le t} dt_1 \dots dt_n [h_{t_1}, [h_{t_2}[\dots a_t]]$$

w/ $a_t = \alpha_t a$

(norm convergent)

Def A primary state, stationary wrt α_t is dynamically stable wrt h iff for small enough λ $\exists$ primary $\omega^{\lambda h}$ in the folium of ω stationary wrt $\alpha_t^{h\lambda}$ which depends continually on h and λ $\lim_{\lambda \to 0} \omega^{\lambda h} = \omega$.

remarks

1) depends on statistical systems

We also need a variant of ergodicity of the dynamics. It shall be that A and $\alpha_t(B)$ commute for $t \to \infty$. We will (following Haag) demand a somewhat strong version

Def (L^1 - Asymptotic Abelianness in Time)
There is a $\|\cdot\|$-dense subset $D \subset \mathfrak{A}$ such that for $A, B \in D$ we have

$$\int_{-\infty}^{\infty} dt \, \| [A, \alpha_t(B)] \| < \infty .$$

This can be checked in concrete examples, e.g. free particles or particles w/ repulsive forces, spin systems w/ short range interaction

NB: This is about $\mathfrak{A}$ and α_t, no state is mentioned.

prop: ω primary and stationary. A-A implies

$$|\omega(A\alpha_t(B)) - \omega(A)\,\omega(B)| \longrightarrow 0$$

as $|t| \to \infty$.

Def "Dynamical stability"

ω primary, stationary wrt α_t. ~~for h∈D~~ is

dynamically stable if for all $h \in D$ there

is primary $\omega^{\lambda h}$ in the folium of ω that

is stationary under $\alpha_t^{\lambda h}$

Cocycle $$\beta_t^{\lambda h} := \alpha_t^{-1} \circ \alpha_t^{\lambda h}$$

compute $$\frac{d}{dt} (\beta_t^{\lambda h})^{-1} A = -i\lambda \, (\alpha_t^{\lambda h})^{-1} [h, \alpha_t A]$$

assume $\omega^{\lambda h}$ is invariant under $\alpha^{\lambda h}$. Integration gives

$$\omega^{\lambda h}\left((\beta_t^{\lambda h})^{-1} A\right) = \omega^{\lambda h}(A) - i\lambda \int_0^t \omega^{\lambda h}\left([h, \alpha_s A]\right) ds$$

$\lim_{t \to \pm\infty}$ exists by assumption. Define the

$$\omega_\pm(A) := \lim_{t \to \pm\infty} \omega^{\lambda h}\left((\beta_t^{\lambda h})^{-1} A\right)$$

As $(\beta_t^{\lambda h})^{-1} \alpha_{t'} = \alpha_{t'}^{\lambda h} (\beta_{t+t'}^{\lambda h})^{-1}$,

$\Rightarrow$ $\omega_\pm$ are α_t invariant.

For small enough λ the $\|\cdot\|$ distance between states is $\leq 2 \Rightarrow$ same folium.

~~primarity + asymptotic abelianness imply that $\exists_1$ invariant state.~~

$\Rightarrow \omega_\pm = \omega$.

So even KMS states are dynamically stable.

Thm A stationary, primary state which is dynamically stable is (α,β)-KMS for some $\beta \in \mathbb{R} \cup \{\pm\infty\}$. (ground states)

technical proof.

Next we want to study a second characterization. "Passivity" (related to second law)

Def A state ω is called <u>passive</u> if for every unitary $U \in D(\delta)$ we have

$\cancel{(\Omega_\omega, H\,\Omega) = \omega(U H U^*) \geq 0 = \omega(H)}$ -

$$-i\omega\,(U^* \delta U) \geq 0$$

Prop passive $\Leftrightarrow \langle u\Omega, H u\Omega\rangle \geq 0$, i.e. after doing a ~~periodic~~ cyclic operation on the system it cannot have lower energy.

Pf $\langle u\Omega, H u\Omega\rangle - \langle \Omega, H\Omega\rangle =$

$\langle \Omega, u^* H u \Omega\rangle - \langle \Omega, u^* u H\Omega\rangle =$

$\langle \Omega, u^*[H, u]\Omega\rangle =: \omega(u^* \delta u)$

D

Thm 1-KMS $\Rightarrow$ passive

Pf $x \geq 1 - e^{-x}$

$\Rightarrow$ $\beta\omega(u H u^{-1}) \geq \omega(u u^{-1}) - \omega(u e^{-\beta H} u^{-1})$

TT?

$\omega(u e^{-\beta H} u^{-1} e^{\beta H})$ as $H\Omega = 0$

$\overset{\text{KMS}}{\omega(u^{-1}u)} = 1$

Thm $\mathfrak{A}$ asymptotically abelian, ω primary (other conditions of temperatures) that is passive. $\Rightarrow$ Then ω is either a gs or a KMS for some β.

Pf. Write $u = e^{i\varepsilon a}$

expand $\omega(u^{-1} H u - H) \geq 0$ in orders of ε.

1st gives $\omega([a, H]) = 0$ (stationary)

2^{nd} arw:

$$\omega([a,[H,a]])\geq 0$$

From black box, we know, there is some τ for which ω is KMS. $\tau = e^{i\bar{H}t}$ has to commute wit H.

$$0 \leq \langle \Omega, 2aHa - Ha^2 - a^2 H | \Omega \rangle$$

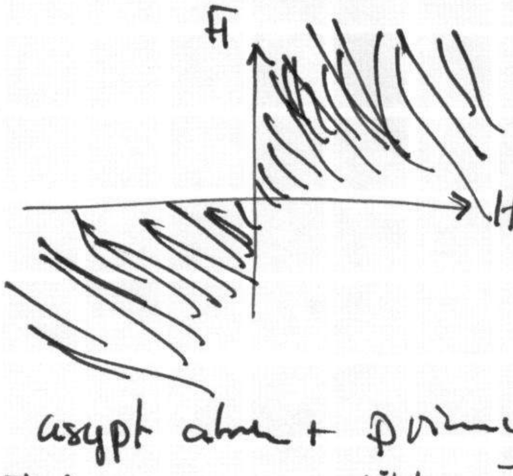
$$= \langle \Omega, 2aHa - a e^{-\bar{H}} Ha - aH e^{-\bar{H}} a | \Omega \rangle$$

$$= 2 \langle \Omega, aH(1-e^{-\bar{H}}) a | \Omega \rangle$$

As this holds for all $a = a^*$, we have

$$H(1-e^{-\bar{H}}) \geq 0$$

since $[H, \bar{H}] = 0$ ⇒ simultaneous spectr



but from asympt abel + positivity => additivity of spectrum. This is compatible only if comm spectrum is a line => $H = \beta\bar{H}$ for some β => β-KMS.

Additivity of spectrum

! H be generator of time evolution, ω primary and asympt. abelian w.r.t. σ w/ $[\sigma, \tau] = 0$, $\omega \circ \sigma = \omega$

Jf $E_1, E_2 \in \sigma(H) \Rightarrow E_1 + E_2 \in \sigma(H)$.

Pf Let U_i be neighbourhoods of E_i and f_i s.th. supp $\hat{f}_i \subset U_i$. Then

$$a_{f_i} |\Omega\rangle := \int_{-\infty}^{\infty} dt \, \alpha_t(a) f_i(t) |\Omega\rangle \neq 0$$

as $\| \sigma_s(a_{f_1}) a_{f_2} |\Omega\rangle \|^2 \xrightarrow[s\to\infty]{} \| a_{f_2} |\Omega\rangle \|^2 \| a_{f_1} |\Omega\rangle \|^2$

from primary asymp. abelian

Thus there is some s such that

$\psi := \sigma_s(a_{f_1}) a_{f_2} |\Omega\rangle \neq 0$

But in spectral representation, ψ has support in $E_1 + E_2 + U_1 + U_2$. But so every neighbourhood of $E_1 + E_2$ has spectral weight $\Rightarrow E_1 + E_2 \in$ Spec.

1D Ising

$S: \{1, 2, \dots, N\} \longrightarrow \{\pm 1\}$ w/ $S_{N+1} = S_1$

Hamilton function $H(S) = -J \sum_{i=1}^{N} S_i S_{i+1} + B \sum_{i=1}^{N} S_i$

We want to compute the partition function

$$Z(J, B) := \sum_{S_1 \in \{\pm 1\}} \cdots \sum_{S_N \in \{\pm 1\}} e^{-H(S)}$$

(absorb β in J and B).

$$= \sum_{S_1 \cdots S_N \in \{\pm 1\}} e^{-\sum_{i=1}^{N} \left(-J S_i S_{i+1} + \frac{B}{2}(S_i + S_{i+1})\right)}$$

Idea: Blockspin transformation: Sum over ~~the even a sub~~ the S_i sitting at the sub-lattice of even i to obtain a new model on a sparser lattice:

Write Z in terms of 'transfer matrix'

$$Z(J, B, N) = \sum_{S_1 \cdots S_N \in \{\pm 1\}} \prod_{i=1}^{N} e^{\left(J S_i S_{i+1} - \frac{B}{2}(S_i + S_{i+1})\right)}$$

view this as a matrix product

$$= \operatorname{tr} T_{J,B}^{N} \quad \text{with} \quad T_{J,B} = \begin{pmatrix} e^{J-B} & e^{-J} \\ e^{-J} & e^{J+B} \end{pmatrix}$$

Combine two sites..

$$\cancel{T_{J,B}^2 = \Big(e^{-2J} + e}$$

~~but first introduce new variables: $x = e^{-4J}$, $y = e^{2B}$~~

$$\cancel{T_{xy} = \Big(1}$$

$$T_{J,B}^2 = \begin{pmatrix} e^{2J-2B} + e^{-2J} & e^{-B} + e^{B} \\ e^{-B} + e^{B} & e^{2J+2B} + e^{-2J} \end{pmatrix}$$

Up to a factor λ (changing the 0-pt of the energy) this has again the same form:

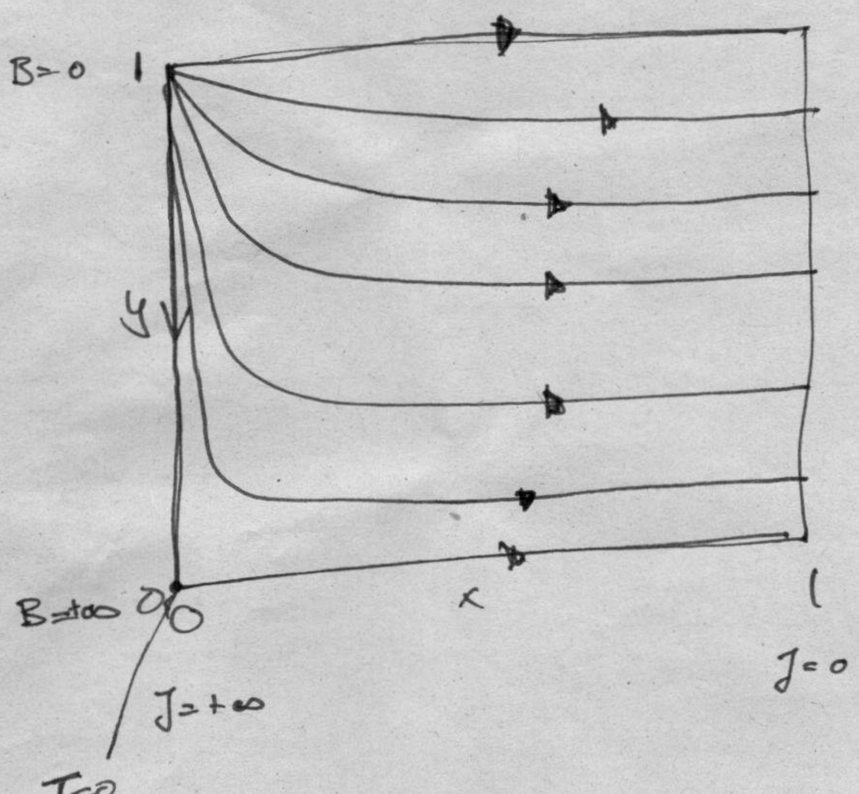
$$T_{J,B}^2 = \lambda T_{J',B'}$$

which we can solve for λ, J', B'. After some algebra and introducing $x := e^{-4J}$, $y := e^{-2B}$ we find

$$x' = \frac{x\,(1+y)^2}{(1+xy)(x+y)} \qquad y' = y\,\frac{x+y}{1+xy}.$$

i.e. we can write the spread out model as an Ising model with new coupling constants J', B'.

By the sparsing we are effectively zooming out, multiplying all length by a factor of 2. Of course, we can do this more than once. We find trajectories in coupling constant space



- y=1 stays y=1
- x=0 stays at x=0
- x=1, y arb. is fixed point

Here we have a discrete renormalization (semi-)group

$$RG: (x, y) \longmapsto \left(x \frac{(1+y)^2}{(1+xy)(x+y)}, \; y \frac{x+y}{1+xy} \right)$$

The long distance (IR) behaviour is governed by the $n \to \infty$ asymptotics of $RG^{\circ n}$: Either $T=0$ or we end up at $J=0$, B finite, uncoupled spins w/ Boltzm.

The discreteness is in fact due to the discreteness of the lattice. In a continuum theory we can expect to have continuous scale transformations

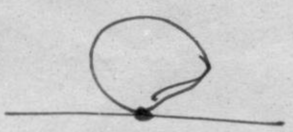
$$l \mapsto z\, l$$

$$g_1(1) \mapsto g_1(z)$$

$$\vdots$$

$$g_n(1) \mapsto g_n(z).$$

For example, in QFT, there is a need for regularization, as diagrams like

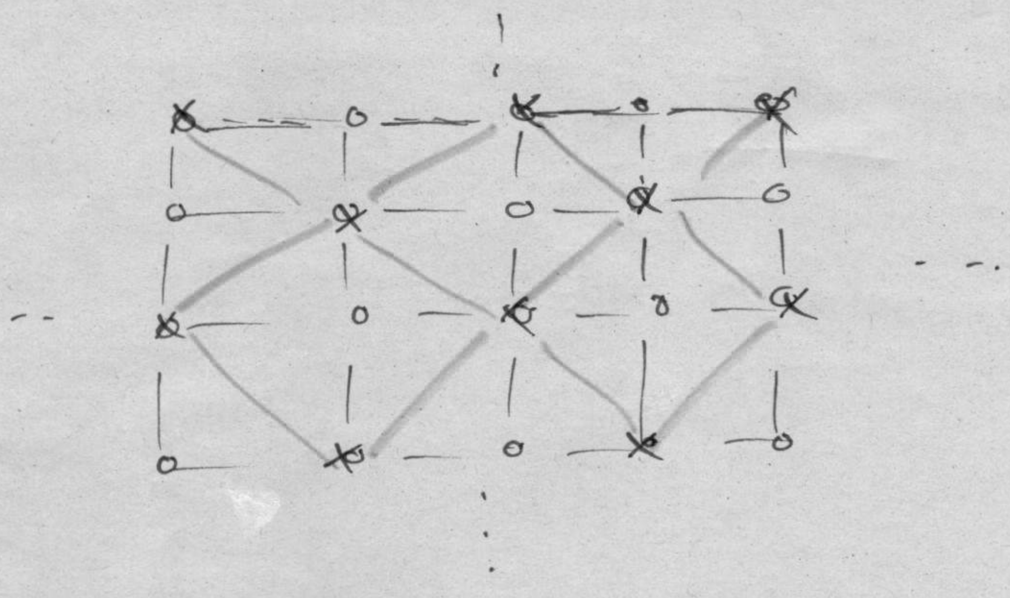
 $\sim \int \frac{d^4p}{p^2+m^2}$

diverge. Any regularization comes at the price of introducing some reference scale μ (e.g. as a cut-off) and then lengths / momenta / energies have to be measured w.r.t. this scale. So

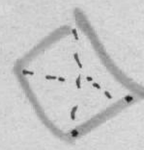
So changing physical lengths by a factor z, is equivalent to changing this reference scale by $\frac{1}{z}$. But the regularized couplings depend on μ.

So, changing μ brings with it a change of couplings $c=(\hbar, m, g, \dots)$. This is often expressed as a flow equation:

What about other models? What about 2D Ising?



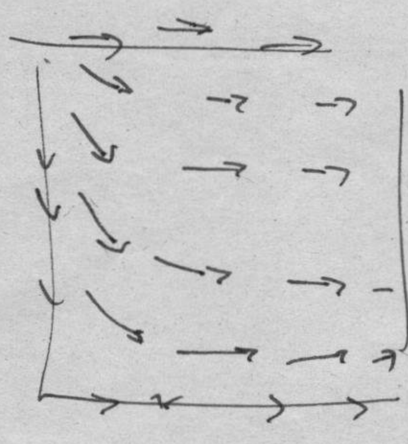
again eliminate every other site. ~~The~~ To preserve a regular lattice a checkerboard one.

But now, besides nearest neighbour , there is as well a next to nearest neighbour interaction . After the transformation, the theory is no longer of the simple Ising form! Besides $\sum_i S_i$ and $\sum_{\langle i,j \rangle} S_i S_j$ there is a new $\sum_{ij, NNN} S_i S_j$.

We could of course include this new dimension in our coupling constant space. But another RG step introduces even larger range interactions! This will only close on some infinite dimensional theory space.

$$\mu \frac{d\vec{c}}{d\mu} = \frac{d\vec{c}}{d\ln\mu} = \vec{\beta}(\vec{c})$$ for some vectorfield

$\vec{\beta}$ in coupling constant space: "The β-function".



Obviously, there are fixed points $\vec{c}_c$ when $\vec{\beta}(\vec{c}_c) = 0$.

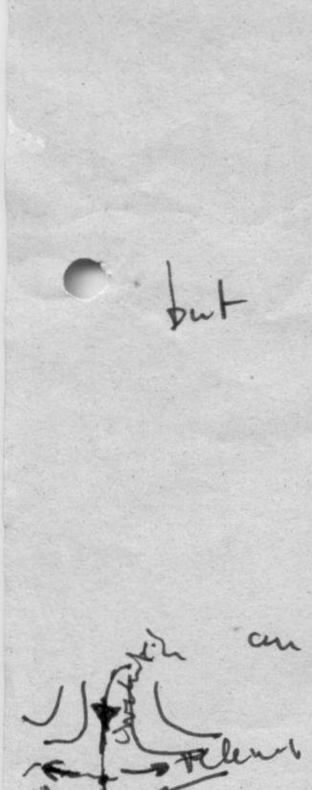
We can expand $\vec{\beta}$ around $\vec{c}_c$ (close to phase transition).

$$\beta(c) \approx \underbrace{\frac{\partial \beta_i}{\partial c_j}}_{M_{ij}} (c-c_c)_j + O((c-c_c)^2)$$

but $$\frac{d(c-c_c)}{d\ln\mu} = M\,(c-c_j)$$ has the solution

$$(c - c_j) = e^{\ln\mu\, M}\,(c(0) - c_c)$$

in an eigenbasis of M (let's see if that exists)



$$(c-c_c)_j = \mu^{m_j}\,(c-c_c)(0)$$ critical exponent

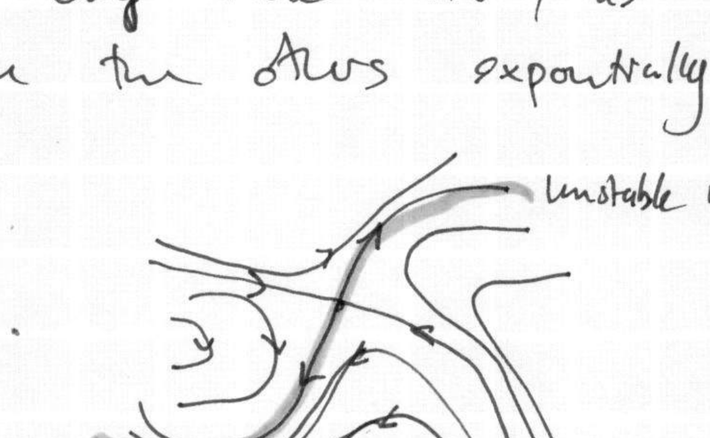
or $$(c-c_c)_j \propto \cancel{\text{const}}\ (c-c_c)_i^{\frac{m_j}{m_i}}$$

So, unless we are dealing with a "renormalizable" theory (for which, be definition, the scale-transformation closes on a finite number of terms in the Hamiltonian) we would have to deal with an infinite dimensional "theory space" of coupling constants.

This of course is ~~far~~ impracticable.

Luckily, however, there is the observation that often ~~fix~~ RG-fix points have only a finite (small) number of relevant directions:

For long distance questions (continuum limit, IR wrt fundamental scale) ~~only the~~ close to the fixpoint, only those matter, as the RG flow has made the others exponentially small.



The finite number of coordinates of the ~~relevant~~ ~~unstable manifold the fulfills scaling relations~~

And physics is detemnd only by those few parameters.

As the ∞-dim Theory space can contain many different looking theories that flow to the same fixedpoint, there is the phenomenon of "universality" that in the IR those theories behave effectively the same and physics is determined in terms of scaling exponents $\mathcal{Z}^{\omega_i}_{\omega_j}$.

This is actually an extremely powerful principle that explains to a large degree why physics is effective: Instead of a complex, real world problem, we can instead study a much simpler model which gives the same predictions. Examples are abundant:

- Ising & water at the triple point
- Fermi liquid theory
- BCS theory
- Standard Model (why renormalizable, gravity)
- AdS / Cond-mat
- traffic flow