HOMEWORK ASSIGNMENT 09

Hand-in deadline: Tuesday 25 June 2013 by 4 p.m. in the "MSP" drop box.

Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 33. (Asymptotics of the elliptic integral K_1)

In this exercise it will be shown that the complete elliptic integral of the first kind $K_1(\kappa)$ has a logarithmic divergence as $\kappa \to 1$. This mathematical property results in the divergence of thermodynamic response functions such as the specific heat capacity in the 2D Ising model (Exercise 34).

Consider the complete elliptic integral of the first kind,

$$K_1(\kappa) \ := \int_0^{\pi/2} \mathrm{d}s \, \frac{1}{\sqrt{1-\kappa^2 \sin^2 s}} \,, \qquad \kappa \in [0,1] \,.$$

(i) Prove that

$$K_1(\kappa) \ = \int_0^1 \mathrm{d}t \, \frac{1}{\sqrt{1-\kappa^2 t^2}\,\sqrt{1-t^2}}$$

(the so-called "Jacobi form" of the elliptic integral).

(ii) Let $\kappa' \in [0,1]$ be such that $\kappa^2 + \kappa'^2 = 1$ (the "complementary modulus" of κ). Prove that

$$K_1(\kappa) \ = \int_{\kappa'}^1 \mathrm{d}u \, \frac{1}{\sqrt{u^2-\kappa'^2}\,\sqrt{1-u^2}} \,.$$

(iii) Prove that

$$K_1(\kappa) \ = \ \frac{1}{\sqrt{1-\kappa'\theta(\kappa')}} \, f(\kappa')$$

where $\theta(\kappa') \in [0,1]$ and $f(\kappa') := \displaystyle\int_{\kappa'}^{\sqrt{\kappa'}} \mathrm{d}u \, \frac{1}{\sqrt{u^2-\kappa'^2}} + \int_{\sqrt{\kappa'}}^1 \mathrm{d}u \frac{1}{u\,\sqrt{1-u^2}}$.

(iv) Prove that $\displaystyle\lim_{\kappa \to 1} \left(K_1(\kappa) - \log\left(\frac{4}{\kappa'}\right)\right) = 0$.

Exercise 34. (Computation of thermodynamic quantities)

In class the free energy density for the 2D Ising model at vanishing magnetic field was proved to be

$$a(B=0,T) \ = \ -\frac{1}{\beta}\log(2\cosh(2\beta\epsilon)) - \frac{1}{2\pi\beta}\int_0^{\pi} \mathrm{d}s \, \log\left(\frac{1}{2}(1+\sqrt{1-\kappa^2\sin^2 s})\right) \,,$$

where $\beta := 1/(k_B T) > 0, \epsilon > 0$, and $\kappa := \frac{2\sinh(2\beta\epsilon)}{\cosh^2(2\beta\epsilon)}$.

(i) Consider the internal energy density $u(B=0,T) := \frac{\partial}{\partial \beta}(\beta a(B=0,T))$. Prove that

$$u(B=0,T) \;=\; -\epsilon \,\coth(2\beta\epsilon)\left(1+\frac{2}{\pi}\kappa' K_1(\kappa)\right)\;,$$

where $\kappa' := 2\tanh^2(2\beta\epsilon) - 1$ and $K_1(\kappa)$ is the complete elliptic integral of the first kind as defined in Exercise 33.

(ii) In class it was shown that the internal energy density has no analytic extensions around $\kappa = 1$. Compute the corresponding critical temperature T_c at which this non-analyticity occurs.

(iii) Consider the specific heat capacity per spin, namely $c(B=0,T) := \frac{\partial u}{\partial T}$. Prove that

$$c(B=0,T) = \frac{2}{\pi}\epsilon^2 k_B \beta^2 \coth^2(2\epsilon\beta)\left(2K_1(\kappa) - 2E_1(\kappa) - (1-\kappa')(\frac{\pi}{2} + \kappa' K_1(\kappa))\right),$$

where $E_1(\kappa) := \int_0^{\pi/2} \mathrm{d}s\;\sqrt{1-\kappa^2\sin^2 s}$ (the complete elliptic integral of the 2nd kind).

Exercise 35 & 36. (RG analysis of the Sierpinski gasket)

This exercise presents one of the few examples where the renormalisation group analysis of the Ising model can be carried out exactly, namely the Ising model on the Sierpinski gasket, a fractal geometry with Hausdorff dimension $d_H = \log 3/\log 2 \approx 1.585$. Interactions are restricted to be explicitly part of the fractal geometry such that the decimation procedure of the real-space RG is exact. In this example criticality occurs at zero temperature, as for the 1D Ising model.

Consider the 2D Sierpinski gasket (SG) Λ, as in Fig. 1. Let N be the total number of lattice sites.

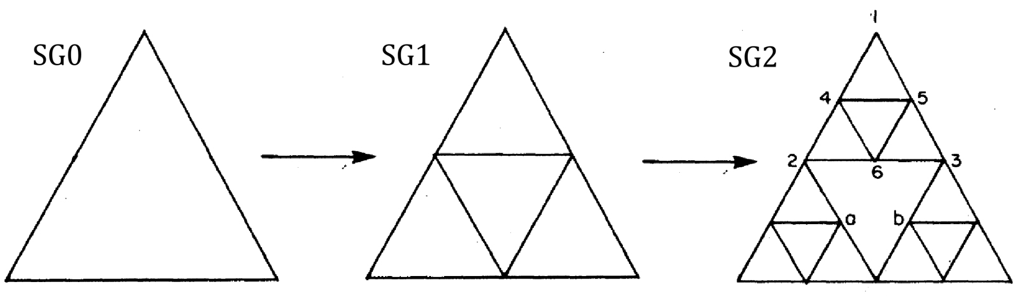

Figure 1: Construction of the Sierpinski gasket. The SG is implemented via repeated construction of the basic shape which is an equilateral triangle with a base parallel to the horizontal axis. It is a union of three copies of itself, each scaled with a factor of 1/2. In the right frame you may see the convention for labelling the spins at the vertices.

Assign to each vertex $p \in \Lambda$ of the SG a spin variable $s_p \in \{-1,+1\}$. Define the energy of an Ising ferromagnet $\mathcal{E}(S)$ (= Hamiltonian) associated with a configuration $S : p \mapsto s_p$ of the system by

$$\mathcal{E}(S|\epsilon, B, K_3) \;:=\; -\epsilon \sum_{\langle p,q\rangle} s_p s_q - B\sum_{p=1}^{N} s_p - K_3 \sum_{\langle p,q,r\rangle} s_p s_q s_r \;,$$

where $\sum_{\langle p,q\rangle}$ denotes the sum over all distinct nearest neighbour pairs and $\epsilon > 0$ denotes the ferromagnetic interaction energy. In addition to the "standard" symmetry-breaking external magnetic field $B \in \mathbb{R}$, consider further a 3-spin coupling $K_3 \in \mathbb{R}$ among the spins at the vertices of each elementary "up-triangle" of the SG (such higher-order interaction would appear in any case in the renormalisation procedure and is imposed here from the very beginning; because of the fractal structure, however, no further higher-order interactions occur during the RG procedure). The parameters ϵ, B, and K_3 are measured in units of $k_B T$.

The Hamiltonian can be rewritten as a sum over non-interacting elementary SGs which form the basic coarse-graining cells. For example, the Hamiltonian of the top unit cell in Fig. 1 is given as

$$\begin{aligned}\mathcal{E}^{\text{cell}}(s_1,\ldots,s_6|\epsilon,B,K_3) \;=\;& -\epsilon(s_1(s_4+s_5)+s_2(s_4+s_6)+(s_3+s_4)(s_5+s_6)+s_5s_6)\\ & - B/2(s_1+s_2+s_3) - B(s_4+s_5+s_6)\\ & - K_3(s_1s_4s_5+s_2s_4s_6+s_3s_5s_6)\,.\end{aligned}$$

Note that a term like $-K_3s_4s_5s_6$ is *not* included as an interaction term in this sum because it is not an "up-triangle".

(i) Perform the decimation for one non-interacting Hamiltonian on one SG's unit cell as depicted in Figure 1, i.e., compute the partial partition sum

$$Q(s_1,s_2,s_3|\epsilon,B,K_3) \;:=\; \sum_{(s_4,s_5,s_6)} e^{-\beta\mathcal{E}(s_1,\ldots,s_6|\epsilon,B,K_3)}$$

by summing over all possible configurations of spins (s_4,s_5,s_6).

(*Hint:* your result can be written as follows:

$$\begin{aligned}Q(s_1,s_2,s_3|\epsilon,B,K_3) \;=\;& e^{B/2(s_1+s_2+s_3)}\big[e^{K_3(s_1+s_2+s_3)+3\epsilon}\cdot 2\cosh(2\epsilon(s_1+s_2+s_3)+3B)\\ & + e^{K_3(s_1-s_2-s_3)-\epsilon}\cdot 2\cosh(2\epsilon s_1+B)\\ & + e^{K_3(-s_1+s_2-s_3)-\epsilon}\cdot 2\cosh(2\epsilon s_2+B)\\ & + e^{K_3(-s_1-s_2+s_3)-\epsilon}\cdot 2\cosh(2\epsilon s_3+B)\big].)\end{aligned}$$

In this decimation step, we have reduced the degrees of freedom of spins on the unit cell from $S=(s_1,\ldots,s_6)$ to $S'=(s_1,s_2,s_3)$ in the decimated cell. We now want to relate the parameters (ϵ,B,K_3) of the unit cell to the corresponding parameters (ϵ',B',K_3',C') of the renormalized lattice such that

$$Q(s_1,s_2,s_3|\epsilon,B,K_3) \;=\; e^{-\beta\mathcal{E}(s_1,s_2,s_3|\epsilon',B',K_3',C')} \;=:\; Q(s_1,s_2,s_3|\epsilon',B',K_3',C')\,.$$

Here C' denotes an additive constant in $\mathcal{E}[S']$ that appears as a contribution to the free energy from tracing out the degrees of freedom in the decimation step.

(ii) Compute $Q(s_1,s_2,s_3|\epsilon,B,K_3)$ and $Q(s_1,s_2,s_3|\epsilon',B',K_3',C')$ for all possible spin configurations (s_1,s_2,s_3).

(*Hint:* you will obtain four different relations $f_i = f_i'$, $i=1,\ldots,4$, from the eight different spin configurations. For example:

$$\begin{aligned}f_1 &:= Q((+1,+1,+1)|\epsilon,B,K_3) \;=\; e^{\frac{3}{2}B}\{2e^{3K_3+3\epsilon}\cosh(6\epsilon+3B)+6e^{-K_3-\epsilon}\cosh(2\epsilon+B)\}\\ f_1' &:= Q((+1,+1,+1)|\epsilon',B',K_3',C') \;=\; e^{3\epsilon'+\frac{3}{2}B'+K_3'+C'}\,.\end{aligned}$$

The other three relations are obtained similarly.)

(iii) Derive the RG map for one RG step as $\mathcal{R} : (\epsilon, B, K_3) \mapsto (\epsilon', B', K_3', C')$. It suffices to express ϵ', B', K_3', and C' in terms of f_1, f_2, f_3, and f_4, the functions found in (ii), which only depend on ϵ, B, K_3.

(*Hint:* Check that $e^{8\epsilon'} = \frac{f_1' f_4'}{f_2' f_3'}$, $e^{4B'} = \frac{f_1' f_2'}{f_3' f_4'}$, $e^{8K_3'} = \frac{f_1'}{f_4'} \left(\frac{f_3'}{f_2'}\right)^3$, and $e^{8C'} = f_1' f_4' (f_2' f_3')^3$.)

Steps (i)-(iii) complete one RG step from one SG unit cell to a renormalized cell (e.g., from SG_2 to SG_1 in Fig. 1). Starting from a Sierpinski gasket SG_m that is constructed in m steps as depicted in Fig. 1, we can iterate this procedure to obtain the nonlinear RG flow in the space of parameters (ϵ, B, K_3) that characterise the Hamiltonian. The nonlinear properties of the RG flow give an insight on the physical behaviour of the system on large length scales.

(iv) Set $B = K_3 = 0$. Prove that in this case the recursion relation for ϵ' can be written as

$$e^{4\epsilon'} = e^{4\epsilon} \frac{1 - e^{-4\epsilon} + 4e^{-8\epsilon}}{1 + 3e^{-4\epsilon}} \,.$$

(v) Prove that $\epsilon \to \infty$ yields a fixed point of this RG map, the so-called Ising fixed point (since it corresponds to the zero temperature fixed point from the 1D Ising model).

(*Hint:* Note that $\cosh(6\epsilon)/\cosh(2\epsilon) = 2\cosh(4\epsilon) - 1$.)

(vi) In the limit of large ϵ, the recursion relations for the other two couplings simplify considerably in the vicinity of the Ising fixed point ($\epsilon \to \infty, B = 0, K_3 = 0$), and are given to linear order by

$$\begin{aligned} \epsilon' &= \epsilon \,, \\ B' &= 4B + 2K_3 \,, \\ K_3' &= -\frac{3}{2} B \,, \end{aligned}$$

as $\epsilon \to \infty$. Determine the eigenvalues of this linear RG map and discuss its values from a physical point of view.

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 33.
(i) Apply the substitution $t := \sin s$.
(ii) Obtain first $K_1(\kappa) = \int_1^{1/\kappa'} dr\ (1-\kappa'^2 r^2)^{-1/2}(r^2-1)^{-1/2}$ by applying the substitution $r := (1-\kappa^2 t^2)^{-1/2}$. Apply another substitution $u := 1/r$ to arrive at the final result.
(iii) Divide and conquer: Split the region of integration over the interval $[\kappa', 1]$ into the two parts $I_1 := [\kappa', \sqrt{\kappa'}]$ and $I_2 := [\sqrt{\kappa'}, 1]$. Deduce that $(1-u^2)$ lies between $1-\kappa'$ and 1 in I_1 and that $u^{-2}(u^2-\kappa'^2)$ lies between $1-\kappa'$ and 1 in I_2. Use these estimates to further estimate the integral from (ii) in the region I_1 and I_2 separately. Show that one obtains $f(\kappa') \leq K_1(\kappa) \leq (1-\kappa')^{-1/2} f(\kappa')$ from which the final result can be derived.
(iv) Use that $f(\kappa') = \log \dfrac{\sqrt{\kappa'}+\sqrt{\kappa'-\kappa'^2}}{\kappa'} - \log \dfrac{\sqrt{\kappa'}}{1+\sqrt{1-\kappa'}} = \cdots = 2\log(1+\sqrt{1-\kappa'}) - \log \kappa'$.
Check that $\lim_{\kappa'\to 0} \left(K_1(\kappa) - \log\left(\dfrac{4}{\kappa'}\right)\right) = 0$.

Hints for Exercise 34.
(i) Introduce $\Delta := \sqrt{1-\kappa^2\sin^2 s}$ for short-hand notation. Compute $u(B=0,T) = -2\epsilon\tanh(2\beta\epsilon) + \dfrac{1}{2\pi\kappa}\dfrac{\partial\kappa}{\partial\beta}\displaystyle\int_0^\pi ds\ \dfrac{\kappa^2\sin^2 s}{\Delta(1+\Delta)}$. Show that $\dfrac{\kappa^2\sin^2 s}{\Delta(1+\Delta)} = -1 + 1/\Delta$ and apply the definition of K_1 from Exercise 33 to arrive at the intermediate expression $u(B=0,T) = -2\epsilon\tanh(2\epsilon\beta) + \dfrac{1}{2\pi\kappa}\dfrac{\partial\kappa}{\partial\beta}(-\pi + 2K_1(\kappa)) = \cdots = -\epsilon\coth(2\epsilon\beta)\left(1 - \dfrac{2}{\pi}K_1(\kappa) + \dfrac{4}{\pi}\tanh^2(2\epsilon\beta)K_1(\kappa)\right)$. Introduce κ' as given in the exercise, note that $\kappa'^2 + \kappa^2 = 1$, and derive the final result.
(ii) -
(iii) Better start out from $\dfrac{\partial u}{\partial T} = \dfrac{\partial\beta}{\partial T}\dfrac{\partial u}{\partial\beta}$. Show that $\dfrac{\partial}{\partial\kappa}(\kappa' K_1(\kappa)) = -\dfrac{\kappa}{\kappa'}K_1(\kappa) + \dfrac{E_1(\kappa)}{\kappa\kappa'} - \dfrac{\kappa'}{\kappa}K_1(\kappa)$ by making use of the equality $\dfrac{\partial K_1(\kappa)}{\partial\kappa} = \dfrac{E_1(\kappa)}{\kappa(1-\kappa^2)} - \dfrac{K_1(\kappa)}{\kappa}$ which is known from the lecture. Make use of $\kappa' = 2\tanh^2(2\beta\epsilon) - 1$ to obtain the final result.

Hints for Exercise 35.

Hints for Exercise 36.