HOMEWORK ASSIGNMENT 06

Hand-in deadline: Tuesday 4 June 2013 by 4 p.m. in the "MSP" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 21. Let $\mathcal{A}$ be a unital C^*-algebra, let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$ (i.e., for any $A\in\mathcal{A}$, $t\to 0 \Rightarrow \|\alpha_t(A)-A\|\to 0$), and let ω be an α_t-invariant state (i.e., $\omega(\alpha_t(A))=\omega(A)$ $\forall A\in\mathcal{A}$ $\forall t\in\mathbb{R}$). Assume further

(1) α_t is asymptotically abelian, i.e., $\lim\limits_{t\to\infty}\|[A,\alpha_t(B)]\|=0 \quad \forall A,B\in\mathcal{A}$,

(2) ω has the "cluster property"

$$\omega(A\alpha_t(B)) \xrightarrow{t\to\infty} \omega(A)\omega(B) \qquad \forall A,B\in\mathcal{A}\,.$$

(This is the typical property shown by KMS states with respect to an asymptotically abelian dynamics, although proving this fact is laborious.)

Consider an arbitrary pure state in the GNS representation $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ of $\mathcal{A}$ associated with ω, i.e., an arbitrary vector $\psi\in\mathcal{H}_\omega$ with $\|\psi\|=1$. Prove that the system prepared in the state ψ exhibit the behaviour of "return to equilibrium" in the following sense

$$\langle\psi,\pi_\omega(\alpha_t(A))\psi\rangle \xrightarrow{t\to\infty} \omega(A) \qquad \forall A\in\mathcal{A}\,.$$

The purpose of the next exercise is to construct a KMS state out of a sequence of KMS states within a scheme that will be then applicable to the construction of an equilibrium state over a quasi-local algebra as a thermodynamic limit of Gibbs states over the local algebras. To this aim you will need the following classical theorem by Paley and Wiener:

A measurable function F on $\mathbb{R}$ is the inverse Fourier transform of a function $\widehat{F}\in C_0^\infty(\mathbb{R})$ with support in $[-R,R]$ $(R>0)$ if and only if the continuation of F to the complex plane is entire analytic and for each $n\in\mathbb{N}$ there exists a constant C_n such that $|F(z)|\leqslant C_n\dfrac{e^{R|\mathfrak{Im}z|}}{(1+|z|)^n}$.

Exercise 22. Let $\mathcal{A}$ be a unital C^*-algebra and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$. Let $\beta\in\mathbb{R}$.

(i) Prove that the following conditions are equivalent:

(1) ω satisfies the (α_t, β)-KMS condition,

(2) the relation

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, f(t)\, \omega(A\alpha_t(B)) \;=\; \int_{-\infty}^{+\infty} \mathrm{d}t\, f(t+\mathrm{i}\beta)\, \omega(\alpha_t(B)A)$$

is valid for all $A, B \in \mathcal{A}$ and all f with Fourier transform $\widehat{f} \in C_0^\infty(\mathbb{R})$.

(ii) Assume in addition that, for each $n \in \mathbb{N}$, $\{\alpha_t^{(n)}\}_{t\in\mathbb{R}}$ is a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$ such that $\|\alpha_t^{(n)}(A) - \alpha_t(A)\| \xrightarrow{n\to\infty} 0$ $\forall A \in \mathcal{A}$ and $\forall t \in \mathbb{R}$, and that $\{\omega_n\}_{n=1}^\infty$ is a sequence of $(\alpha_t^{(n)}, \beta)$-KMS states on $\mathcal{A}$ which converges in the weak-$*$ topology to a state ω. Prove that ω is a (α_t, β)-KMS state on $\mathcal{A}$.

The purpose of the next exercise is to recapitulate the exact solution of the 1D Ising model. In the lecture we will discuss Onsager's exact solution of the 2D Ising model by applying the transfer matrix method. Here we review this method for the 1D Ising model. The crucial outcome of this calculation is that the Ising model in one dimension does not exhibit a phase transition for non-zero temperature as opposed to the two-dimensional case.

Exercise 23. (1D Ising ferromagnet)

Consider the lattice $\Lambda = \{1, 2, \dots, N\}$. Assign to each $p \in \Lambda$ a spin variable $s_p \in \{-1, +1\}$ (spin up / spin down). Define the energy $\mathcal{E}(S)$ associated with a generic configuration $S = \{s_p \,|\, p \in \Lambda\}$ of the system by

$$\mathcal{E}(S) \;:=\; -\epsilon \sum_{p=1}^{N} s_p s_{p+1} - B \sum_{p=1}^{N} s_p$$

(with the convention of taking the boundary condition $s_{N+1} = s_1$), where $\epsilon > 0$ denotes the interaction energy between neighbouring spins ("ferromagnetic coupling") and $B \in \mathbb{R}$ denotes the external magnetic field.

(i) Compute the partition function $Q_N(B,T) := \sum_S e^{-\beta\mathcal{E}(S)}$, with $\beta := 1/(k_B T)$, $T \geqslant 0$.
(*Hint:* "transfer matrix method", namely, factorise each exponential in $Q_N(B,T)$ into terms involving only two neighbouring spins, each of which has the form $P(s_p, s_{p+1}) = \exp(\beta\epsilon s_p s_{p+1} + \frac{1}{2}\beta B(s_p + s_{p+1}))$.)

(ii) Compute the free energy density in the thermodynamic limit, namely compute the limit $a(B,T) := \lim_{N\to\infty} a_N(B,T)$ where $a_N(B,T) := -\frac{1}{\beta N} \log Q_N(B,T)$.

(iii) Is $a(B,T)$ analytic in B and T?

(iv) Compute the mean magnetization $m(B,T) := \lim_{N\to\infty} \big\langle \frac{1}{N} \sum_{p=1}^{N} s_p \big\rangle$, where $\langle\cdot\rangle$ denotes the average over all possible configurations.

(v) Plot $B \mapsto m(B,T)$ for fixed $T > 0$ and $T = 0$ (it is convenient to use the re-scaled variable βB).
(*Remark:* you should argue from this plot that the system is *not* ferromagnetic for $T > 0$.)

Exercise 24. (Tensor product of matrices)

Notation: $\mathcal{M}_m(\mathbb{C})$ is the algebra of $m \times m$ complex matrices, $\{|j\rangle \mid j = 1, \ldots, m\}$ is the canonical basis of $\mathbb{C}^m$, $\langle i|A|j\rangle$ is the (ij)-entry of $A \in \mathcal{M}_m(\mathbb{C})$. Given $A_1, \ldots, A_k \in \mathcal{M}_m(\mathbb{C})$ define their *tensor product* $A_1 \otimes \cdots \otimes A_k \in \mathcal{M}_{m^k}(\mathbb{C})$ by

$$\big\langle i_1, \ldots, i_k | A_1 \otimes \cdots \otimes A_k | j_1, \ldots, j_k \big\rangle \;:=\; \prod_{l=1}^{k} \big\langle i_l | A_l | j_l \big\rangle,$$

where $|i_1, \ldots, i_k\rangle = |i_1\rangle \otimes \cdots \otimes |i_k\rangle \in \mathbb{C}^{m^k}$ and each $i_p, j_p \in \{1, \ldots, m\}$.

(i) Prove that $(A_1 \otimes \cdots \otimes A_k)(B_1 \otimes \cdots \otimes B_k) = (A_1B_1) \otimes (A_2B_2) \otimes \cdots \otimes (A_kB_k)$.

(ii) Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $t \in \mathbb{R}$. Define $e^{tX} := \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n$. Prove that $e^{tX} = (\cosh t)\mathbb{1} + (\sinh t)X$.

(iii) Let $t \in \mathbb{R}$. Define $A(t) := e^t \mathbb{1} + e^{-t} X = \begin{pmatrix} e^t & e^{-t} \\ e^{-t} & e^t \end{pmatrix}$. Find two real-valued functions $\theta(t)$ and $f(t)$ such that $A(t) = f(t) e^{\theta(t) X}$.

(iv) Set $X_\alpha := \mathbb{1} \otimes \cdots \otimes X \otimes \cdots \otimes \mathbb{1} \in \mathcal{M}_{2^n}(\mathbb{C})$, the tensor product consisting of n factors with X as the α-th factor. Prove that $e^X \otimes \cdots \otimes e^X = e^{(X_1 + X_2 + \cdots + X_n)}$.

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 21. A density argument to replace ψ with $\pi_\omega(B)\Omega_\omega$ in $\langle\psi, \pi_\omega(\alpha_t(A))\psi\rangle$, for some $B \in \mathcal{A}$. Then a straightforward application of assumptions (1) and (2).

Hints for Exercise 22. (i) Apply Paley-Wiener to $F(z) := f(z)\omega(\alpha_z(B)A)$, $z \in \mathbb{C}$. (ii) Establish property (2) for the limiting state ω. Writing (2) for each ω_n and exploiting the convergence of the ω_n's and the $\alpha_t^{(n)}$'s, yields the result by dominated convergence.

Hints for Exercise 23. (i) Each exponential in $Q(B,T)$ can be factorised into terms involving only two neighbouring spins, each of the form $P(s_p, s_{p+1}) = \exp(\beta\epsilon s_p s_{p+1} + \frac{1}{2}\beta B(s_p + s_{p+1})$. Since $s_p, s_{p+1} \in \{-1, +1\}$, P can be regarded as a $2{\times}2$ matrix and $Q(B,T)$ reads as the trace over the N-th power of P. Note that P is symmetric and express the partition function in terms of the eigenvalues of P.

Hints for Exercise 24. (Straightforward computations.)