

# Functional Analysis – Problems in the class, sheet 5

Mathematisches Institut der LMU – SS2010  
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*The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at [www.math.lmu.de/~michel/SS10\\_FA.html](http://www.math.lmu.de/~michel/SS10_FA.html).*

**Problem 17.** Let  $g \in L^1(\mathbb{R})$ , nonnegative, and  $p \geq 1$ . Consider the map  $T : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  such that  $f \xrightarrow{T} f * g$  (the map of taking the convolution of  $L^p$ -functions with  $g$ ). Show that  $\|T\| = \|g\|_1$ .

**Problem 18.** Consider the set

$$S = \left\{ f \in L^p([0, 1]) \mid \int_0^1 f^2 = 1 \right\}$$

Find the  $L^p$ -distance from  $S$  of the function  $f(x) = x$  for

- (i)  $p = 1$ ;
- (ii)  $p = 2$ ;
- (iii)  $p = \infty$ .

**Problem 19.** Generalise the Cauchy-Schwarz inequality  $|\int fg \, d\mu| \leq \|f\|_2 \|g\|_2$  to the form

$$\left| \int fg \, d\mu \right| \leq \frac{1}{2} \left( \alpha \|f\|_2^2 + \alpha^{-1} \|g\|_2^2 \right) \quad \forall \alpha > 0.$$

**Problem 20.** For any  $a_1, a_2 > 0$  compute the convolution

- (i)  $\mathbb{1}_{[0, a_1]} * \mathbb{1}_{[0, a_2]}$  ( $\mathbb{1}_A$  is the characteristic function of the set  $A$ )
- (ii)  $g_{a_1} * g_{a_2}$ , where  $g_a(x) = e^{-x^2/a^2}$ ,  $x \in \mathbb{R}$ .