Functional Analysis – Exercise sheet 5

Mathematisches Institut der LMU – SS2010
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Due: Monday 7 June 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 7 June tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 17. Let $p \in (1,\infty)$. Consider the map $T : L^p(\mathbb{R}^+, \mathrm{d}x) \to L^p(\mathbb{R}^+, \mathrm{d}x)$ defined by

$$(Tf)(x) \;:=\; \frac{1}{x}\int_0^x f(t)\mathrm{d}t\,.$$

Show that $\|T\| = \dfrac{p}{p-1}$. (*Hint:* you may exploit two alternative strategies:

1. find an isomorphism $L^p(\mathbb{R}^+, \mathrm{d}x) \xrightarrow{U} L^p(\mathbb{R}, \mathrm{d}y)$ of the form $(Uf)(y) = f(e^y)\phi(y)$ where ϕ is some appropriate fixed function, then identify UTU^{-1} and compute its norm using Problem in class no. 17;
2. or show by the Bounded Extension principle that it is enough to prove the statement for suitably regular, non-negative functions f and then use integration by parts in the integral for $\|Tf\|_p^p$; last, show that the constant is optimal.)

Exercise 18. Consider the function f defined for almost every $x \in \mathbb{R}^3$ as $f(x) = \dfrac{1}{|x|(|x|+1)^3}$.

(i) Decide for which $p \in [1,\infty]$ one has $f \in L^p(\mathbb{R}^3)$.

(ii) Decide for which $q \in [1,\infty]$ one has $\dfrac{\mathbb{1}_{\mathcal{B}_1}}{|\cdot|^{3/2}} * f \in L^q(\mathbb{R}^3)$. (Here $\mathbb{1}_{\mathcal{B}_1}$ is the characteristic function of the unit ball $\{x \in \mathbb{R}^3 \,|\, |x| \leqslant 1\}$ and $\frac{1}{|\cdot|}$ is the function $x \mapsto \frac{1}{|x|}$.)

(iii) Show that $\dfrac{1}{|\cdot|^{3/2}} * f \in L^q(\mathbb{R}^3)$ for every $q \in (2,\infty]$. (*Hint:* notice that here you *cannot* apply any of the inequalities seen in class. Derive instead a suitable pointwise bound on $\dfrac{1}{|\cdot|^{3/2}} * f$ and show that this bound is in L^q.)

Exercise 19. Let $f \in L^p(\mathbb{R})$ with $1 \leqslant p < \infty$. Show that

$$\int_{-\infty}^{+\infty} |f(x+t) - f(x)|^p \, \mathrm{d}x \to 0 \qquad \text{for } t \to 0\,.$$

Exercise 20.

(i) Show that in any inner-product space the *parallelogram law* for the corresponding norm is satisfied:

$$\|x+y\|^2 \; + \; \|x-y\|^2 \; = \; 2\,\|x\|^2 \; + \; 2\,\|y\|^2 \qquad (\text{where } \|x\|^2 = \langle x, x \rangle)\,.$$

(ii) Conversely, let $(X, \| \; \|)$ be a normed vector space on $\mathbb{R}$ or $\mathbb{C}$ in which the parallelogram law holds. Show that X is an inner product space, i.e., one can find an inner product $\langle \, , \rangle$ on X such that $\|x\|^2 = \langle x, x \rangle$.

(*Hint:*

- *Define* a map $X \times X \ni (x, y) \mapsto \langle x, y \rangle$ in terms of norms of the arguments, which held true if X would be an inner product space. In doing so you have to distinguish whether the vector space is over the real or complex field.
- Then deduce from the parallelogram law that $\langle \; , \; \rangle$ thus defined satisfies the properties of a scalar product. For the proof of $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ use parallelograms in the three-dimensional parallelepiped spanned by the vectors x, y, z. For the proof of $\langle x, \lambda y \rangle = \lambda \langle x.y \rangle$, λ being a scalar, first assume that λ is integer, then rational, then consider the general case.)