Functional Analysis II – Problem sheet 10

Mathematisches Institut der LMU – SS2009
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Handout: 30.06.2009
Due: Tuesday 7.07.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Exercise 27. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ ($d \geqslant 1$, integer). Prove that

$$\int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x = 0 \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d) \qquad \Rightarrow \qquad f = 0 \quad \text{a.e.}$$

(Recall the notation: L^1_{loc} is the family of (equivalence classes of) functions that, once restricted to any compact K, are in $L^1(K)$, while C_c^∞ is the family of infinitely differentiable and compactly supported functions.) *Hint:* introduce the mollifiers $j_m(x) := m^d j(mx)$, $m \in \mathbb{N}$, for some positive $j \in C_c^\infty(\mathbb{R}^d)$ supported in the ball of radius 1 centred at the origin and with $\int_{\mathbb{R}^d} j(x)\mathrm{d}x = 1$. By means of Lemmas 47 and 48 in the Funktionalanalysis class last semester you may show that the identity $\int_{\mathbb{R}^d} f\varphi\,\mathrm{d}x = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ implies $f * j_m = 0$ as a smooth function and then you may exploit the L^1-limit as $m \to \infty$.

Exercise 28. (This exercise proves Lemma 2.31 stated in the class.) Let Ω be an open, non-empty set of $\mathbb{R}^d$ ($d \geqslant 1$, integer). Let $T : \mathcal{D}(\Omega) \to \mathbb{C}$ be a linear complex-valued functional on the space of test functions over Ω. Prove that

$$T \in \mathcal{D}'(\Omega) \qquad \Leftrightarrow \qquad \begin{cases} \forall\, K \subset \Omega \quad \exists\, C > 0 \quad \exists\, m \in \mathbb{N}_0 \ : \\[2ex] |T(\varphi)| \leqslant C \displaystyle\sum_{|\alpha| \leqslant m} \widehat{p}_{K,\alpha}(\varphi) \quad \forall \varphi \in \mathcal{D}_K(\Omega) \end{cases}$$

Recall that $\mathcal{D}_K(\Omega) = \{\varphi \in \mathcal{D}(\Omega) : \mathrm{supp}(\varphi) \subseteq K\}$ and that $\widehat{p}_{K,\alpha}(\varphi) = \sup\limits_{x \in K} |D^\alpha \varphi(x)|$.