Advanced Mathematical Quantum Mechanics – Homework 7

Mathematisches Institut der LMU – SS2009
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To be discussed on: 26.06.2009, 10 a.m. – 12 p.m., lecture room B-132 (tutorial session)
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Exercise 7.1. Let $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ be the "vector" whose components are the three Pauli matrices. This notation is to express a sum like $v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$ with weights $v_j \in \mathbb{C}$ in the compact symbolic form $\overrightarrow{v} \cdot \boldsymbol{\sigma}$ where $\overrightarrow{v} := (v_1, v_2, v_3) \in \mathbb{C}^3$. With this convention, prove that

$$(\overrightarrow{v} \cdot \boldsymbol{\sigma})(\overrightarrow{w} \cdot \boldsymbol{\sigma}) \;=\; (\overrightarrow{v} \cdot \overrightarrow{w})\mathbb{1} \;+\; i\,(\overrightarrow{v} \times \overrightarrow{w}) \cdot \boldsymbol{\sigma}$$

for every $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{C}^3$.

Exercise 7.2. [*The Aharonov-Casher theorem.*] Consider a Schrödinger particle with spin $\frac{1}{2}$ confined on a plane and coupled with a magnetic field B orthogonal to the plane (set $2m = \hbar = 1$). With the notation discussed in the class, the Hamiltonian is

$$H = (-i\nabla + A)^2\mathbb{1} + \sigma_3 B \;=\; \big[\boldsymbol{\sigma} \cdot (p + A)\big]^2$$

acting on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. Here $A = (A_1, A_2)$, $B = \partial_{x_1} A_2 - \partial_{x_2} A_1$, $\mathbb{1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $p = -i\nabla$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$.[1] Assume that $B \in C_0^\infty(\mathbb{R}^2)$ and denote by $\Phi_B := \int_{\mathbb{R}^2} B(x)\mathrm{d}x$ the flux of B. Prove that if $|\Phi_B| > 2\pi$ then 0 is an *eigenvalue* of H with degeneracy equal to the the largest integer strictly less than $\frac{1}{2\pi}|\Phi_B|$, otherwise the zero-eigenvalue is absent. Prove also that if $\Phi_B > 0$ then the zero-eigenstates have the form $\left(\begin{smallmatrix} 0 \\ \psi \end{smallmatrix}\right)$ ("spin down") while if $\Phi_B < 0$ then the zero-eigenstates have the form $\left(\begin{smallmatrix} \psi \\ 0 \end{smallmatrix}\right)$ ("spin up"). That is, the ground state is always anti-parallel to the (flux of the) magnetic field.

Here are some hints:

- Given B, show that as a vector potential A you may choose $A = (A_1, A_2) = (\partial_{x_2}\phi, -\partial_{x_1}\phi)$ where $\phi(x) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|x - x'|)B(x')\mathrm{d}x'$.
- To solve the zero-energy equation $H\Psi = \big[\boldsymbol{\sigma} \cdot (p + A)\big]^2\Psi = 0$, look for solutions $\Psi = \left(\begin{smallmatrix} \psi_+ \\ \psi_- \end{smallmatrix}\right)$ in $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ of the simpler equation $\boldsymbol{\sigma} \cdot (p + A)\Psi = 0$. This yields two distinct PDE's for $\psi_+(x_1, x_2)$ and $\psi_-(x_1, x_2)$ that you can read as Cauchy-Riemann conditions of analyticity.
- Such conditions will force one of the component of Ψ to be zero (recall that there are no analytic functions in L^2) and the other component to have the expected multiplicity.

Exercise 7.3. Warm-up: let T be a $n \times n$ matrix and prove that TT^* and T^*T have the same eigenvalues with the possible exception of the eigenvalue 0. Let H be as in Exercise 7.2. Prove that $H = H_+ \oplus H_-$ with $H_\pm = (-i\nabla + A)^2 \pm B$ acting on $L^2(\mathbb{R}^2)$ and that H_+ and H_- have the same spectrum except perhaps at 0.

[1] It is understood that A is chosen so to guarantee that H is essentially self-adjoint on a suitable subspace.