Advanced Mathematical Quantum Mechanics – Homework 3

Mathematisches Institut der LMU – SS2009
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Exercise 3.1. Let $E(\lambda) := \inf\{\mathcal{E}^{\rm TF}(\rho) : \rho \in \mathcal{D}_\lambda\}$ be the lowest energy of the Thomas-Fermi functional $\mathcal{E}^{\rm TF}$ on its natural domain $\mathcal{D}_\lambda$.[1] Prove the following:

- For every $\lambda > 0$ the function $\lambda \mapsto E(\lambda)$ is differentiable.
- If $0 < \lambda \leqslant \lambda_c$ then $\dfrac{\mathrm{d}E}{\mathrm{d}\lambda} = -\mu(\lambda)$, where $\mu(\lambda)$ is the constant (the "*chemical potential*") entering the Thomas-Fermi equation $\gamma\rho^{2/3}(x) = \big[-V(x) - \frac{1}{|x|} * \rho - \mu(\lambda)\big]_+$ that determines the minimiser ρ_λ.
- If $\lambda \geqslant \lambda_c$ then $\dfrac{\mathrm{d}E}{\mathrm{d}\lambda} = 0$.
- The function $\lambda \mapsto \dfrac{\mathrm{d}E}{\mathrm{d}\lambda}$ is continuous. In other words, $\lambda \mapsto E(\lambda)$ is continuously differentiable.

[1] Recall from the class: the energy $E(\lambda)$ is a monotone decreasing function. It is strictly monotone for $0 \leqslant \lambda \leqslant \lambda_c$ and constant for $\lambda \geqslant \lambda_c$. It is strictly convex for $0 \leqslant \lambda \leqslant \lambda_c$. If $0 \leqslant \lambda \leqslant \lambda_c$ there exists a unique minimiser ρ_λ satisfying $\int \rho_\lambda \mathrm{d}x = \lambda$. If $\lambda > \lambda_c$ there is no minimiser satisfying $\int \rho_\lambda \mathrm{d}x = \lambda$: in other words, any minimiser with $\int \rho_\lambda \mathrm{d}x \leqslant \lambda$ is given by ρ_{λ_c}.