Advanced Mathematical Quantum Mechanics – Homework 2

Mathematisches Institut der LMU – SS2009
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Exercise 2.1. Let $\varrho \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$. Prove that the function

$$\frac{1}{|x|} * \varrho \,:\, \mathbb{R}^3 \to \mathbb{R}$$

is a bounded continuous function vanishing as $|x| \to \infty$ and with

$$\Big\| \frac{1}{|x|} * \varrho \Big\|_\infty \leqslant (12/5)(5\pi^2)^{1/6} \|\rho\|_{5/3}^{5/6} \|\rho\|_1^{1/6} \,.$$

(*Hint:* density argument with approximating C_0^∞-functions to prove boundedness, continuity, and vanishing at infinity; Hölder for the estimate.)

Exercise 2.2. Let $\mathcal{B}_R$ be the ball in $\mathbb{R}^d$ centred at the origin ad with radius $R > 0$ and let $\Omega := \mathbb{R}^d \setminus \mathcal{B}_R$. Let $f, g : \Omega \to \mathbb{R}$ be two functions such that

- ◇ f and g are positive and continuous
- ◇ $f(x) \to 0$ and $g(x) \to 0$ as $|x| \to \infty$
- ◇ $\Delta f \geqslant 4\pi f^{3/2}$ (in the distributional sense)
- ◇ $\Delta g \leqslant 4\pi g^{3/2}$ (in the distributional sense)
- ◇ $g \geqslant f$ on $\partial\Omega$ (i.e., as $|x| = R$).

Prove that as a consequence of the above assumptions $g(x) \geqslant f(x)$ for all $x \in \Omega$. (*Hint:* use a maximum principle argument.)

Exercise 2.3. Let $E(\lambda) := \inf\{\mathcal{E}^{\mathrm{TF}}(\rho) \mid \rho \in \mathcal{D}_\lambda\}$ be the lowest energy of the Thomas-Fermi functional $\mathcal{E}^{\mathrm{TF}}$ on its natural domain $\mathcal{D}_\lambda = \{\rho \,|\, \rho \geqslant 0, \rho \in L^1 \cap L^{5/3}, \int \rho \leqslant \lambda\}$, and let ρ_λ be its unique minimiser. Recall that $\lambda \mapsto E(\lambda)$ is convex, non-increasing, and bounded below, and that the domain $\mathcal{D}_\lambda$ is convex and $\mathcal{E}^{\mathrm{TF}}$ is strictly convex on it. Define then

$$\lambda_c := \inf\{\lambda \mid E(\lambda) = E(\infty)\} \leqslant +\infty \,.$$

(*i*) Prove that on $[0, \lambda_c]$ the function $\lambda \mapsto E(\lambda)$ is *strictly* convex and *strictly* decreasing and that the unique minimiser ρ_λ of $\mathcal{E}^{\mathrm{TF}}$ on $\mathcal{D}_\lambda$ has L^1-norm given exactly by $\int \rho_\lambda(x)\mathrm{d}x = \lambda$.

(*ii*) Assume that $\lambda_c < \infty$.[1] Prove that on $(\lambda_c, +\infty)$ the function $\lambda \mapsto E(\lambda)$ is constant and that the minimiser ρ_λ *never* satisfies $\int \rho_\lambda(x)\,\mathrm{d}x = \lambda$, instead it is *always* $\rho_\lambda = \rho_{\lambda_c}$.

[1] In principle λ_c could be $+\infty$ and the condition $\lambda_c < \infty$ has to be *assumed*. Actually this is always the case in the TF theory, as it can be proved.