Self-adjoint extensions of positive symmetric operators: the Krein-Vishik-Birman (KVB) theory and applications

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A fundamental question in quantum mechanics concerns the existence and description of self-adjoint operators defined by the background Hamiltonian; in fact, this is generally the case in boundary-value problems for elliptic differential equations with conditions to be satisfied on the boundary of some domain. Such a problem usually boils down to the determination of self-adjoint extensions of some underlying symmetric operator.

Krein proved that a positive, densely defined symmetric operator T in a Hilbert space H has 2 distinguished self-adjoint extensions, namely, the Friedrichs (or hard) extension T_F and an operator T_K, which had earlier been uncovered by von Neumann and is now called the Krein-von Neumann (or soft) extension. Furthermore, Krein showed that the set of positive self-adjoint extensions of T coincides with the set of self-adjoint operators S satisfying

$$T_K \leq S \leq T_F,$$

in the form sense. The Friedrichs extension has an acknowledged importance in quantum mechanics while Grubb has shown that the Krein-von Neumann extension has a role to play in elasticity theory. Krein initiated the task of characterising all the self-adjoint extensions S of T, his work being further developed by Vishik and then Birman. Subsequently, important generalisations were made by Grubb and by Arlinskii and co-authors. The talk will describe the main elements of the theory, and then work of Grubb (and later authors) in applying the KVB theory to boundary-value problems for elliptic differential operators. If time allows, recent work with Malcolm Brown and Ian Wood will be discussed and the Brown-Ravenhall operator considered.