Tree Ordinals and Proof-Theoretic Bounding Functions

or

"What I learned from WB".

Stan Wainer (Leeds)

Buchholz papers that influenced me

1980 3 Contribs. "Recent Advances in Proof Thy" Oxford.

1981 + Fef. Pohlers & Sieg "Iterated Ind. Defns & Subsystems of Anal" LNM 897

1987 "Indep. Result for $(\Pi^1_1\text{CA})+(\text{BI})$" APAL

1994 + Cichon & Weiermann "Unif. Approach to fund. seqns & subrec. hierarchies" MLQ

Ordinal Presentations

$$\alpha = \bigcup_n \alpha[0] \subset \alpha[1] \subset \ldots \subset \alpha[n] \subset \ldots$$

where $\alpha[n]$ finite

$$\beta + 1 \in \alpha[n] \Rightarrow \beta \in \alpha[n]$$

$$\beta \in \alpha[n] \Rightarrow \beta + 1 \in \alpha[n+1]$$

(B-C-W : β has α-norm $\leq n$)

<u>Predecessors</u> : $P_n(\alpha) = \max \alpha[n]$

<u>Rank Function</u> : $G_\alpha(n) = \text{size } \alpha[n]$

<u>Fast-Growing</u> : $B_\alpha(n) = B^2_{P_n(\alpha)}(n)$

Tree Ordinals (Ω_1)

- $0 \in \Omega_1$ $\qquad 0[n] = \phi$
- $\alpha \in \Omega_1 \implies \alpha+1 \in \Omega_1$ $\qquad \alpha+1[n] = \alpha[n] \cup \{\alpha\}$
- $\lambda : N \to \Omega_1 \implies \lambda \in \Omega_1$ $\qquad \lambda[n] = \lambda_n[n]$

  $\lambda_i \in \lambda_{i+1}[i+1]$

Then $\alpha[n]$ is a presentation of $\|\alpha\|$

(+) $\beta+0=\beta$, $\beta+(\alpha+1)=(\beta+\alpha)+1$, $\beta+\lambda = \langle \beta+\lambda_i \rangle_{i\in N}$

(exp) $2^0 = 1$, $2^{\alpha+1} = 2^\alpha + 2^\alpha$, $2^\lambda = \langle 2^{\lambda_i} \rangle_{i\in N}$.

Tree Ordinals (Ω_2)

- $0 \in \Omega_2$ $\qquad 0[\delta,n] = \phi$
- $\alpha \in \Omega_2 \implies \alpha+1 \in \Omega_2$ $\qquad \alpha+1[\delta,n] = \alpha[\delta,n] \cup \{\alpha\}$
- $\lambda : N \to \Omega_2 \implies \lambda \in \Omega_2$ $\qquad \lambda[\delta,n] = \lambda_n[\delta,n]$
- $\lambda : \Omega_1 \to \Omega_2 \implies \lambda \in \Omega_2$ $\qquad \lambda[\delta,n] = \lambda_\delta[\delta,n]$

  $\xi \in \delta[n] \Rightarrow \lambda_\xi \in \lambda_\delta[\delta,n]$

Fast-Growing $\varphi : \Omega_2 \times \Omega_1 \to \Omega_1$

$$\begin{cases} \varphi_0(\beta) = \beta + 1 \\ \varphi_{\alpha+1}(\beta) = \varphi_\alpha \circ \varphi_\alpha(\beta) \\ \varphi_\lambda(\beta) = \langle \varphi_{\lambda_i}(\beta) \rangle_{i \in N} & \lambda : N \to \Omega_2 \\ \varphi_\lambda(\beta) = \varphi_{\lambda_\beta}(\beta) & \lambda : \Omega_1 \to \Omega_2 . \end{cases}$$

Then with $\omega = id_N$ and $\omega_1 = id_{\Omega_1}$:

$$\varphi_{\omega_1}(\beta) = \beta + 2^\beta$$

$$\varphi_{\omega_1 \cdot 2}(\omega) = \varphi_{\omega_1}^{2^\omega}(\omega) = \dot{\varepsilon}_0$$

$$\varphi_{\dot{\varepsilon}_{\omega_1+1}}(\omega) = \text{Howard}$$

G-Collapse : $G_{\varphi_\alpha(\beta)}(n) = B_{G_\alpha(n)}(G_\beta(n))$

Infinitary Arithmetic $\quad n : N \vdash^{\alpha} \Gamma$

- Logic :-

(Ax) $\quad n : N \vdash^{\alpha} \Gamma$ if a true atom $\in \Gamma$

$$(\exists) \quad \frac{n : N \vdash^{\beta} m : N \qquad n : N \vdash^{\beta} \Gamma, A(m)}{n : N \vdash^{\alpha} \Gamma, \exists x A(x)}$$

$\beta \in \alpha[n]$

$$(\forall) \quad \frac{\max(n,i) : N \vdash^{\beta_i} \Gamma, A(i) \text{ all } i}{n : N \vdash^{\alpha} \Gamma, \forall x A(x)}$$

$\beta_i \in \alpha[\max n, i]$

+ Propositional rules + Cut.

- Computation :-

(N1) $\quad n : N \vdash^{\alpha} \Gamma, m : N$ if $m \leq n+1$

$$(N2) \quad \frac{n : N \vdash^{\beta} m : N \qquad m : N \vdash^{\beta} \Gamma}{n : N \vdash^{\alpha} \Gamma}$$

Embedding of PA

$$PA \vdash_r^k A(x) \Rightarrow n : N \vdash_r^{\omega.k} A(n)$$

Proof

Suppose $\forall x A(x)$ proven by Induction;

from $\vdash^{k-1} A(0)$ and $\vdash^{k-1} \forall x (A(x) \to A(x+1))$

$$\text{Cut}\ \cfrac{A(0) \quad A(0) \to A(1)}{\text{Cut}\ \cfrac{A(1) \quad A(1) \to A(2)}{\text{Cut}\ \cfrac{A(2) \quad A(2) \to A(3)}{A(3)}}}$$

...etcetera

$\therefore$ by successive A-Cuts :-

$$n : N \vdash_r^{\omega.(k-1)+n} A(n)$$

$\therefore$ By $(\forall)^\infty$ $\quad \vdash_r^{\omega.(k-1)+\omega} \forall x A(x)$

Cut Elimination (Gentzen, Schütte, ...)

1. $n \vdash_r^{\beta} \neg C, B \; ; \; n \vdash_r^{\gamma} C, B \implies n \vdash_r^{\beta+\gamma} B$

provided $\gamma[\;] \subseteq \beta[\;]$ and $|C| \leq r+1$.

2. Hence $n \vdash_{r+1}^{\alpha} B \implies n \vdash_r^{2^\alpha} B$.

Proof by induction on γ.

Suppose $C \equiv \exists x D(x)$, and

$$\delta \in \gamma[n]: \quad \frac{n \vdash^{\delta} m \qquad n \vdash^{\delta} C, D(m), B}{n \vdash^{\gamma} C, B} \exists$$

$\forall$-Invert $n \vdash^{\beta} \neg C, B$ to get $m \vdash^{\beta} \neg D(m), B$.

Then by the ind. hypoth. at $\delta < \gamma$,

$$\text{D-Cut} \; \frac{(N2)\dfrac{n \vdash^{\delta} m \qquad m \vdash^{\beta} \neg D(m), B}{n \vdash^{\beta+1} \neg D(m), B} \qquad n \vdash^{\beta+\delta} D(m), B}{n \vdash_r^{\beta+\gamma} B}$$

Bounding Functions B_α

$$(\exists)\quad \frac{n:N \vdash^{\beta}_{0} m:N \qquad n:N \vdash Spec(n,m)}{n:N \vdash^{\alpha}_{0} \exists y\, Spec(n,y)}$$

Let $B_\beta(n) = \max\{m \mid n:N \vdash^{\beta}_{0} m:N\}$

If $$\frac{n:N \vdash^{\gamma} k:N \qquad k:N \vdash^{\gamma} m:N}{n:N \vdash m:N}$$

then $m \leq B_\gamma(k) \wedge k \leq B_\gamma(n)$

So $B_{\gamma+1}(n) = B_\gamma \circ B_\gamma(n)$.

To complete the definition add

$B_0(n) = n+1$, $B_\lambda(n) = B_{\lambda_n}(n)$.

Theorem If $PA \vdash \forall x \exists y\, Spec(x,y)$

$\forall n\ \exists m < B_\alpha(n)\ Spec(n,m)$

for some $\alpha < \varepsilon_0$.

Fast-Growing Hierarchy

$$B_0(n) = n+1$$

$$B_{\alpha+1}(n) = B_\alpha \circ B_\alpha(n)$$

$$B_\lambda(n) = B_{\lambda_n}(n)$$

Can be written as

$$B_\alpha(n) = n \oplus 2^\alpha$$

$$n \oplus 2^0 = n+1$$

$$n \oplus 2^{\alpha+1} = n \oplus 2^\alpha \oplus 2^\alpha$$

$$n \oplus 2^\lambda = n \oplus 2^{\lambda_n}$$

Infinitary ID_1 $\quad \gamma : \Omega_1, n : N \overset{\alpha \in \Omega_2}{\vdash} \Gamma$

$$ID_1(W)^\infty = \text{Inf. Arith} + (W) + (\Omega_1)$$

(W) $$\frac{\gamma, n \overset{\beta}{\vdash} m : N \qquad \gamma, n \overset{\beta}{\vdash} \Gamma, \mathfrak{A}(W,m)}{\gamma, n \overset{\alpha}{\vdash} \Gamma, W(m)}$$

where $\beta \in \alpha[\gamma, n]$ and $\mathfrak{A}(W,m)$ is
$m = 0 \vee \exists k \in W (m = k \oplus 1) \vee \forall i\, \exists k \in W (\{m\}(i) \simeq k)$

(Ω)

$$\omega, \ell \overset{\delta}{\vdash_0} \Delta, W(m)$$
$$\delta \in \Omega_1,\ \gamma \in \delta[\ell],\ \Delta \subset \text{Pos}W$$
$$\Downarrow$$
$$\frac{\gamma, n \overset{\beta}{\vdash} \Gamma_0, W(m) \qquad \delta, \max(n,\ell) \overset{\beta}{\vdash} \Delta, \Gamma_1}{\gamma, n \overset{\alpha}{\vdash} \Gamma_0, \Gamma_1}$$

$\beta \in \alpha[\gamma, n]$.

Embedding $ID_1(W)$

$$ID_1 \vdash \Gamma(\vec{x}) \implies \omega, n \vdash^{\omega_1+\omega} \Gamma(\vec{n})$$

Proof To derive the W axioms :-

(W1) $\forall x\,(\mathfrak{A}(W, x) \to W(x))$
Straightforward by W-Rule and $\forall^\infty$

(W2) $\forall x\,(\mathfrak{A}(A, x) \to A(x)) \to W(m) \to A(m)$

Apply Ω-Rule with $\beta = \omega_1$ and
$\Gamma_0 = \overline{W}(m)$
$\Gamma_1 = \neg\forall x\,(\mathfrak{A}(A, x) \to A(x)),\ A(m)$

The point is that $\omega, l \vdash^{\delta}_{0} \Delta, W(m)$
can be transformed into
$\delta, l \vdash^{\delta} \Delta, \Gamma_1$ hence $\delta, l \vdash^{\omega_1} \Delta, \Gamma_1$
by weakening since $\omega_1[\delta, l] = \delta[\delta, l]$.

Collapsing

If $\gamma, n \vdash_0^{\alpha} \Gamma \subset \mathrm{Pos}\, W_1$ in $ID_1(W)^{\infty}$

then $n \vdash_0^{\varphi_\alpha(\gamma)} \Gamma$ without (Ω_1)

Proof

$$l \vdash_0^{\delta} \Delta, W(m)$$

$$\Downarrow$$

$$(\Omega_1)\ \frac{\gamma, n \vdash_0^{\beta} \Gamma_0, W(m) \qquad \delta, l \vdash_0^{\beta} \Delta, \Gamma_1}{\gamma, n \vdash_0^{\alpha} \Gamma_0, \Gamma_1}$$

IHyp (LHP) $\quad n \vdash_0^{\varphi_\beta(\gamma)} \Gamma_0, W(m)$

Apply RHP $\quad \varphi_\beta(\gamma), n \vdash_0^{\beta} \Gamma_0, \Gamma_1$

IHyp (RHP) $\quad n \vdash_0^{\varphi_\beta \varphi_\beta(\gamma)} \Gamma_0, \Gamma_1$

Note $\beta \in \alpha[\gamma, n] \Rightarrow \varphi_\beta(\gamma) \in \varphi_\alpha(\gamma)[n]$

$\Rightarrow \varphi_\beta^2(\gamma)[n] \subseteq \varphi_\alpha(\gamma)[n]$

Bounding

$$ID_1(W) \vdash \exists y\, Spec(x, y)$$

$$\omega, n \vdash^{\omega_1 \cdot 2} \exists y\, Spec(n, y)$$

$$\omega, n \vdash_0^{\alpha} \exists y\, Spec(n, y) \qquad \alpha = 2^{\cdot^{\cdot^{\cdot^{2^{\omega_1 \cdot 2}}}}}$$

$$n \vdash_0^{\varphi_\alpha(\omega)} \exists y\, Spec(n, y)$$

$$\models \exists y < B_{\varphi_\alpha(\omega)}(n).\ Spec(n, y)$$

Fast-Growing below "Howard".

And so on – $ID_{<\omega}(W) \equiv \Pi^1_1\text{-}CA_0$