The Strength of Martin-Löf Type Theory with the Logical Framework

(Work in Progress)

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1. Motivation

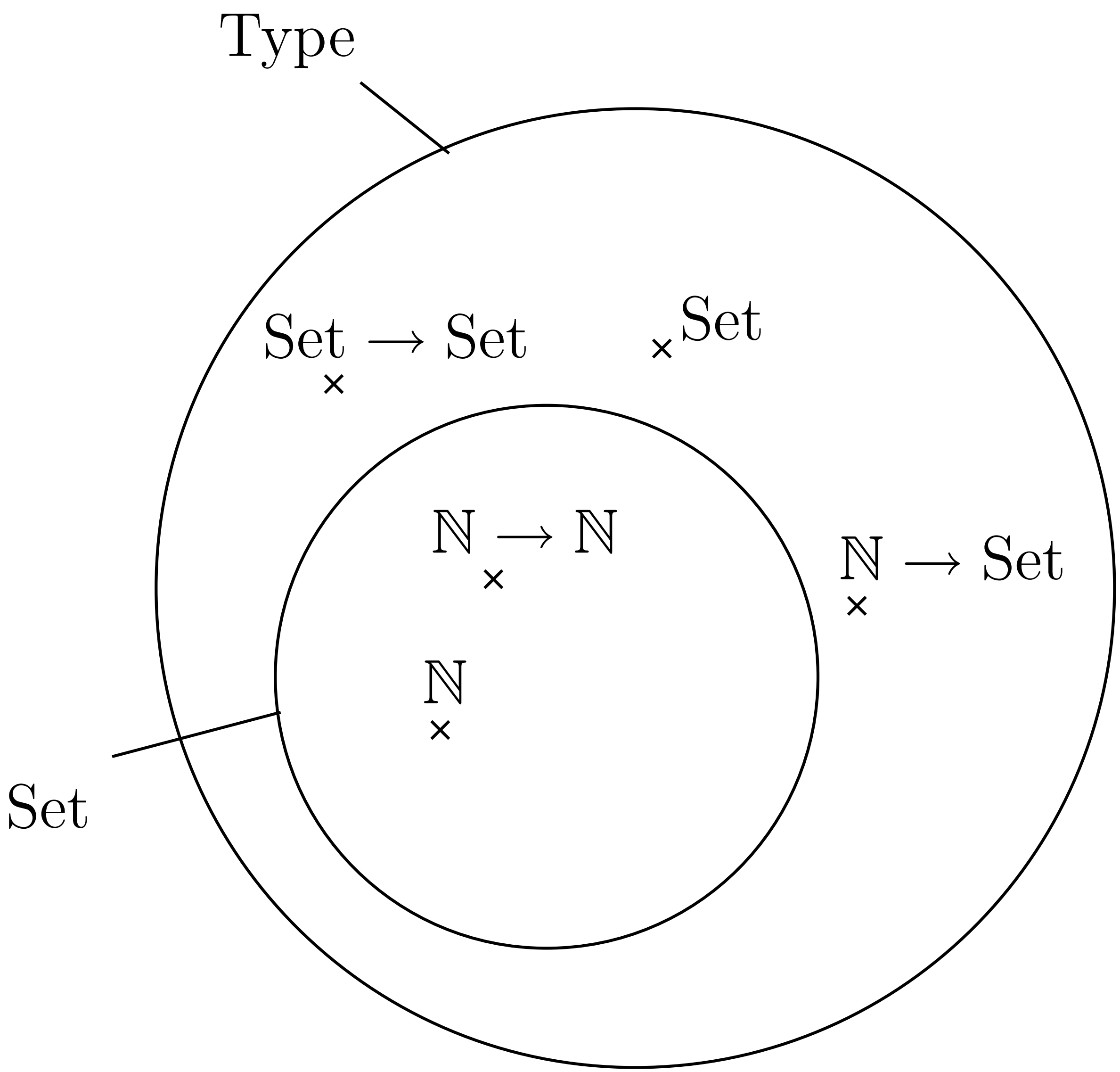
- Logical framework (**LF**) added to Martin-Löf Type Theory (**MLTT**) in order to provide an infrastructure for defining set constructions.
- LF obtained by adding
  - one type level **Type** on top of the standard type level **Set**,
  - s.t. $\mathrm{Set} \cup \{\mathrm{Set}\} \subseteq \mathrm{Type}$,
  - and by closing both Set and Type under the **dependent function type**

$$(x : A) \to B$$

and (possibly) the **dependent product**

$$(x : A) \times B$$

Logical Framework



Simplification by LF

- Without the LF, elimination for $\mathbb{N}$ is given by

$$\frac{\begin{array}{c}\Gamma, x : \mathbb{N} \Rightarrow C[x] : \mathrm{Set} \\ \Gamma \Rightarrow step0 : C[0] \\ \Gamma, x : \mathbb{N}, y : C[x] \Rightarrow stepS[x,y] : C[\mathrm{S}(x)] \\ \Gamma \Rightarrow n : \mathbb{N}\end{array}}{\Gamma \Rightarrow \mathrm{P}(step0, (x,y)stepS[x,y], n) : C[n]}$$

  together with an equality version of it,
- With the LF, it is given by

$$\begin{array}{l}\mathrm{P} : (C : \mathbb{N} \to \mathrm{Set}) \\ \quad \to (step0 : C\ 0) \\ \quad \to (stepS : (n : \mathbb{N}) \to C\ n \to C\ (\mathrm{S}\ n)) \\ \quad \to (n : \mathbb{N}) \to C\ n\end{array}$$

Syntax for the Logical Framework

- Most theorem provers for dependent type theory based on the LF.
- In order to simplify our interpretation in KPI^{+} we use a version where we have

$$\frac{A : \mathrm{Set}}{\mathcal{E}l(A) : \mathrm{Type}}$$

rather than

$$\frac{A : \mathrm{Set}}{A : \mathrm{Type}}$$

Problem

- LF amounts to **adding a universe** (namely Set) to type theory.
  - Why doesn't this increase its strength?
- Because of this we **avoided** until now the LF in proof theoretic analyses of extensions of MLTT.
- **Goal:** Extend the methodology of proof theoretic analyses so that LF is included.
  - Aim: show $|\mathrm{ML_1W} + \mathrm{LF}| = |\mathrm{ML_1W}|$ similarly for other variants of MLTT.

2. Models of $\mathrm{ML_1W}$ without LF

- Let CTerm = set of closed terms.
- **Environment** η = finite functions $\mathrm{Var} \to \mathrm{CTerm}$.
- Model of $\mathrm{ML_1W}$ without LF introduced by defining a **PER model** in $\mathrm{KPI}^+ := \mathrm{KPI}^r + \exists \mathrm{I}.$ "I inaccessible".
- For certain terms A corresponding to set expressions we define for environments η s.t. $\mathrm{FV}(A) \subseteq \mathrm{dom}(\eta)$

$$[\![A]\!]_\eta \subseteq \mathrm{CTerm}^2$$

- Then we show by induction on derivations that, if

$$\mathrm{ML_1W} \vdash \Gamma \Rightarrow \theta$$

then

$$\mathrm{KPI}^+ \vdash \mathrm{Correct}(\Gamma \Rightarrow \theta)$$

Models of $\mathrm{ML}_1\mathrm{W}$ (no LF)

- For simplicity we treat $[\![A]\!]$ as a set of terms rather than a set of pairs of terms.
- For instance

$$\begin{aligned}
\mathrm{Correct}(x : A \Rightarrow B : \mathrm{Set}) :=& \\
&\mathrm{Correct}(\emptyset \Rightarrow A : \mathrm{Set}) \\
&\wedge \forall r \in [\![A]\!].\mathrm{PER}([\![B]\!]_{[x \mapsto r]}) \wedge \mathrm{Closure}([\![B]\!]_{[x \mapsto r]})
\end{aligned}$$

$$\begin{aligned}
\mathrm{Correct}(x : A \Rightarrow b : B) :=& \\
&\mathrm{Correct}(x : A \Rightarrow B : \mathrm{Set}) \\
&\wedge \forall r \in [\![A]\!].b[x := r] \in [\![B]\!]_{[x \mapsto r]}
\end{aligned}$$

3. Models of $\mathrm{ML}_1\mathrm{W}$ + LF

- With the LF the judgement $A : \mathrm{Set}$ is no longer special.
  - $A : \mathrm{Set}$ has the same status as $a : A$.
  - Instead “$A : \mathrm{Type}$” is special.
- We need to define $[\![A]\!]_\eta$ for **type expressions** rather than set-expressions.
- Correctness statements as before, but with **Set** replaced by **Type**.
- Need to define $[\![$**Set**$]\!]$.

Interpretation of Elements of Type

- Idea: $[\![\mathrm{Set}]\!] = \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Set}^{\alpha}$ which is a proper class.
- Problem: If we interpret

$$[\![\mathbb{N} \to \mathrm{Set}]\!] := \{a \mid \forall n \in [\![\mathbb{N}]\!].a\ n \in [\![\mathrm{Set}]\!]\}$$

  we will interpret large elimination, which increases the proof theoretical strength.

  - Large elimination means that for $C := \mathrm{W}x : A.B$ or $C := \mathbb{N}$ we can define $f : C \to D$ by induction over C for any $D : \mathrm{type}$.
  - Small elimination means that we require $D : \mathrm{Set}$.

Interpretation of Elements of Type

- We need to make sure that

$$[\![\mathbb{N} \to \mathrm{Set}]\!] = \bigcup_{n \in \mathbb{N}} ([\![\mathbb{N}]\!] \; [\![\to]\!] \; \mathrm{Set}^{\kappa_n})$$

  (where $\kappa_n = n$th admissible above I).
- For this we define

$$[\![\mathbb{N} \to \mathrm{Set}]\!]^n = [\![\mathbb{N}]\!] \; [\![\to]\!] \; \mathrm{Set}^{\kappa_n}$$

Interpretation of Elements of Type

- What is $[\![\mathrm{Set} \longrightarrow \mathrm{Set}]\!]$?
- Cannot restrict it to $\mathrm{Set}^{\kappa_n} \longrightarrow \mathrm{Set}^{\kappa_n}$.
  - E.g. for any $n \in \mathbb{N}$ we have $\lambda x.(\mathrm{W}y : \mathcal{E}l(x).x) \in \mathrm{Set}^{\kappa_n} \, [\![\longrightarrow]\!] \, \mathrm{Set}^{\kappa_{n+1}}$.
- We can define $[\![\mathrm{Set} \longrightarrow \mathrm{Set}]\!]^e$ for any $e :: \mathrm{nat} \longrightarrow \mathrm{nat}$ e.g.

$$\lambda x.(\mathrm{W}y : \mathcal{E}l(x).x) \in [\![\mathrm{Set} \longrightarrow \mathrm{Set}]\!]^{\lambda n.n+1}$$

- $[\![(\mathrm{Set} \longrightarrow \mathrm{Set}) \longrightarrow \mathrm{Set}]\!]^e$ defined for $e :: (\mathrm{nat} \longrightarrow \mathrm{nat}) \longrightarrow \mathrm{nat}$.

Functionals of Finite Types

- Let the finite types be $\epsilon, \mathrm{nat}, \alpha \to \beta, \alpha \times \beta$.
- Let $e :: \alpha$ mean that e is a Kleene index for a functional of finite type α.
  - ϵ is the trivial type (contains only element 0).
  - We can contract $\epsilon \times \alpha$, $\alpha \times \epsilon$, $\epsilon \to \alpha$ to α and $\alpha \to \epsilon$ to ϵ.
- $\mathrm{Btype}(A)$ is defined as a finite type as follows:
  - $\mathrm{Btype}(\mathrm{Set}) := \mathrm{nat}$.
  - $\mathrm{Btype}(\mathcal{E}l(t)) := \epsilon$.
  - $\mathrm{Btype}((x : A) \overset{\to}{\underset{\times}{}} B) := \mathrm{Btype}(A) \overset{\to}{\underset{\times}{}} \mathrm{Btype}(B)$.

Ctype

- We need to guarantee as well that if e.g.

$$\mathrm{ML_1W} \vdash x : A, y : B \Rightarrow \mathrm{Context}$$

  then $[\![A]\!]\downarrow \wedge \forall a \in [\![A]\!].[\![B]\!]_{[x \mapsto a]}\downarrow$.
- This will require that certain $\alpha = \kappa_n$ do exist.
  E.g. $[\![\mathcal{E}l(t)]\!]\downarrow$ if $t \in \mathrm{Set}^{\kappa_n}$.
- $\mathrm{Ctype}(A)$ is defined as a sequence of finite types:
  - $\mathrm{Ctype}(\mathrm{Set}) := \emptyset$.
  - $\mathrm{Ctype}(\mathcal{E}l(t)) := \mathrm{nat}$.
  - $\mathrm{Ctype}((x : A) \begin{smallmatrix}\to \\ \times\end{smallmatrix} B)$

$$:= \mathrm{Ctype}(A) +\!\!+ (\mathrm{Btype}(A) \to \mathrm{Ctype}(B)).$$

$[\![A]\!]^{\vec{\mathrm{g}}}\!\downarrow$

- We define for $\vec{\mathrm{g}} :: \mathrm{Ctype}(A)$ whether $[\![A]\!]^{\vec{\mathrm{g}}}\!\downarrow$:
  - $[\![\mathrm{Set}]\!]^{\emptyset}\!\downarrow \;:=\; \top$.
  - $[\![\mathcal{E}l(t)]\!]^{n}\!\downarrow \;:=\; \exists\alpha.(\alpha = \kappa_n \wedge t \in \mathrm{Set}^{\alpha})$.
  - $[\![(x : A) \overset{\longrightarrow}{\underset{\times}{}} B]\!]^{\vec{\mathrm{f}},\vec{\mathrm{g}}}\!\downarrow$

    $$:= [\![A]\!]^{\vec{\mathrm{f}}}\!\downarrow \wedge \forall h :: \mathrm{Btype}(A).\forall a \in [\![A]\!]^{\vec{\mathrm{f}};h}.[\![B]\!]^{\vec{\mathrm{g}}(h)}_{[x \mapsto a]}\!\downarrow.$$

$[\![A]\!]^{\vec{\mathrm{g}},h}$

- We define $[\![A]\!]^{\vec{\mathrm{g}},h}$ for $\vec{\mathrm{g}} :: \mathrm{Ctype}(A)$, $h :: \mathrm{Btype}(A)$:
  - $[\![\mathrm{Set}]\!]^{\emptyset;n}$
    $$:= \{a \mid \exists \alpha.\alpha = \kappa_n \wedge a \in \mathrm{Set}^{\alpha}\}.$$
  - $[\![\mathcal{E}l(t)]\!]^{n;\epsilon}$
    $$:= \{a \mid \exists \alpha.\alpha = \kappa_n \wedge a \in \mathcal{E}l^{\alpha}(t)\}.$$
  - $[\![(x : A) \to B]\!]^{\vec{\mathrm{f}},\vec{\mathrm{g}};h}$
    $$:= \{a \mid \forall k :: \mathrm{Btype}(A).\forall b \in [\![A]\!]^{\vec{\mathrm{f}};k}.a\ b \in [\![B]\!]^{\vec{\mathrm{g}}(k);h\ k}_{[x\mapsto b]}\}.$$
  - $[\![(x : A) \times B]\!]^{\vec{\mathrm{f}},\vec{\mathrm{g}};h}$
    $$:= \{a \mid \pi_0(a) \in [\![A]\!]^{\vec{\mathrm{f}};\pi_0(h)} \wedge \pi_1(a) \in [\![B]\!]^{\vec{\mathrm{g}}(\pi_0(h));\pi_1(h)}_{[x\mapsto\pi_0(a)]}\}.$$

Example

- $$[\![\mathrm{Set} \to \mathrm{Set}]\!]^{\emptyset;f}$$
  $$:= \{a \mid \forall k :: \mathrm{nat}.\forall b.(\exists \alpha.\alpha = \kappa_n \wedge b \in \mathrm{Set}^{\alpha}) \to (\exists \alpha.\alpha = \kappa_{f\ n} \wedge a\ b \in \mathrm{Set}^{\alpha})\}$$
- Especially

$$\lambda x.(\mathrm{W} y : \mathcal{E}l(x).x) \in [\![\mathrm{Set} \to \mathrm{Set}]\!]^{\emptyset;\lambda n.n+1}$$

Correct($\Gamma \Rightarrow \theta$)

- $\mathrm{Btype}(x_1 : A_1, \ldots, x_n : A_n \Rightarrow A : \mathrm{Type})$
  $:= \mathrm{Btype}((x_1 : A_1) \to \cdots \to (x_n : A_n) \to A : \mathrm{Type})$.
- Similarly for Ctype.
- For $\vec{\mathrm{f}}, \vec{\mathrm{g}} :: \mathrm{Ctype}(\Gamma \Rightarrow A : \mathrm{Type})$, we define

$$\begin{array}{l}
\mathrm{Correct}(\Gamma \Rightarrow A : \mathrm{Type})^{\vec{\mathrm{f}},\vec{\mathrm{g}}} := \\
\quad \mathrm{Correct}(\Gamma \Rightarrow \mathrm{Context})^{\vec{\mathrm{f}}} \\
\quad \wedge \forall \vec{\mathrm{k}} :: \mathrm{Btype}(\Gamma). \forall \vec{\mathrm{r}} \in [\![\Gamma]\!]^{\vec{\mathrm{f}};\vec{\mathrm{k}}}. \\
\qquad [\![A]\!]^{\vec{\mathrm{g}}(\vec{\mathrm{k}})} \downarrow \\
\qquad \wedge \forall l :: \mathrm{Btype}(A). \mathrm{PER}([\![A]\!]^{\vec{\mathrm{g}}(\vec{\mathrm{k}});l}) \wedge \mathrm{Closure}([\![A]\!]^{\vec{\mathrm{g}}(\vec{\mathrm{k}});l}).
\end{array}$$

Correct($\Gamma \Rightarrow \theta$)

- For $\vec{\mathrm{f}}, \vec{\mathrm{g}} :: \mathrm{Ctype}(\Gamma \Rightarrow A : \mathrm{Type})$, we define

$$\mathrm{Correct}(\Gamma \Rightarrow a : A)^{\vec{\mathrm{f}},\vec{\mathrm{g}};l} :=$$
$$\quad \mathrm{Correct}(\Gamma \Rightarrow A : \mathrm{Type})^{\vec{\mathrm{f}}}$$
$$\quad \wedge \forall \vec{\mathrm{k}} :: \mathrm{Btype}(\Gamma).\forall \vec{\mathrm{r}} \in [\![\Gamma]\!]^{\vec{\mathrm{f}};\vec{\mathrm{k}}}.$$
$$\qquad a[\vec{x} \mapsto \vec{\mathrm{r}}] \in [\![A]\!]^{\vec{\mathrm{g}}(\vec{\mathrm{k}});l\ \vec{\mathrm{k}}}.$$

- Now prove by Meta-induction on the derivation that if

$$\mathrm{ML_1W} \vdash \Gamma \Rightarrow \theta$$

then there Meta-exist $\vec{\mathrm{f}}, g$ s.t.

$$\mathrm{KPI}^{+} \vdash \mathrm{Correct}(\Gamma \Rightarrow \theta)^{\vec{\mathrm{f}};g}$$

Conclusion

- LF doesn't add strength, but very difficult to deal with it (unless one treats it as a proper universe).
- From a foundational point of view this means that the logical framework adds a lot of syntactic complexity to type theory (meaning explanation).
  - $\Rightarrow$ LF is too "strong" for just providing an infrastructure for defining type theories.
  - Approach by P. Aczel to provide a "weaker" form of the LF.
- Methodology for upper bounds seems to work for many variants of MLTT.