The modal μ-calculus Hierarchy on Restricted Classes of Transition Systems

Luca Alberucci[1] Alessandro Facchini[2]

[1] IAM, University of Berne

[2] Universities of Lausanne and Bordeaux 1

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... is an extension of modal logic allowing least and greatest fixpoint constructors for any (syntactically) monotone formula. containing "all" extensions of modal logic with fixpoint constructors.

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- *PDL*: $\langle\alpha^*\rangle\psi = \mu x.\psi \vee \langle\alpha\rangle x$
- *CTL*: $\mathbf{EG}\varphi = \nu x.\varphi \wedge \Diamond x$ and $\mathbf{E}(\varphi\mathcal{U}\psi) = \mu x.\psi \vee (\varphi \wedge \Diamond x)$

Some expressible properties

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Allways eventually "p":

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A formula with ad $= 3$:

$$\varphi \equiv \mu x.\nu y.\mu z.\big((d_1 \wedge \Diamond x) \vee (d_2 \wedge \Diamond y) \vee (d_3 \wedge \Diamond z) \vee \ldots$$

$$\ldots \vee (c_1 \wedge \Box x) \vee (c_2 \wedge \Box y) \wedge (c_3 \wedge \Box z)\big)$$

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Syntactical modal μ-calculus hierarchy

The alternation depth implies a "strict" syntactical hierarchy on the class of all μ-formulae.

The modal μ-calculus hierarchy

Bradfield (1996): Stictness of semantical modal μ-calculus hierarchy

The semantical modal μ-calculus hierarchy is strict on the class of all transition systems.

$\Rightarrow$ For each *n* there is a formula φ with $\mathrm{ad}(\varphi) = n$ such that for all formulae ψ with $\mathrm{ad}(\psi) < n$ we do **not** have

$$\text{for all transition systems } \mathcal{T}: \quad (\mathcal{T} \models \varphi \quad \Leftrightarrow \quad \mathcal{T} \models \psi).$$

We answer the three following questions:

Strictness of the semantical modal μ-calculus hierarchy on the class of all...

1. ...reflexive transition systems?
2. ...transitive and symmetric transition systems?
3. ...transitive transition systems?

Overview

$\mathcal{L}_\mu$-formulae

$$\varphi ::\equiv p \mid \ \sim p \mid \top \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi \ldots$$

$$\ldots \mid \mu x.\varphi \mid \nu x.\varphi$$

where $p, x \in \mathrm{P}$ and x occurs only positively in $\eta x.\varphi$ $(\eta = \nu, \mu)$.

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$$\dots \mid \mu x.\varphi \mid \nu x.\varphi$$

where $p, x \in \mathrm{P}$ and x occurs only positively in $\eta x.\varphi$ ($\eta = \nu, \mu$).

$\neg\varphi$ is defined by using de Morgan dualities for boolean connectives, the usual modal dualities for $\Diamond$ and $\Box$, and

$$\neg\mu x.\varphi(x) \equiv \nu x.\neg\varphi(x)[x/\neg x] \quad \text{and} \quad \neg\nu x.\varphi(x) \equiv \mu x.\neg\varphi(x)[x/\neg x].$$

- $x \in \text{bound}(\varphi)$ then φ_x is subformula of φ of the form $\eta x.\alpha$.
- φ *well-named* if no two distincts occurrences of fixed point operators in φ bind the same variable, no variable has both free and bound occurrences in φ and if for any subformula $\eta x.\alpha$ of φ we have that x appears once in α.

Syntactical modal μ-calculus hierarchy

Let $\Phi \subseteq \mathcal{L}_\mu$. $\nu(\Phi)$ is the smallest class of formulae such that:

- $\Phi, \neg\Phi \subset \nu(\Phi)$;
- If $\psi(x) \in \nu(\Phi)$ and x occurs only positively, then $\nu x.\psi \in \nu(\Phi)$;
- If $\psi, \varphi \in \nu(\Phi)$, then $\psi \wedge \varphi, \psi \vee \varphi, \Diamond\psi, \Box\psi \in \nu(\Phi)$;
- If $\psi, \varphi \in \nu(\Phi)$ and x is bound in ψ, then $\varphi[x/\psi] \in \nu(\Phi)$

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- If $\psi, \varphi \in \nu(\Phi)$ and x is bound in ψ, then $\varphi[x/\psi] \in \nu(\Phi)$

similarly for $\mu(\Phi)$

For all $n \in \mathbb{N}$, we define the class of μ-formulae Σ_n^μ and Π_n^μ inductively as follows:

- $\Sigma_0^\mu := \Pi_0^\mu := \mathcal{L}_\mathsf{M}$;
- $\Sigma_{n+1}^\mu = \mu(\Pi_n^\mu)$;
- $\Pi_{n+1}^\mu = \nu(\Sigma_n^\mu)$.

$$\Delta_n^\mu := \Sigma_n^\mu \cap \Pi_n^\mu$$

Alternation depth:

$$\mathrm{ad}(\varphi) := \inf\{k : \varphi \in \Delta_{k+1}^\mu\}.$$

Transition Systems

Transition Systems

A *transition system* $\mathcal{T}$ is a triple $(\mathsf{S}, \rightarrow^{\mathcal{T}}, \lambda)$ consisting of

- a set S of *states*,
- a binary relation $\rightarrow^{\mathcal{T}} \subseteq \mathsf{S} \times \mathsf{S}$ called *transition relation*,
- the *valuation* $\lambda : P \rightarrow \wp(\mathsf{S})$ assigning to each propositional variable p a subset $\lambda(p)$ of S.

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A *pointed transition system* $(\mathcal{T}, s_0)$ consists of a transition system $\mathcal{T}$ and a distinguished state s_0.

Denotation of a formula

$\|\varphi\|_{\mathcal{T}}$ is defined as usual by induction on the complexity of $\varphi \in \mathcal{L}_\mu$. Simultaneously for all transition systems $\mathcal{T}$ we set:

- ...
- $\|\nu x.\alpha\|_{\mathcal{T}} = \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|\alpha(x)\|_{\mathcal{T}[x \mapsto \mathsf{S}']}\}$
- $\|\mu x.\alpha\|_{\mathcal{T}} = \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha(x)\|_{\mathcal{T}[x \mapsto \mathsf{S}']} \subseteq \mathsf{S}'\}$

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$$\|\nu x.\varphi(x)\|_{\mathcal{T}} = \mathit{GFP}(\|\varphi(x)\|_{\mathcal{T}}) \quad \text{and} \quad \|\mu x.\varphi(x)\|_{\mathcal{T}} = \mathit{LFP}(\|\varphi(x)\|_{\mathcal{T}})$$

Some equivalences

- If x is not in the scope of a modality in $\varphi(x)$ then for all $\mathcal{T}$

$$\|\nu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi(\top)\|_{\mathcal{T}} \quad \text{and} \quad \|\mu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi(\bot)\|_{\mathcal{T}}$$

- For all $\varphi(x, y)$ and all $\mathcal{T}$

$$\|\nu x.\nu y.\varphi(x, y)\|_{\mathcal{T}} = \|\nu x.\varphi(x, x)\|_{\mathcal{T}}$$

$$\|\mu x.\mu y.\varphi(x, y)\|_{\mathcal{T}} = \|\mu x.\varphi(x, x)\|_{\mathcal{T}}.$$

- Every formula φ is equivalent to well-named formula $\mathrm{nf}(\varphi)$.

Classes of Transition Systems

- $\|\varphi\| = \{(\mathcal{T}, s) \ ; \ s \in \|\varphi\|_{\mathcal{T}}\}$
- $\|\varphi\|^r = \{(\mathcal{T}, s) \ ; \ s \in \|\varphi\|_{\mathcal{T}} \text{ and } \mathcal{T} \text{ reflexive}\}$
- Similarly form $\|\varphi\|^t, \|\varphi\|^{st}, \|\varphi\|^{rst}$.

For all $n \in \mathbb{N}$, we define the following classes pointed transisition systems

- $\Sigma_n^{\mu,\mathbb{T}} = \{\|\varphi\| \;;\; \varphi \in \Sigma_n^{\mu}\}$
- $\Pi_n^{\mu,\mathbb{T}} = \{\|\varphi\| \;;\; \varphi \in \Pi_n^{\mu}\}$
- $\Delta_n^{\mu,\mathbb{T}} = \{\|\varphi\| \;;\; \varphi \in \Delta_n^{\mu}\}$

Similarly for $\mathbb{T}^r, \mathbb{T}^t, \mathbb{T}^{st}$ and $\mathbb{T}^{rst}$.

How do we decide if $s \in \|\varphi\|_{\mathcal{T}}$?

Games for the modal μ-calculus

Evaluation game for classical propositional logic

$\mathcal{E}((q \vee r) \wedge p, (\mathcal{T}, s_1))$:

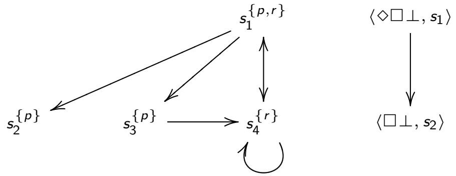
$$\begin{tikzcd}
 & & s_1^{\{p,r\}} \arrow[dll] \arrow[dl] \arrow[d] \\
s_2^{\{p\}} & s_3^{\{p\}} \arrow[r] & s_4^{\{r\}} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\langle (q \vee r) \wedge p, s_1 \rangle \arrow[d] \arrow[dr] & \\
\langle p, s_1 \rangle & \langle q \vee r, s_1 \rangle \arrow[d] \\
 & \langle r, s_1 \rangle
\end{tikzcd}$$

position	player	next position
$\langle p_i, s\rangle$	-	-
$\langle \psi \vee \phi, s\rangle$	**V** chooses between $\langle \psi, s\rangle$ and $\langle \phi, s\rangle$	**V** choice
$\langle \psi \wedge \phi, s\rangle$	**F** chooses between $\langle \psi, s\rangle$ and $\langle \phi, s\rangle$	**F** choice

Evaluation game for modal logic

$\mathcal{E}(\Diamond\Box\bot, (\mathcal{T}, s_1))$:



position	player	next position
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$\langle \psi \vee \phi, s\rangle$	**V** chooses between $\langle \psi, s\rangle$ and $\langle \phi, s\rangle$	**V** choice
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$\langle \Diamond\psi, s\rangle$	**V** chooses a point s' s.t. $s \rightarrow s'$	$\langle \psi, s'\rangle$
$\langle \Box\psi, s\rangle$	**F** chooses a point s' s.t. $s \rightarrow s'$	$\langle \psi, s'\rangle$

Evaluation game for the modal μ-calculus

position	player	next position
$\langle p_i, s\rangle$	-	-
$\langle \psi \vee \phi, s\rangle$	**V** chooses between $\langle \psi, s\rangle$ and $\langle \phi, s\rangle$	**V** choice
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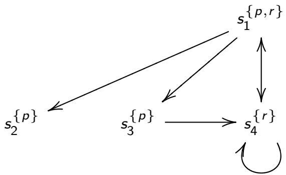
$\langle \Diamond\psi, s\rangle$	**V** chooses a point s' s.t. $s \rightarrow s'$	$\langle \psi, s'\rangle$
$\langle \Box\psi, s\rangle$	**F** chooses a point s' s.t. $s \rightarrow s'$	$\langle \psi, s'\rangle$
$\langle \mu x.\psi, s\rangle$	-	$\langle \psi, s\rangle$
$\langle \nu x.\psi, s\rangle$	-	$\langle \psi, s\rangle$
$\langle x, s\rangle$	-	$\langle \psi_x, s\rangle$

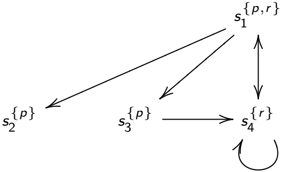
"There is an infinite branch" $\mathcal{E}(\nu x.\Diamond x, (\mathcal{T}, s_1))$:



$$\begin{tikzcd}
\langle \nu x.\Diamond x, s_1 \rangle \arrow[d] \\
\langle \Diamond x, s_1 \rangle \arrow[d] \\
\langle x, s_3 \rangle \arrow[d] \\
\langle \nu x.\Diamond x, s_3 \rangle \arrow[d, dashed] \\
\langle x, s_4 \rangle
\end{tikzcd}$$

"There is branch with infinitely often "p"

$\mathcal{E}(\nu x.\mu y.(p \wedge \Diamond x) \vee \Diamond y, (\mathcal{T}, s_1))$:



$$\langle \nu x.\mu y.(p \wedge \Diamond x) \vee \Diamond y, s_1 \rangle$$

$$\langle p \wedge \Diamond x, s_1 \rangle$$

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$$\langle \mu y.(p \wedge \Diamond x) \vee \Diamond y, s_3 \rangle$$

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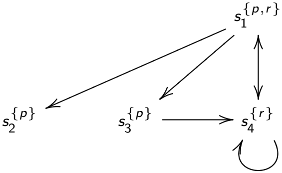

$$\langle \Diamond x, s_4 \rangle$$

$$\langle \nu x.\mu y.(p \wedge \Diamond x) \vee \Diamond y, s_4 \rangle$$

$$\langle \Diamond y, s_4 \rangle$$

$$\langle \mu y.(p \wedge \Diamond x) \vee \Diamond y, s_1 \rangle$$

$$\vdots$$

Game-theoretical version of the “fundamental theorem”

Theorem [Streett Emerson 89]

$s \in ||\varphi||_{\mathcal{T}}$ iff **V** has a winning strategy in $\mathcal{E}(\varphi, (\mathcal{T}, s))$.

Game Formulae

For all $n \geq 1$ we define the Σ_n^μ Game formula $W_{\Sigma_n^\mu}$ and the Π_n^μ Game formula $W_{\Pi_n^\mu}$ such that (n even):

$$W_{\Sigma_n^\mu} :\equiv \mu x_{n+1}.\nu x_n. \ldots \nu/\mu x_2 \Big(\bigvee_{i=2}^{n+1} (d_i \wedge \Diamond x_i) \vee \bigvee_{i=2}^{n+1} (c_i \wedge \Box x_i) \Big)$$

$$W_{\Pi_n^\mu} :\equiv \nu x_{n+2}.\mu x_{n+1}. \ldots \mu/\nu x_3 \Big(\bigvee_{i=3}^{n+2} (d_i \wedge \Diamond x_i) \vee \bigvee_{i=3}^{n+2} (c_i \wedge \Box x_i) \Big)$$

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$$W_{\Pi_n^\mu} :\equiv \nu x_{n+2}.\mu x_{n+1}. \ldots \mu/\nu x_3 \Big(\bigvee_{i=3}^{n+2} (d_i \wedge \Diamond x_i) \vee \bigvee_{i=3}^{n+2} (c_i \wedge \Box x_i) \Big)$$

$W_{\Sigma_n^\mu} \in \Sigma_n^\mu$ and $W_{\Pi_n^\mu} \in \Pi_n^\mu$.

Theorem [Emerson,Jutla (91), Walukiewicz (00)]

Let φ be a Π^μ_n-formula and $(\mathcal{T}, s)$ be a pointed transition system. Player **V** has a winning strategy for $\mathcal{E}(\varphi, (\mathcal{T}, s))$ if and only if $\mathcal{T}(\mathcal{E}(\varphi, (\mathcal{T}, s))) \in \|W_{\Pi^\mu_n}\|$; similarly for Σ^μ_n-formulae.

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Corollary

$$(\mathcal{T}, s) \in \|\varphi\| \quad \Leftrightarrow \quad \mathcal{T}(\mathcal{E}(\varphi, (\mathcal{T}, s))) \in \|W_{\Pi^\mu_n}\|;$$

similarly for Σ^μ_n-formulae.

The Hierarchy on Reflexive Transition Systems

Construct $\mathcal{E}^r(\varphi, (\mathcal{T}, s))$ by making the "moves" relation E reflexive and adapting Ω to Ω^r:

$$\Omega^r(\langle \psi, s \rangle) = \Omega(\langle \psi, s \rangle) \quad \psi \equiv \eta x.\alpha$$

$$\Omega^r(\langle \psi, s \rangle) = \begin{cases} 0 & \text{if } \langle \psi, s \rangle \in V_1 \\ 1 & \text{if } \langle \psi, s \rangle \in V_0. \end{cases} \quad \psi \not\equiv \eta x.\alpha$$

Lemma

Player **V** has a winning strategy for $\mathcal{E}^r(\varphi, (\mathcal{T}, s))$ iff Player **V** has a winning strategy for $\mathcal{E}(\varphi, (\mathcal{T}, s))$.

Reflexive Game formula

For all $n \geq 0$ we define the Σ_n^μ Walukiewicz formula $W_{\Sigma_n^\mu}$ and the Π_n^μ Walukiewicz formula $W_{\Pi_n^\mu}$ such that (n even):

$$W^r_{\Sigma_n^\mu} :\equiv \mu x_{n+1}.\nu x_n.\dots.\nu/\mu x_0\Big(\bigvee_{i=0}^{n+1}(d_i \wedge \Diamond x_i) \vee \bigvee_{i=0}^{n+1}(c_i \wedge \Box x_i)\Big)$$

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$W^r_{\Sigma^\mu_n} \in \Sigma^\mu_{n+2}$ and $W^r_{\Pi^\mu_n} \in \Pi^\mu_{n+2}$.

Proposition

Let $(\mathcal{T}, s)$ be an arbitrary pointed transition system. For all $\varphi \in \Pi_n^\mu$ we have that:

$$\mathcal{T}(\mathcal{E}^r(\varphi, (\mathcal{T}, s))) \in \|W^r_{\Pi_n^\mu}\| \text{ if and only if } (\mathcal{T}, s) \in \|\varphi\|.$$

and analogously for $W^r_{\Sigma_n^\mu}$.

Theorem

For all natural numbers $n \in \mathbb{N}$ we have that

$$\Sigma_n^{\mathbb{T}^r} \subsetneq \Sigma_{n+1}^{\mathbb{T}^r} \quad \text{and} \quad \Pi_n^{\mathbb{T}^r} \subsetneq \Pi_{n+1}^{\mathbb{T}^r}.$$

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Proof

Else for all k

$$\Sigma_n^{\mathbb{T}^r} = \Sigma_{n+k}^{\mathbb{T}^r} = \Pi_n^{\mathbb{T}^r} = \Pi_{n+k}^{\mathbb{T}^r}$$

and $\|W^r_{\Sigma_n^\mu}\|^r \in \Pi_n^{\mathbb{T}^r}$ or $\|\neg W^r_{\Sigma_n^\mu}\|^r \in \Sigma_n^{\mathbb{T}^r}$.

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$$(\mathcal{T}^F, s^F) \in \|\neg W^r_{\Sigma_n^\mu}\| \quad \text{iff} \quad \mathcal{T}(\mathcal{E}^r(\neg W^r_{\Sigma_n^\mu}, (\mathcal{T}^F, s^F))) \in \|W^r_{\Sigma_n^\mu}\|$$

The Hierarchy on transitive and symmetric Transition Systems

Lemma

Let $\mathcal{T}$ be a transitive transition system and let $s' \in \mathrm{scc}(s)$. For all μ-formulae φ we have that

$$s \in \| \triangle \varphi \|_{\mathcal{T}} \quad \text{iff} \quad s' \in \| \triangle \varphi \|_{\mathcal{T}}$$

where $\triangle \in \{\Box, \Diamond\}$.

Theorem

Let $\mathcal{T}$ be a transitive and symmetric transition system. We have that

$$\|\nu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi(\varphi(\top))\|_{\mathcal{T}}.$$

The syntactical translation $(.)^t : \mathcal{L}_\mu \to \mathcal{L}_\mathrm{M}$ is defined as:

- ...
- $(\mu x.\varphi)^t = (\varphi(\varphi(\bot)))^t$
- $(\nu x.\varphi)^t = (\varphi(\varphi(\top)))^t$

Corollary

On transitve and symmetric (and reflexive) transition systems we have that

$$\|\varphi\|_\mathcal{T} = \|\varphi^t\|_\mathcal{T}.$$

The Hierarchy on transitive Transition Systems

Lemma

Let $\mathcal{T}$ be a transitive transition system and let s, s' be two states such that $s \rightarrow^{\mathcal{T}} s'$. For all μ-formulae φ we have that

$$s \in \|\Box\varphi\|_{\mathcal{T}} \quad \Longrightarrow \quad s' \in \|\Box\varphi\|_{\mathcal{T}} \quad \text{and}$$

$$s' \in \|\Diamond\varphi\|_{\mathcal{T}} \quad \Longrightarrow \quad s \in \|\Diamond\varphi\|_{\mathcal{T}}.$$

Theorem

Let $\mathcal{T}$ be a transitive transition system and let $\nu x.\varphi(x)$ be a formula such that x is in the scope of a $\Box$ modality. We have that

$$\|\nu x.\varphi(x)\|_{\mathcal{T}} = \|\varphi(\varphi(\top))\|_{\mathcal{T}}.$$

$\tau : \mathcal{L}_\mu \to \mathcal{L}_\mu$ is defined as:

- ...
- $\tau(\mu x.\varphi) = \tau(\varphi(\varphi(\bot)))$, x is in the scope of a $\Diamond$ in φ
- $\tau(\mu x.\varphi) = \mu x.\tau(\varphi)$, x is not in the scope of a $\Diamond$ in φ
- $\tau(\nu x.\varphi) = \tau(\varphi(\varphi(\top)))$, x is in the scope of a $\Box$ in φ
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- $\tau(\nu x.\varphi) = \nu x.\tau(\varphi)$, x is not in the scope of a $\Box$ in φ

Corollary

On transitve transition systems we have that

$$\|\varphi\|_{\mathcal{T}} = \|\tau(\varphi)\|_{\mathcal{T}}.$$

Notation and Definitions

- Let $\varphi(x_1, \ldots, x_n)$ be a formula by φ^{x_i} we denote the fomrmula obtained by cutting all branches except x_i.

$$(\nu x.\mu y.\mu z.(\Box x \wedge \Diamond y) \vee (\Diamond z \wedge p))^x \equiv \nu x.\Box x \vee p$$

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$$(\nu x.\mu y.\mu z.(\Box x \wedge \Diamond y) \vee (\Diamond z \wedge p))^x \equiv \nu x.\Box x \vee p$$

- For all set of variables X the formula $\varphi^{\mathrm{free}(X)}$ is the formula obtained from φ by eliminating all quantifiers binding a variable $x \in X$ but leaving the previously bound variable x as a free occurrence.

$$(\nu x.\mu y.\mu z.(\Box x \wedge \Diamond y) \vee (\Diamond z \wedge p))^{\mathrm{free}(x,y,z)} \equiv (\Box x \wedge \Diamond y) \vee (\Diamond z \wedge p)$$

For each sequence of $\langle x_1, \ldots, x_k \rangle$ with $x_j \in \mathrm{bound}(\varphi)$, we define the formula $\varphi^{\langle x_1, \ldots, x_k \rangle}$ as follows:

$$\varphi^{\langle x_1 \rangle} :\equiv \varphi^{x_1}$$

and

$$\varphi^{\langle x_1, \ldots, x_k, x_{k+1} \rangle} :\equiv \varphi^{\langle x_1, \ldots, x_k \rangle}[x_k / \varphi^{x_{k+1}}_{x_k}].$$

- Let φ be a μ-formula and $X, Y \subset \mathrm{bound}(\varphi)$. $\mathrm{Path}^{X \to Y}(\varphi)$ is the smallest set such that for all $x \in X$

$$\{\langle x, y\rangle \ ; \ \ y \in \mathrm{free}(\varphi_x) \text{ and } y \in Y\} \subseteq \mathrm{Path}^{X \to Y}(\varphi)$$

and such that if $\langle x_1, \dots, x_m, y\rangle \in \mathrm{Path}(\varphi)$, if $x' \in X$, if $x' \notin \{x_1, \dots, x_m\}$ and if $x_1 \in \mathrm{free}(\varphi_{x'})$ then

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- For all $x \in X$ we define

$$\mathrm{Path}^{x \to Y}(\varphi) = \{\langle x_1, \dots, x_k, y\rangle \in \mathrm{Path}^{X \to Y}(\varphi) \ ; \ x_1 \equiv x\}.$$

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- The formula $\varphi^{x_i \to Y}$ is defined such that
$$\varphi^{x_i \to Y} \equiv \bigvee_{\mathbf{s} \in \text{Path}^{x_i \to Y}} \varphi^{\mathbf{s}}_{x_i}.$$

The *unfolding of X in ψ as subformula of φ*, $\mathrm{unf}_{\varphi}^{X}(\psi)$, is the formula defined recursively such that

$$\mathrm{unf}_{\varphi}^{\{x_1\}}(\psi) \equiv \psi[x/\varphi_x]$$

and such that if $X = \{x_1, \ldots, x_n\}$ then

$$\mathrm{unf}_{\varphi}^{X}(\psi) \equiv \psi[x_1/\mathrm{unf}_{\varphi}^{X^{-1}}(\varphi_{x_1}), \ldots, x_n/\mathrm{unf}_{\varphi}^{X^{-n}}(\varphi_{x_n})]$$

where $X^{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$.

The Translation

$\varphi \in \Sigma_2^\mu$ with $\{x_1, \ldots, x_n\} = X$ all μ-variables and $\{y_1, \ldots, y_m\} = Y$ all ν-variables. We define $\rho(\varphi) \in \Delta_2^\mu$ as

$$\varphi^{\mathrm{free}(X)}[x_1/\varphi^{x_1 \to Y} \vee \mathrm{unf}^X_{\varphi^{-Y}}(\varphi^{-Y}_{x_1}), \ldots, x_n/\varphi^{x_n \to Y} \vee \mathrm{unf}^X_{\varphi^{-Y}}(\varphi^{-Y}_{x_n})].$$

Lemma

Let $\mathcal{T}$ be a transitive transition system, and let $\varphi \in \Sigma_2^\mu$ such that all ν-variables (resp. μ-variables) x are in the scope of only $\Diamond$ (resp. $\Box$). Then we have

$$\|\varphi\|_\mathcal{T} = \|\rho(\varphi)\|_\mathcal{T}$$

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Proof

Show the existence of a normal form for winning plays for player **V** of $\mathcal{E}(\varphi, (\mathcal{T}, s))$ and show that these plays are winning for **V** in $\mathcal{E}(\rho(\varphi), (\mathcal{T}, s))$; and vice versa.

$R : \mathcal{L}_\mu \to \Delta_2^\mu$ is defined as

- ...
- $R(\mu x.\varphi) = \rho(\mathrm{nf}(\mu x.(R(\varphi))))$
- $R(\nu x.\varphi) = \neg(R(\mu x.\neg\varphi[x/\neg x]))$

Theorem

For all $\varphi \in \mathcal{L}_\mu$ and all transitive transition systems $\mathcal{T}$ we have that

$$\|\varphi\|_{\mathcal{T}} = \|R(\tau(\varphi))\|_{\mathcal{T}}.$$

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Questions or Remarks?