Elite Master Course: Theoretical and Mathematical Physics

Dressing of Charges in Classical Field Theory

Master Thesis

Submitted by

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Abstract

In this thesis we carry out a study of the dressing of point charges in classical field theory. We therefore provide a detailed analysis of the massless wave equation as a differential equation of distributions on space-time. Our main focus lies on the regularity structure of certain types of solutions. This leads us to introduce the notion of time foilable distributions and to develop tools to analyze them. Our formalism provides a rigorous understanding of the "germination" of asymptotic fields and offers insights into the complications that arise in the construction of scattering states in the quantized version of the theory.

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Chapter 1

Introduction

This thesis is dedicated to the analysis of the regularity of fields and potentials in classical electrodynamics. It is performed within the theory of distributions on space-time, which, in addition to smooth charge distributions, also enables the treatment of singular sources such as point particle trajectories. A thorough understanding of these fields and their potentials is crucial for the construction of theories that couple them dynamically to charges. So far, no fully satisfactory formulation has been found for point charges which incorporates the empirical effects one would like to address within the range of the presented theory. Furthermore, the insights in this thesis go beyond classical theories as the regularity of the potentials plays an important role in quantum field theoretical models. We survey established results in various models in the following section to outline the motivation for the questions addressed in the subsequent chapters. These will be summarized in Section 1.2 followed by an overview of the structure of the thesis.

1.1 Survey of Models in Classical Electrodynamics

We consider models of classical electrodynamics involving the coupled dynamics of charge distributions and fields or potentials. In these models, the charge distributions are restricted to the form of a finite number of charged particles. These are considered as either point-like or extended but compactly confined clouds without any internal or rotational degree of freedom. Before stating our objectives in Section 1.2 below, we provide a short introduction to the models considered in this study.

1.1.1 Naive Maxwell-Lorentz Electrodynamics: This is the text-book theory of relativistic point-charges coupled to electromagnetical fields which is not well-defined. This fact is not just a mathematical curiosity but indeed challenges the formulation of theories of this character in physics.

1.1.2 Pseudo-Relativistic Abraham's Model: A model that replaces the point particles by extended but compactly supported charge clouds. The dynamical equations are well-defined and many features of their solutions are rigorously understood. In particular, an effective self-interaction leads to the effect of radiational damping. However, indications exist that the dynamics is not empirically adequate when considering charge clouds of radii below a certain threshold.

1.1.3 Maxwell's Fields: We consider the charge distribution as given and regard only the dynamics of the electromagnetical fields. The equations of motion can be regarded in a generalized setting allowing for a rigorous discussion of the regularity of solutions even in the case of point particles. This provides insights into the challenges occurring when constructing coupled theories.

1.1.4 Scalar Abraham Model: By replacing the interaction via electromagnetical fields or respectively via their gauge potentials by one scalar potential while considering non-relativistic particles, the pseudo-relativistic Abraham model reduces to a much simpler coupled theory of charged extended particles interacting with one potential. Still, the solution carries much resemblance with the former model.

1.1.5 Scalar Potential: Similar to the case of Maxwell's fields the dynamics of the scalar potential for point particle sources can be regarded in some generalized setting. This renders their discussion valuable for the coupled theory.

1.1.1 Naive Maxwell-Lorentz Electrodynamics

As typical for a theory with its origin in the 19th century, Maxwell-Lorentz electrodynamics does not only deal with observable quantities but proposes a realm of beables and their dynamics in a static spacetime.

Matter is modeled via point particles which are assigned positions in three dimensional space $\mathbf{r}(t)$ for any given time point t. Since the law of motion is not governed by some static potential but via entities $\mathbf{E}(t)$ and $\mathbf{B}(t)$ which are themselves dynamic with regard to time, they are also usually attributed ontological character. Thus, electrodynamics in its orthodox formulation is a theory about particles and fields. The full dynamical law of these entities is constructed by coupling the Lorentz und the Maxwell equations and considering them as an initial value problem in the following way:

Theory 1 Naive Maxwell-Lorentz Electrodynmaics

The **ontological basis** of naive Maxwell-Lorentz electrodynamics consists of

- a finite number of $N \in \mathbb{N}$ point particles, each individually characterized by an index $i \in [N]$, a mass m_i, a charge q_i and time dependent position $\mathbf{r}_i : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}_i(t)$
- and two fields $(\mathbf{E}, \mathbf{B}) : \mathbb{R} \times \mathbb{R}^3 \to (\mathbb{R}^3 \times \mathbb{R}^3), (t, \boldsymbol{x}) \mapsto \big(\mathbf{E}(t, \boldsymbol{x}), \mathbf{B}(t, \boldsymbol{x})\big)$.

Its **dynamics** is governed by the following set of differential equations

Maxwell's equations

$$\begin{aligned} \partial_t \mathbf{E}(t, \mathbf{x}) &= -\nabla \times \mathbf{B}(t, \mathbf{x}) \\ \partial_t \mathbf{B}(t, \mathbf{x}) &= \nabla \times \mathbf{E}(t, \mathbf{x}) - \mathbf{j}(t, \mathbf{x}) \\ \nabla \cdot \mathbf{E}(t, \mathbf{x}) &= \rho(t, \mathbf{x}) \\ \nabla \cdot \mathbf{B}(t, \mathbf{x}) &= 0 \end{aligned}$$

Lorentz' equations

$$\frac{d}{dt} m_i \gamma\big(\dot{\mathbf{r}}_i(t)\big)\dot{\mathbf{r}}_i(t) = q_i \begin{pmatrix} \mathbf{E}\big(t, \mathbf{r}_i(t)\big) \\ +\dot{\mathbf{r}}_i(t) \times \mathbf{B}\big(t, \mathbf{r}_i(t)\big) \end{pmatrix}$$

for all particles $i \in [N]$ with $\gamma(\mathbf{v}) := 1/\sqrt{1 - \mathbf{v}^2} \in \mathbb{R}_{>0}$

which are coupled by $\rho(t, \boldsymbol{x}) = \sum_{i=0}^{N} q_i \delta\big(\mathbf{x} - \mathbf{r}_i(t)\big)$ and $\mathbf{j}(t, \boldsymbol{x}) = \sum_{i=0}^{N} q_i \dot{\mathbf{r}}_i(t) \delta\big(\boldsymbol{x} - \mathbf{r}_i(t)\big)$.

This theory is usually understood as an **initial value problem**, which, by providing at some fixed time $t_0 \in \mathbb{R}$ initial

- particle positions and momenta $\big(\mathbf{r}_i(t_0), \mathbf{v}_i(t_0)\big) \in \mathbb{R}^3 \times \mathbb{R}^3$ for all particles $i \in [N]$
- and field configurations $\big(\mathbf{E}(t_0, \cdot), \mathbf{B}(t_0, \cdot)\big) : \mathbb{R}^3 \to (\mathbb{R}^3 \times \mathbb{R}^3), \boldsymbol{x} \mapsto \big(\mathbf{E}(t_0, \boldsymbol{x}), \mathbf{B}(t_0, \boldsymbol{x})\big)$

fulfilling the conditions $|\mathbf{v}_i(t_0)| < 1$, $\nabla \cdot \mathbf{E}(t_0, \boldsymbol{x}) = \rho(t_0, \boldsymbol{x})$ and $\nabla \cdot \mathbf{B}(t_0, \boldsymbol{x}) = 0$, is expected to yield $\boldsymbol{r}_i$, $\mathbf{E}$ and $\mathbf{B}$ at all other time instances.

Breakdown

This vague idea of a theory cannot be solidified without major modifications. A broad overview of the occuring problems in such an attempt is given in [Spo04, Chapter 2.3]. In a nutshell: Since in this model the fields are sourced by the charged particles, the point like nature of their charge distribution necessitate singularities of the fields. The Lorentz equation determines the particle evolution via evaluating the fields at these points in space. There is, thus, no canonical way to grasp the acceleration of the charged particle by those fields.

An additional mechanism needs to be put in place to complement the ideas sketched above to a coherent picture. Some ideas result in a substantive shift in perception even on the ontology (like Wheeler-Feynmann-Electrodynmaics) while others deviate less from the painted picture. In the present study,we follow extentions of the ideas of [Abr14].

1.1.2 Pseudo-Relativistic Abraham's Model

Abraham's idea is to eliminate the generation of the singularities occurring in the laws of dynamics by substituting the point like particles with extended charged clouds.[1] Their charge density distribution is modelled by a common fixed function $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R}_{\geq 0})$ spherically symmetric with $\int_{\mathbb{R}^3} d^3x \varrho(\boldsymbol{x}) = 1$ in some fixed reference frame not susceptible to any dynamics. This breaks special-relativistic covariance and opens up a huge parameter space $(C_c^\infty(\mathbb{R}^3, \mathbb{R}_{\geq 0}))$ but results in a theory whose complicated coupled dynamics can be understood in certain cases.
In the context of other more drastic modifications, this quite minimal replacement leaves us with a relatively similar ontology. The equations of motions are modified to resemble interaction of rigid bodies with electrodynmaical fields. Neither internal nor rotational degrees of freedom are implemented.

Theory 2 Pseudo-relativistic Abraham's Model

The **ontological basis** of the relativistic Abraham Model consists of

- a finite number of $N \in \mathbb{N}$ ϱ-shaped particles each individually characterized by an index $i \in [N]$, a mass $m_i > 0$, a charge q_i and time dependent position $\mathbf{r}_i : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}_i(t)$
- and two fields $(\mathbf{E}, \mathbf{B}) : \mathbb{R} \times \mathbb{R}^3 \to (\mathbb{R}^3 \times \mathbb{R}^3), (t, \boldsymbol{x}) \mapsto \big(\mathbf{E}(t, \boldsymbol{x}), \mathbf{B}(t, \boldsymbol{x})\big)$.

The **dynamics** is governed by the following set of differential equations

Maxwell's equations

$$\begin{aligned} \partial_t \mathbf{E}(t, \mathbf{x}) &= -\nabla \times \mathbf{B}(t, \mathbf{x}) \\ \partial_t \mathbf{B}(t, \mathbf{x}) &= \nabla \times \mathbf{E}(t, \mathbf{x}) - \mathbf{j}(t, \mathbf{x}) \\ \nabla \cdot \mathbf{E}(t, \mathbf{x}) &= \rho(t, \mathbf{x}) \\ \nabla \cdot \mathbf{B}(t, \mathbf{x}) &= 0 \end{aligned}$$

Lorentz' equations

$$\frac{d}{dt} m_i \gamma\big(\dot{\mathbf{r}}_i(t)\big)\dot{\mathbf{r}}_i(t) = q_i \begin{pmatrix} \varrho * \mathbf{E}\big(t, \mathbf{r}_i(t)\big) \\ +\dot{\mathbf{r}}_i(t) \times \varrho * \mathbf{B}\big(t, \mathbf{r}_i(t)\big) \end{pmatrix}$$

for all particles $i \in [N]$ with $\gamma(\mathbf{v}) := 1/\sqrt{1 - \mathbf{v}^2} \in \mathbb{R}_{>0}$

which are coupled by $\rho(t, \boldsymbol{x}) = \sum_{i=1}^N q_i \varrho\big(\boldsymbol{x} - \mathbf{r}_i(t)\big)$ and $\mathbf{j}(t, \boldsymbol{x}) = \sum_{i=1}^N q_i \dot{\mathbf{r}}_i(t) \varrho\big(\boldsymbol{x} - \mathbf{r}_i(t)\big)$.

This theory is usually understood as an **initial value problem** which by providing at some fixed time $t_0 \in \mathbb{R}$ initial

- particle positions and momenta $\big(\mathbf{r}_i(t_0), \mathbf{v}_i(t_0)\big) \in \mathbb{R}^3 \times \mathbb{R}^3$ for all particles $i \in [N]$
- and field configurations $\big(\mathbf{E}(t_0, \cdot), \mathbf{B}(t_0, \cdot)\big) : \mathbb{R}^3 \to (\mathbb{R}^3 \times \mathbb{R}^3), \boldsymbol{x} \mapsto \big(\mathbf{E}(t_0, \boldsymbol{x}), \mathbf{B}(t_0, \boldsymbol{x})\big)$

fulfilling the conditions $|\mathbf{v}_i(t_0)| < 1$, $\nabla \cdot \mathbf{E}(t_0, \boldsymbol{x}) = \rho(t_0, \boldsymbol{x})$ and $\nabla \cdot \mathbf{B}(t_0, \boldsymbol{x}) = 0$ is expected to yield $\boldsymbol{r}_i$, $\mathbf{E}$ and $\mathbf{B}$ at all other time instances.

Despite the complexity there are several properties of this model proved around the 2000s with mathematical rigor. Many restrict to the one particle model which we will focus on in the following. A short overview will follow which is a subset of the survey in [Spo04, Chapter 2.4, Chapter 5].
The statements are formulated in a setting where the fields $\mathbf{E}$ and $\mathbf{B}$ are understood as $\mathbf{E}, \mathbf{B} : \mathbb{R} \to \mathscr{M}^2 \subset L^2(\mathbb{R}^3)^2, t \mapsto (\mathbf{E}_t, \mathbf{B}_t) := \big(\boldsymbol{x} \mapsto (\mathbf{E}(t, \boldsymbol{x}), \mathbf{B}(t, \boldsymbol{x}))\big)$ such that the spatial derivatives of Maxwell's equations are understood in a weak sense in $\mathscr{M}$ and time derivatives are constructed by covering the linear superset of $\mathscr{M}$ with a suitable method-dependent topology.

Initial Value Dynamics

For a single particle and large classes of initial configurations $\big(\mathbf{r}(t_0), \mathbf{v}(t_0), \mathbf{E}(t_0, \cdot), \mathbf{B}(t_0, \cdot)\big)$ fulfilling the constraints in the initial value formulation and having finite field energy global existence of solutions in the mentioned sense is granted by the laws of dynamics. Proofs using different techniques can be found in [KS00], [BD01] and [BDD13].

[1]This dates back to before Einstein's special relativity. Abraham's non relativistic model of extended electrons is presented in his book "Theorie der Elektrizität II" [Abr14, Chapter 3].

Radiation Damping

A prominent feature of classical electrodynamics is Bremsstrahlung. The by energy-loss expected friction force counteracting acceleration of the charged particle is a result of Abraham's model. A reliable rigorous result on the relaxation to constant motion $\lim_{t\to\infty} \ddot{\mathbf{r}}(t) = 0$ has been achieved in [KS00] by the implementation of additional restrictions of the initial field configurations and the charge density. Scale separation (overview in [Spo04, Chapter 7]) arguments allow to extract the Abraham–Lorentz force used in heuristic approaches to model self-force.

Dressing: The Soliton

Due to the coupling of the naked charge and the electromagnetical fields, a freely moving charge is expected to carry along an accompanying field. Dynamics in the Abraham model with one charge following straight lines can be found in [Spo04, Chapter 4.1]. Considered in the rest frame of the electron, the accompanying fields are the usual, by the transformed charge distribution smeared, Coulomb fields of a point particle at rest. This dressed charge moving constantly on a straight line is called soliton.
Based on the observation of the relaxation to constant movement by radiational damping, [IKM04] proofed a rigorous result about the convergence of the dynamics to the soliton when initiated by field configurations with additional decay behavior and the before mentioned condition on the charge distribution. By considering situations of slowly varying external potentials, one can extract an effective dynamics of the soliton. This is achieved in [Spo04, Chapter 7] by taking advantage of the scale separation between the resulting external force and the dimension of the charge distribution.

Mass-Renormalization and Runaways

In the before mentioned effective dynamics, the mass of the soliton is constituted by two terms. One part is due to the bare mass, i.e. the parameter m in Abraham's model. The second contribution can be traced back to the inertia of the electromagnetical dressing field which gets carried along. It increases when the charge distribution shrinks towards a point it increases and limits at an infinite value. But even before the limiting point there are hints for an arising dynamical instability of the model. Since the effective mass needs to be compared with experimental data, the bare mass is adjusted accordingly. If the charge distribution has a characteristic diameter below the classical electron radius, the field contribution forces the bare mass to become negative. The methods in [BD01] still guarantee the existence of the dynamics under the stated assumptions on the initial conditions but there seem to occur so called runaways, situations in which the electron exponentially accelerates, which can be considered unphysical.

1.1.3 Maxwell's Fields

The analysis of the Abraham model is very challenging. Many aspects about the complications of the fully coupled theory can be studied by restricting oneself to the scenario where the dynamics of the charge is considered given and restricting one analysis to the generated fields. We will see in the following that in this case we do not need to restrict ourselves to extended charges but are also able to cope with the point particle. Thus, we state the model for now in a vague way.

Theory 3 **Particle-Sourced Maxwell Fields**

The **ontological basis** in this model consists of

- one ϱ-shaped particle (either extended or point like) with charge q and a given smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ such that there exists $v_{max} < 1$ with $|\dot{\mathbf{r}}(t)| \leq v_{max}$ for all $t \in \mathbb{R}$
- and two fields $(\mathbf{E}, \mathbf{B}) : \mathbb{R} \times \mathbb{R}^3 \to (\mathbb{R}^3 \times \mathbb{R}^3), (t, \boldsymbol{x}) \mapsto \big(\mathbf{E}(t, \boldsymbol{x}), \mathbf{B}(t, \boldsymbol{x})\big)$.

The **dynamics** of the particle is considered as externally fixed while the fields are governed by the following set of differential equations

Maxwell's equations

$$\partial_t \mathbf{E}(t, \mathbf{x}) = -\nabla \times \mathbf{B}(t, \mathbf{x})$$

$$\partial_t \mathbf{B}(t,\mathbf{x}) = \nabla \times \mathbf{E}(t,\mathbf{x}) - \mathbf{j}(t,\mathbf{x})$$
$$\nabla \cdot \mathbf{E}(t,\mathbf{x}) = \rho(t,\mathbf{x})$$
$$\nabla \cdot \mathbf{B}(t,\mathbf{x}) = 0$$

which are sourced by $\rho(t,\boldsymbol{x}) = \sum_{i=1}^{N} q_i \varrho\big(\boldsymbol{x} - \mathbf{r}_i(t)\big)$ and $\mathbf{j}(t,\boldsymbol{x}) = \sum_{i=1}^{N} q_i \dot{\mathbf{r}}_i(t) \varrho\big(\boldsymbol{x} - \mathbf{r}_i(t)\big)$.

The theory is usually understood as an **initial value problem** which, by providing at some fixed time $t_0 \in \mathbb{R}$ initial field configurations $\big(\mathbf{E}(t_0,\cdot), \mathbf{B}(t_0,\cdot)\big) : \mathbb{R}^3 \to (\mathbb{R}^3 \times \mathbb{R}^3), \boldsymbol{x} \mapsto \big(\mathbf{E}(t_0,\boldsymbol{x}), \mathbf{B}(t_0,\boldsymbol{x})\big)$ fulfilling the conditions $\nabla \cdot \mathbf{E}(t_0,\boldsymbol{x}) = \rho(t_0,\boldsymbol{x})$ and $\nabla \cdot \mathbf{B}(t_0,\boldsymbol{x}) = 0$, is expected to yield $\mathbf{E}$ and $\mathbf{B}$ at all other time instances.

Note that ϱ, $\mathbf{E}$ and $\mathbf{B}$ will not turn out to be functions in an ordinary sense in the case of an point particle. Thus consider the introduction to the model rather as an heuristic sketch.

Extended Particle

In the case of a smooth charge distribution $\varrho \in C_c^\infty(\mathbb{R}^3)$ and differentiable initial field configurations, Maxwell's equations can be understood in the regular sense as partial derivatives of functions.

Initial Value Dynamics: The dynamics of the $\mathbf{E}$ and $\mathbf{B}$ fields initiated by smooth configurations is unfolded by Kirchhoff's formula as can be seen by a reformulation of the equations of motion. This has been shown in [Dec10, Theorem 4.14].

Dressing: In [Dec10, Theorem 4.14] Deckert proved in addition, that solutions pointwisely approach some fixpoint of the dynamics with increasing temporal distance to the initial time t_0 if the initial fields obey some decay behavior at infinity ([Dec10, Theorem 4.18]). In physics, this solution is referred to as the Liénard-Wiechert fields. As discussed in the coupled Abraham model, the radiational damping forces the particle on straight lines resulting in an relaxation to the soliton. Considering particle trajectories in this model resembling such kind of behavior one can discuss a relaxation to the boosted and smeared Coulomb fields.

Germination of the Dressing: In particular [Dec10, Theorem 4.18] does not impose the constraints of the initial value problem, i.e. $\nabla \cdot \mathbf{E}(t_0,\boldsymbol{x}) = \rho(t_0,\boldsymbol{x})$ and $\nabla \cdot \mathbf{B}(t_0,\boldsymbol{x}) = 0$, on the initial fields. Thus, it includes the special case of considering $\mathbf{E}(t_0,\boldsymbol{x}) = \mathbf{0}$ and $\mathbf{B}(t_0,\boldsymbol{x}) = \mathbf{0}$ at $t_0 \in \mathbb{R}$. In this case the theorem additionally provides the pointwise convergence towards the Liénard-Wiechert fields. This germination of the dressing out of the vacuum can be used as a scheme of generating asymptotic fields. Especially, when considering theories of complicated coupled dynamics without any exact solutions available, this method becomes fruitful.

Point Particle

One can also say a lot about this model in the case of a point particle. The linearity of Maxwell's equations in $\mathbf{E}, \mathbf{B}$ and ϱ allows to assign them a concrete meaning beyond evaluatable functions on $\mathbb{R}^4$ which gives insights into the challenges of the construction of an completely coupled model. A detailed analysis has been performed in [Har18] where the charge distribution is considered as a δ distribution $\varrho = \delta \in \mathscr{D}'(\mathbb{R}^3)$ and the time dependent fields as time dependent functions of spatial distributions $(\mathbf{E}, \mathbf{B}) : \mathbb{R} \to \big((\mathscr{D}'(\mathbb{R}^3))^3 \times \mathscr{D}'(\mathbb{R}^3)^3\big)$ with a certain strong notion of $\mathfrak{C}$-differentiabilityDef. 6.1.3 with respect to the time parameter.

Initial Value Dynamics: The existence of the dynamics initiated by initial values of a special form has been proven in [Har18, Lemma 4.2.3]. There, initial field configurations were regarded which are retarded Liénard-Wiechert fields sourced by an arbitrary charge trajectory in the past that is only constraint to coincide with $\mathbf{r}(t_0)$ at t_0.

Shocks: [Har18] actually treats a coupled system of finitely many point charges coupled via electrodynamical fields. In the model, self-interaction terms expected to lead to radiational damping are neglected. This model raises concerns towards the assumption that electrodynamics of point particles can be understood as an initial value problem. The argument is built upon the observation that most initial configurations lead to reproducing singular wavefronts resulting in non differentiable particle trajectories.

Gauge Potentials

When regarding implications of the preceding discussions in quantum theories of electrodynamics, the dynamical laws should be rewritten in their hamiltonian form. The orthodox way of doing so is by introducing of gauge potentials $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\Phi \quad \mathbf{B} = \nabla \times \mathbf{A}\,,$$

and the following lagrangian:

$$L(\mathbf{r}, \Phi, \mathbf{A})(t, \boldsymbol{x}) = -\sum_{i=1}^{N} \left(m_i \sqrt{1 - \dot{\mathbf{r}}(t)^2} - q_i \Big(\varrho * \Phi\big(t, \mathbf{r}(t)\big) - \dot{\mathbf{r}}(t) \cdot \varrho * \mathbf{A}\big(t, \mathbf{r}(t)\big) \Big) \right) + \frac{1}{2} \int_{\mathbb{R}^3} d^3x \left(\Big(\nabla\Phi(t, \boldsymbol{x}) + \partial_t \mathbf{A}(t, \boldsymbol{x}) \Big)^2 - \big(\nabla \times \mathbf{A} \big)^2 \right).$$

The Euler-Lagrange equation agree with the equations of motion of Theory 2, i.e. the pseudo-relativistic Abraham model. Due to the so called gauge ambiguity in the potentials Φ and $\mathbf{A}$, they do not directly represent physical degrees of freedom. Thus the transition to the hamiltonian formulation needs additional treatment which we will circumvent by regarding a simpler model.

1.1.4 Scalar Abraham Model

From now on we will consider a simpler model which resembles many features of classical electrodynamics in the sense of its mathematical structure. This model consists of particles interacting with just a single potential. The particles are going to be treated non-relativistically and, in anticipation of otherwise ill-defined dynamics, as ϱ-shaped. The interaction is mediated by some scalar field which plays the role of a potential for the particle.

Theory 4 Scalar Abraham Model

The **ontological basis** of this model with scalar interactions consists of

- a finite number of $N \in \mathbb{N}$ ϱ-shaped particles each individually characterized by an index $i \in [N]$, a mass $m_i > 0$, a charge q_i and time dependent position $\mathbf{r}_i : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}_i(t)$
- and one field $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, (t, \boldsymbol{x}) \mapsto \phi(t, \boldsymbol{x})$.

The **dynamics** is governed by the following set of differential equations

Wave equation	**Particle equation**
$\Box\phi(t, \boldsymbol{x}) = \rho(t, \boldsymbol{x})$	$\frac{d}{dt} m_i \dot{\mathbf{r}}_i(t) = -q_i \nabla \varrho * \phi(t, \mathbf{r}_i)$ for all particles $i \in [N]$

which are coupled by $\rho(t, \boldsymbol{x}) = \sum_{i=1}^{N} q_i \varrho\big(\boldsymbol{x} - \mathbf{r}_i(t)\big)$.

This theory is usually understood as an **initial value problem** which, by providing at some fixed time $t_0 \in \mathbb{R}$ initial

- particle positions and momenta $\big(\mathbf{r}_i(t_0), \mathbf{v}_i(t_0)\big) \in \mathbb{R}^3 \times \mathbb{R}^3$ for all particles $i \in [N]$
- and field configuration $\big(\phi(t_0, \cdot), \dot{\phi}(t_0, \cdot)\big) : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}, \boldsymbol{x} \mapsto \big(\phi(t_0, \boldsymbol{x}), \dot{\phi}(t_0, \boldsymbol{x})\big)$,

is expected to yield $\boldsymbol{r}_i$ and ϕ at all other time instances.

Comparison to Electrodynamics

In this theory of scalar interaction, the field ϕ is to be compared with the potentials Φ and $\mathbf{A}$ in classical electrodynamics. When applying the Lorenz partial gauge fix $\partial_t\Phi + \nabla \cdot \mathbf{A} = 0$, the equations of motion for the fully coupled Abraham model (with non-relativistic particles) in term of the potentials reduce to

$$\textbf{Maxwell's equations:} \quad \begin{aligned} \Box\Phi(t,\boldsymbol{x}) &= \rho(t,\boldsymbol{x}) \\ \Box\mathbf{A}(t,\boldsymbol{x}) &= \mathbf{j}(t,\boldsymbol{x}) \end{aligned}$$

$$\textbf{Lorentz' equations:} \quad \frac{d}{dt} m_i \dot{\mathbf{r}}_i(t) = -q_i \nabla \varrho * \Phi(t, \mathbf{r}_i) + q_i \begin{pmatrix} -\varrho * \partial_t \mathbf{A}\big(t, \mathbf{r}_i(t)\big) \\ +\dot{\mathbf{r}}_i(t) \times \big(\varrho * \nabla \times \mathbf{A}\big)\big(t, \mathbf{r}_i(t)\big) \end{pmatrix}$$

which "contains" the scalar model when restricting to $\mathbf{A} = 0$ and setting $\phi = \Phi$. This, however, is a questionable "limit" of the theory since $\mathbf{A}_0 \approx 0 \approx \dot{\mathbf{A}}_0$ at t_0 does not need to imply that $\mathbf{A}$ and its derivatives stay small since it is dynamically coupled to Φ. Further, there is a left over gauge freedom which allows to shovel over parts from Φ to $\mathbf{A}$. Still, this model, and in particular its quantized version, provides a valuable setting to test ideas which are to be applied to electrodynamics later. We will argue towards the end of the thesis that for the again simplified model of analysis the connection is quite close and the characterized effects are expected to carry over.

1.1.5 Scalar Potential

In section 1.1.3 we have seen that it is possible to learn a lot about the problems occurring in the coupled system of particles and fields by analyzing the fields sourced by some given charge trajectory of a single particle. Similarly, we expect insights on the scalar Abraham model by restricting ourselves towards the analysis of the scalar field in the following model:

Theory 5 Particle-Sourced Scalar Potential

The **ontological basis** in this model consists of

- one ϱ-shaped particle (either extended or point like) with charge q and a given smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ with some maximal velocity $v_{\max} < 1$ such that $\|\dot{\mathbf{r}}(t)\| \leq v_{\max}$ and
- one field $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, (t, \boldsymbol{x}) \mapsto \phi(t, \boldsymbol{x})$.

The **dynamics** of the particle is considered externally fixed while the potential is governed by the **wave equation** $\Box\phi(t,\boldsymbol{x}) = \rho(t,\boldsymbol{x})$ which is sourced by $\rho(t,\boldsymbol{x}) = q\varrho\big(\boldsymbol{x} - \mathbf{r}(t)\big)$.

This theory is usually understood as an **initial value problem** which, by providing at some fixed time $t_0 \in \mathbb{R}$ initial field configuration $\big(\phi(t_0,\cdot), \dot{\phi}(t_0,\cdot)\big) : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}, \boldsymbol{x} \mapsto \big(\phi(t_0,\boldsymbol{x}), \dot{\phi}(t_0,\boldsymbol{x})\big)$, is expected to yield ϕ at all other time instances.

1.2 Objectives

The rest of thesis is concerned with the analysis of solutions of the dynamics of the wave equation with specific kinds of sources, in particular including point particle trajectories. Our main objective is the understanding of the dressing of charged particles with their Liénard-Wiechert fields. Special interest lies in the generalization of the statements concerning the germination of the dressing Maxwell fields for extended particles. We summarize:

Distributions and Green's Functions: We familiarize ourselves with the theory of distributions on space-time which provides for linear partial differential equations, like the wave equation, a formalism of generalized functions capable of treating singular sources. The extension exceeds the setting of distribution on space valued functions of time used in [Har18] and provides a comfortable ground for the method of Green's functions. We prove that the advanced and retarded Green's functions can indeed be formulated as distributions on space-time and provide their, for our purposes relevant, properties.

Sources and Solutions: The sources of the potentials are also stated as distributions on space-time. We provide large classes of sufficient conditions for the construction of solutions to the dynamics by the convolution product of advanced and retarded Green's functions and the sources.

Regularity of the Potentials: Both for the discussion of the initial value formulation of the scalar field and for their asymptotics, a certain kind of regularity with respect to time is necessary. We provide, therefore, a notion of time-foilable distributions which are distributions that can be regarded as distributions on space valued functions of time. The formalism of [Har18] is included within this setting. In the special case of charge trajectory sources, an explicit form of the solution can be provided.

Selection of the Potential: We discuss the special role of the constructed solutions of the dynamics in the context of initial value constraints and constraints on the asymptotic behavior of solutions. Special interest is placed on the germination of the dressing potentials.

Structure of the Thesis

Chapter 1 lays out the instruction and provides an overview on the question which we address in this thesis. In the following Chapter 2, we dedicate one section to each of the objectives. Finally, the main body of this thesis closes with Chapter 3, where we provide an outlook. The mathematically rigorous groundwork of the introduced formalism is laid out in a comprehensive manner in Chapters 4 to 7.

Chapter 2

Scalar Potentials

We dedicate ourselves towards the analysis of the massless wave equation

$$\Box\phi = \rho \tag{2.1}$$

as a differential equation of generalized functions in the sense of distributions on space-time. The development of tools and their application for the construction of specific solutions and the analysis of their regularity are executed in Sections 2.1 to 2.3. Section 2.4 provides a discussion of the relevance of these solutions in the context of Theory 5 of particle-sourced scalar potentials with special emphasis on dressing potentials.

2.1 Distributions and Green's Functions

In the case of a sufficiently smooth charge-distribution ϱ of the particle, the wave equation can be understood in the conventional sense of convergent differential quotients of pointwisely defined functions. This notion, however, falls apart when regarding sources representing moving point particles. A suitable formalism is provided by distributions, generalizations of pointwisely defined functions with a weak notion of derivatives. We present a didactic introduction to these tools in Section 4.1 and limit ourselves to mentioning only the most relevant definitions and statements in this section. Furthermore, certain elementary inhomogeneities can be understood as generalized functions with support at a single space-time point allowing for a linear decomposition of the sources. Due to the linearity and translational invariance[Def. 4.1.1] of the partial differential operator $\Box$, one can construct solutions ϕ of Equation (2.1) for many sources out of solutions for these elementary inhomogeneities. Additionally, despite the given source, the wave equation is covariant with respect to Poincaré-transformations. The formalism of distributions on space-time respects this symmetry of time and space.

2.1.1 Distributions

Distributions on space-time, a specific setting of generalized functions, are linear forms[Def. 4.1.8], denoted by $\mathsf{LF}[C_c^\infty(\mathbb{R}^4,\mathbb{C}),\mathbb{C}]$, on compactly supported and smooth functions $C_c^\infty(\mathbb{R}^4,\mathbb{C})$ which are continuous with respect to a specific topology[Def. 4.1.19] on C_c^∞ denoted by $\tau_{C_c^\infty}$. The compactly supported and smooth functions are referred to as test functions. The set of distributions is a topological vector space with its weak topology[Def. 4.1.10], denoted by $\tau_{\mathsf{LF}[C_c^\infty]}$, with respect to the test functions:

Definition Distributions on Space-Time

Given $\mathcal{D} := \left(C_c^\infty(\mathbb{R}^4,\mathbb{C}), \tau_{C_c^\infty}\right)$ we call the following topological vector space the space of distributions on space-time:

$$\mathcal{D}'(\mathbb{R}^4) := \Big(\big\{u \in \mathsf{LF}[C_c^\infty,\mathbb{C}] \,|\, u : \mathcal{D} \to (\mathbb{C}, |\cdot|) \text{ is continuous}\big\}, \tau_{\mathsf{LF}[C_c^\infty]}\Big).$$

This notion of generalized functions provides a particular rich framework for our discussion due to the following reasons:

- Any locally Lebesgue integrable function on space-time is canonically embedded in $\mathscr{D}'(\mathbb{R}^4)$. Such a function is called a representation of the corresponding distribution by functions, or in short, a representation.
- Since the test functions are closed under the application of any linear partial differential operator[Def. 4.1.1], ∂ the action of ∂ can be defined on $\mathscr{D}'(\mathbb{R}^4)$ by duality[Def. 4.1.14]. It coincides with the conventional and weak derivatives of (weakly) differentiable functions along the canonical embedding. Thus, distributions offer a generalized setting for partial differential equations like our wave equation.
- The compactness of the support of the test functions allows for the "arbitrary growth at infinity" of the generalized functions. An additional benefit lies in the fact that the Fourier-Laplace transformation[Def. 4.2.5] of these test functions is holomorphic, allowing for the construction of distributions via complex surface integrals introduced in Section 4.2. We will use these methods in Section 2.1.3.
- Requiring the continuity of the linear forms with respect to the special topology $\tau_{C_c^\infty}$ on the test functions limits the pointwise irregularity of the distribution to a certain degree. We argue in Section 4.1 that this degree is sufficient for the discussion of linear partial differential equations. Furthermore, we see in Section 2.2 that these kinds of pointwise irregularity still allow for a generalization of the convolution on these generalized functions which is essential for our discussion.
- The weak topology on the space of the generalized functions is weak enough such that operations like the application of partial differential operators or the convolution with other distributions are (sequentially) continuous.

The mentioned pointwise irregularities and the global behavior of distributions become clear by the following theorem (Theorem 4.1.25):

Theorem **Regularity Theorem for Distributions on Space-Time**

Given $u \in \mathscr{D}'(\mathbb{R}^4)$, there exists $(u_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}$ in $C(\mathbb{R}^4, \mathbb{C})$ such that for all compact $\mathsf{K} \subset \mathbb{R}^4$, the set $\{\boldsymbol{\alpha} \in \mathbb{N}_0^N \mid \operatorname{sp} u_{\boldsymbol{\alpha}} \cap \mathsf{K} \neq \varnothing\}$ is finite and the following equality holds true:

$$u = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N} \partial^{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}} .$$

Put loosely, distributions are locally finite weak partial derivatives of continuous functions.

Tempered Distributions

When considering linear partial differential equations, the Fourier-Laplace transformation is especially useful since it turns a differential equation into an algebraic one if the involved partial differential operator is translation invariant. A generalization of these methods can be implemented in the contest of linear forms on test functions as expanded in Section 4.2. Furthermore, it comes with advantages to limit oneself to the more restrictive setting of generalized functions called tempered distribution on space-time. They are themselves linear forms of Schwartz functions $S(\mathbb{R}^4, \mathbb{C})$[Def. 4.2.17], i.e. smooth functions whose partial derivatives of any order decay faster than any inverse polynomial at infinity, which are continuous with respect to the topology τ_S[Def. 4.2.17] on S and equipped with the weak topology[Def. 4.1.10]. $S(\mathbb{R}^4, \mathbb{C})$ are then called tempered test functions.

Definition **Tempered Distributions on Space-Time**

Given $\mathscr{S} := \big(S(\mathbb{R}^4, \mathbb{C}), \tau_s\big)$, we call the following topological vector space the space of tempered distributions:

$$\mathscr{S}'(\mathbb{R}^4) := \Big(\big\{u \in \mathsf{LF}[S(\mathbb{R}^4, \mathbb{C}), \mathbb{C}] \,|\, u : \mathscr{S} \to (\mathbb{C}, |\cdot|) \text{ is continuous}\big\}, \tau_{\mathsf{LF}[S]}\Big).$$

Indeed is $\mathscr{S}'(\mathbb{R}^4)$, according to Proposition 4.2.18, canonically identified as a subset of $\mathscr{D}'(\mathbb{R}^4)$. For these generalized functions, there exists a powerful generalization of the theory of Fourier transformations:

- The tempered test functions are closed as well under the application of linear partial differential operators allowing for the formulation of weak differential equations.

- The Fourier transformation in Definition/Theorem 4.2.10 is according to Theorem 4.2.20 a continuous bijection on $\big(S(\mathbb{R}^N, \mathbb{C}), \tau_S\big)$. Thus, the Fourier transformation defined on $\mathcal{S}'(\mathbb{R}^4)$ by duality is bijective.

- Due to its definition by duality, the Fourier transform and its inverse on $\mathcal{S}'(\mathbb{R}^4)$ are continuous with respect to the weak topology (Theorem 4.2.20).

- By loosening the decay behavior of the test functions, the "growth at infinity" of tempered distributions is restricted by polynomial growth.

- The continuity requirement of the linear forms with respect to τ_S limits the local, pointwise irregularity of these generalized function to the same degree as of distributions.

The last two statements are again supported by the following theorem (Theorem 4.2.23):

Theorem Regularity Thm. for Tempered Distributions on Space-Time

$\mathcal{S}'(\mathbb{R}^4) = \left\{\partial^{\boldsymbol{\alpha}} g \,\middle|\, g \in C(\mathbb{R}^4, \mathbb{C}) \text{ and polynomially bounded}, \boldsymbol{\alpha} \in \mathbb{N}_0^N\right\}$ with weak derivatives in the sense of Def. 4.1.14.

2.1.2 Green's Functions

As already stated in the introduction of the section, we want to linearly decompose the source ρ into elementary inhomogeneities supported at a single space-time point. These elementary inhomogeneities are translations[Def. 4.1.2 and 4.2.3] of the delta distributions[Definition/Proposition 4.1.11] on space-time:

Definition Lemma Delta Distributions on Space-Time

We call $\delta \in \mathcal{S}'(\mathbb{R}^4)$ defined by $\delta : \mathcal{S} \to \mathbb{C}, \big((t, \boldsymbol{x}) \mapsto f(t, \boldsymbol{x})\big) \mapsto \delta[f] := f(0, \mathbf{0})$ the delta distribution on space-time.

It is according to the characterization of continuity of linear forms on $S(\mathbb{R}^4, \mathbb{C})$ in Theorem 4.2.24 indeed continuous.

Our objective is now to find a solution of Equation (2.1) with ρ replaced by δ within the setting of distributions. We call such solutions Green's functions[Def. 4.1.26] and denote them by G. In Section 4.2.1, two lines of thought are provided on how to find such solutions. Both construct a linear decomposition map $\mathcal{LC}^*$ (Def. 4.2.2 and Lemma 4.2.4), the elementary inhomogeneity, into plane waves $(t, \boldsymbol{x}) \mapsto e^{\mathrm{i}(\omega t - \langle \mathbf{k} | \boldsymbol{x} \rangle)}$ with $k \in \mathbb{C}^4$ turning Equation (2.1) into the algebraic equation

$$m_{ph,\partial^t} \cdot \mathcal{LC}^*(G) = \mathcal{LC}^*(\delta) \tag{2.2}$$

where m_{ph,∂^t} is the characteristic polynomial[Definition/Lemma 4.2.1] of the transposition[Def. 4.1.14] of ∂ on certain vector spaces of linear forms. Since the set of zero points $\{m_{ph,\Box} = 0\}$ of $m_{ph,\Box}$ is quite large, it is involved to retrieve a distribution which is indeed a distribution.

2.1.3 Green's Functions of the Wave Equation

Section 4.4 performs the two constructions of Section 4.2 for the advanced and retarded Green's functions of the massless wave equation[Def. 4.4.1]:

Definition Advanced and Retarded Green's Functions

We will call the following linear forms the advanced and retarded Green's function

$$G^{\text{adv/ret}} : S(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, f \mapsto \int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\Theta(\mp t)\frac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big)$$

which are abbreviated by writing

$$G^{\text{adv/ret}}(t, \boldsymbol{x}) = \Theta(\mp t)\frac{1}{4\pi|\boldsymbol{x}|}\delta(|\boldsymbol{x}| \pm t)$$

which is just a formal expression not to be understood as the evaluation of a function or the composition of its ingredients.

For the sake of clarity, we will limit ourselves here to a sketch of the construction and refer to Section 4.4 for details:

Construction by avoiding $\{m_{ph,\square} = 0\}$: We choose a linear decomposition map $\mathcal{LC}^*$ involving just functions $(t, \boldsymbol{x}) \mapsto e^{\mathrm{i}(\omega t - \langle \mathbf{k}|\boldsymbol{x}\rangle)}$ such that $(\omega, \boldsymbol{k}) \notin \{m_{ph,\square} = 0\}$. This leads to

$$G^{\text{adv/ret}} : C_c^\infty(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, f \mapsto G^{\text{adv/ret}}[f] = (2\pi)^{-4/2} \int d\omega d^3k \,\frac{(\mathcal{FL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2}$$

where $\mathcal{FL}$ is the Fourier-Laplace transformation Def. 4.2.5 and $\int d\omega d^3k$ is understood as a complex surface integral Def. 4.2.14 on a domain determined by the curves Def. 4.4.2  respectively . These curves are constructed in such a way that the resulting complex surface avoids $\{m_{ph,\square} = 0\}$. Now, the task consists of showing that this linear form indeed coincides with the representation given in the definition and proving its continuity.

Construction without avoiding $\{m_{ph,\square} = 0\}$: We restrict ourselves by regarding Equation (2.1) as an equation of tempered distributions and use the Fourier transform $\mathcal{F}_{ph}$ on $\mathcal{S}'(\mathbb{R}^4)$ as the linear decomposition map $\mathcal{LC}^*$. Equation (2.2) is then also an equation of tempered distributions due to bijectivity and bicontinuity of the Fourier transform on tempered test functions. This comes at the advantage that solving Equation (2.2) for a continuous linear form on $S(\mathbb{R}^4, \mathbb{C})$ implies that the corresponding G is ensured to be continuous, too. However, since this linear decomposition involves plane waves $(t, \boldsymbol{x}) \mapsto e^{\mathrm{i}(\omega t - \langle \mathbf{k}|\boldsymbol{x}\rangle)}$ with $(\omega, \boldsymbol{k}) \in \mathbb{R}^4$ and $\mathbb{R}^4 \cap \{m_{ph,\square} = 0\} \neq \emptyset$, it complicates the solution of Equation (2.2). We adapt by replacing $m_{ph,\square}$ with $(\omega, \boldsymbol{k}) \mapsto -(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2$, and solve the resulting equation leading to

$$\mathcal{F}_{ph} G_\varepsilon^{\text{adv/ret}} : \mathbb{R}^4 \to \mathbb{C}, (\omega, \boldsymbol{k}) \mapsto \frac{(2\pi)^{-4/2}}{-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2}$$

and perform the limit $\varepsilon \to 0$ of the solutions in the weak topology. Similar to the other construction, it remains to be shown that the resulting tempered distribution agrees with the definition of the advanced and retarded Green's function.

These discussions lead to Theorem 4.4.3, which also includes a statement about the support of $G^{\text{adv/ret}}$ where $\Gamma^{\text{light},\mp}_{(0,\mathbf{0})}$ indicates the backwards, and respectively, forwards light cone Def. 4.3.2 of the space-time point $(0, \mathbf{0})$:

Theorem Properties of the Advanced and Retarded Green's Functions

The following properties all hold true for $G^{\text{adv/ret}}$:

1. $G^{\text{adv/ret}} \in \mathcal{S}'(\mathbb{R}^4)$.
2. $\square G^{\text{adv/ret}} = \delta$.
3. When restricting $G^{\text{adv/ret}}$ to $C_c^\infty(\mathbb{R}^4, \mathbb{C})$ they take the form:

$$f \mapsto G^{\text{adv/ret}}[f] = (2\pi)^{-4/2} \int d\omega d^3k \,\frac{(\mathcal{FL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2}\,.$$

4. The Fourier transform of $G^{\mathrm{adv/ret}}$ is given by the limit of the tempered distributions

$$\mathcal{F}_{ph} G_\varepsilon^{\mathrm{adv/ret}} : \mathbb{R}^4 \to \mathbb{C}, (\omega, \boldsymbol{k}) \mapsto \frac{(2\pi)^{-4/2}}{-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2}$$

i.e. $\mathcal{F}_{ph} G^{\mathrm{adv/ret}} = \lim_{\varepsilon\to 0} \mathcal{F}_{ph} G_\varepsilon^{\mathrm{adv/ret}}$ exists in $\mathcal{S}'$.

5. $\mathrm{sp}(G^{\mathrm{adv/ret}}) = \Gamma^{\mathrm{light},\mp}_{(0,\mathbf{0})}$.

2.2 Sources and Solutions

Considering Equation (2.1) as an equation of generalized functions, i.e. of distributions, we restrict ourselves to the case $\rho \in \mathcal{D}'(\mathbb{R}^4)$. According to the method explained in Section 2.1 we want to linearly decompose the source into translated elementary inhomogeneities, i.e. $\mathsf{T}_{(t',\boldsymbol{x}')}\delta$, which is supported at the space-time point $(t', \boldsymbol{x}') \in \mathbb{R}^4$.

2.2.1 Linear Decomposition of the Source

The aim of the linear decomposition of ρ into $\mathsf{T}_{(t',\boldsymbol{x}')}\delta$ will lead us to the convolution of generalized functions. We start the discussion for the special case where the source is a test function leading to the convolution of distributions and test functions. In the second step of sources of arbitrary regularity within the setting of distributions, the convolution is once more generalized to the case of a product of a distribution with another of compact support.

Convolution of Distributions and Test Functions

If ρ is a test function, then by the definition of the delta distribution and the translation, we find $(\mathsf{T}_{(t,\boldsymbol{x})}\delta)[\widetilde{\rho}] := \delta[\mathsf{T}_{(t,\boldsymbol{x})}\widetilde{\rho}] = \rho(t,\boldsymbol{x})$ when defining $\widetilde{\rho}(t',\boldsymbol{x}') := \rho(-t',-\boldsymbol{x}')$ for $(t',\boldsymbol{x}') \in \mathbb{R}^4$ since then $\big(\mathsf{T}_{(t,\boldsymbol{x})}\widetilde{\rho}\big)(t',\boldsymbol{x}') = \widetilde{\rho}(t'-t,\boldsymbol{x}'-\boldsymbol{x}) = \rho(t-t',\boldsymbol{x}-\boldsymbol{x}')$. Falsely assuming for a moment that δ is the canonical embedding of an ordinary function in $\mathcal{D}'(\mathbb{R}^4)$, we find

$$\begin{aligned}
\rho(t,\boldsymbol{x}) &= (\mathsf{T}_{(t,\boldsymbol{x})}\delta)[\widetilde{\rho}] \\
&= \int_{\mathbb{R}^4} dt' d^3x'\, \delta(t',\boldsymbol{x}')\widetilde{\rho}(t'-t,\boldsymbol{x}'-\boldsymbol{x}) \\
&= \int_{\mathbb{R}^4} dt' d^3x'\, \delta(t',\boldsymbol{x}')\rho(t-t',\boldsymbol{x}-\boldsymbol{x}') \\
&= \int_{\mathbb{R}^4} dt' d^3x'\, \delta(t-t',\boldsymbol{x}-\boldsymbol{x}')\rho(t',\boldsymbol{x}') \\
&\overset{(i)}{=} \int_{\mathbb{R}^4} dt' d^3x'\, \rho(t',\boldsymbol{x}')\big(\mathsf{T}_{(t',\boldsymbol{x}')}\delta(t,\boldsymbol{x})\big)
\end{aligned}$$

which motives the generalization of the convolution (Definition/Lemma 4.1.22) of a distribution and a test function:

Definition Convolution of Distributions and Test Functions

For $u \in \mathcal{D}'(\mathbb{R}^4)$ and $f \in C_c^\infty(\mathbb{R}^4,\mathbb{C})$, we define their convolution by the following expression:

$$u * f : \mathbb{R}^4 \to \mathbb{C}, (t,\boldsymbol{x}) \mapsto \big(u * f\big)(t,\boldsymbol{x}) := u[\mathsf{T}_{(t,\boldsymbol{x})}\widetilde{f}]\,.$$

Regarding the formal expression on the right-hand side of (i) in the preceding calculation, we interpret $\rho = \delta * \rho$ as the linear decomposition of ρ via "an integral" in the variables $(t',\boldsymbol{x}')$ of the generalized functions $\mathsf{T}_{(t',\boldsymbol{x}')}\delta$ with coefficients given by $\rho(t',\boldsymbol{x}') \in \mathbb{R}$.

Convolution of Distributions and Compactly Supported Distributions

The preceding discussion motivates the desire of defining the convolution $\delta * \rho$ for arbitrary distributions ρ. Since we, again, warnt that the equation $\rho = \delta * \rho$ holds true, we expect the result of such an convolution to be a distribution, i.e. we need to assign a meaning to $(\delta * \rho)[f]$ for all test functions $f \in C_c^\infty(\mathbb{R}^4, \mathbb{C})$. If $\delta * \rho$ is indeed a distribution, this expression can be reformulated as the convolution in the following expression and the expectation of being able to lift the associativity of convolutions of pointwisely defined functions motivates the subsequent manipulations:

$$(\delta * \rho)[f] = (\delta * \rho) * \widetilde{f}(0, \mathbf{0}) = \delta * (\rho * \widetilde{f})(0, \mathbf{0}) \,.$$

We have granted the well-defined of the term $\rho * \widetilde{f}$ as a convolution of the distribution ρ and the test function $\widetilde{f}$. Two important properties provide the ability of defining the second convolution:

- The limitedness of the pointwise irregular behavior of distributions is sufficient such that the convolution of ρ with the smooth test function $\widetilde{f}$ leads to a smooth function (Proposition 4.1.23).
- The compactness of the support of δ implies that it can be canonically regarded as a linear form on $C^\infty(\mathbb{R}^4, \mathbb{C})$ (Definition/Proposition 5.1.4).

Furthermore, there exists a topological τ_{C^∞} such that the space of continuous linear form on $C^\infty(\mathbb{R}^4, \mathbb{C})$ can be identified with compactly supported distributions (Definition/Proposition 5.1.4):

Definition **Compactly Supported Distributions on Space-Time**

Given $\mathcal{E}(\mathbb{R}^4) := \big(C^\infty(\mathbb{R}^4, \mathbb{C}), \tau_{C^\infty}\big)$, we call the following topological vector space the space of compactly supported distributions on space-time:

$$\mathcal{E}'(\mathbb{R}^4) := \Big(\big\{u \in \mathsf{LF}[C^\infty(\mathbb{R}^4, \mathbb{C}), \mathbb{C}] \,|\, u : \mathcal{E} \to (\mathbb{C}, |\cdot|) \text{ is continuous}\big\}, \tau_{\mathsf{LF}[C^\infty]}\Big) \,.$$

Proposition **Embedding of Compactly Supported Distributions**

$\mathcal{E}(\mathbb{R}^4)$ can be canonically identified with the set of distributions on space-time having compact support.

The interplay of the topologies $\tau_{C_c^\infty}$ and τ_{C^∞} and the continuity of (compactly supported) distributions implies the continuity of the two maps

- $\rho * [\,\widetilde{\cdot}\,] : \mathcal{D} \to \mathcal{E}, f \mapsto \rho * \widetilde{f}$ (Item 1 of Proposition 5.1.6)
- $\delta * [\,\cdot\,](0, \mathbf{0}) : \mathcal{E} \to (C, |\cdot|), g \mapsto \delta * g(0, \mathbf{0})$ (Definition/Proposition 5.1.4)

resulting in the continuity of $\delta * \rho := \delta * (\rho * [\,\widetilde{\cdot}\,])$, i.e. $\delta * \rho \in \mathcal{D}'(\mathbb{R}^4)$. Moreover, we have strengthened the continuity properties of the pairing of distributions with their test function. These pairings are additionally sequential continuous with respect to the involved convolution as proven in Proposition 5.1.6. We showed that this also implies sequential continuity in both arguments of the convolution of distributions with distribution of compact support within their respective weak topology. Furthermore, by the density of $C_c^\infty(\mathbb{R}^4, \mathbb{C})$ in the distribution spaces, this allows to lift properties like commutativity and statements about partial derivatives and the support of the convolution. All this leads to Definition/Theorem 5.1.8:

Definition **Theorem** **Convolution of $\mathcal{E}'$ and $\mathcal{D}'$**

Given $u \in \mathcal{E}'$, $v \in \mathcal{D}'$, there exists a unique way to define their convolution $u * v \in \mathcal{D}'$ and $v * u \in \mathcal{D}'$ such that

1. $u * v = v * u \in \mathcal{D}'$
2. $\mathrm{sp}(u * v) \subset \mathrm{sp}(u) + \mathrm{sp}(v)$
3. $\partial(u * v) = (\partial u) * v = u * (\partial v)$ for all $\partial \in \mathfrak{D}$

4. $(u_i)_{i\in\mathbb{N}} \in \mathscr{E}'$, $(v_i)_{i\in\mathbb{N}} \in \mathscr{D}'$ with $u_i \xrightarrow{\mathscr{E}'} u$, $v_i \xrightarrow{\mathscr{D}'} v$ implies
   - $u_i * v_i \xrightarrow{\mathscr{D}'} u * v$
   - $v_i * u_i \xrightarrow{\mathscr{D}'} v * u$

   for $i \to \infty$.

Summarizing: These generalization of the convolution allow for a "integral-like" linear decomposition of sources $\rho \in \mathscr{D}'(\mathbb{R}^4)$ by towards space-time points $(t', \boldsymbol{x}') \in \mathbb{R}^4$ translated elementary inhomogeneities $\mathsf{T}_{(t',\boldsymbol{x}')}\delta$ with coefficients "$\rho(t', \boldsymbol{x}')$" via $\rho = \delta * \rho$.

2.2.2 Solutions via Linear Combination

Let us assume to have a Green's function G of the massless wave operator $\Box$ at hand and we consider one space-time point $(t', \boldsymbol{x}') \in \mathbb{R}^4$. By the translational invariance of $\Box$ the translation of the Green's function, $\mathsf{T}_{(t',\boldsymbol{x}')}G$ provides a solution of the massless wave equation sourced by the translated elementary inhomogeneity $\mathsf{T}_{(t',\boldsymbol{x}')}\delta$, i.e. $\Box\mathsf{T}_{(t',\boldsymbol{x}')}G = \mathsf{T}_{(t',\boldsymbol{x}')}\delta$ or in the abusive notation convention of generalized functions:

$$\Box\mathsf{T}_{(t',\boldsymbol{x}')}G(t, \boldsymbol{x}) = \mathsf{T}_{(t',\boldsymbol{x}')}\delta(t, \boldsymbol{x}) \,.$$

Performing again purely formal calculations motivated by the false assumption that G, δ and ρ are canonical embeddings of ordinary functions in $\mathscr{D}'(\mathbb{R}^4)$ result in the following manipulations:

$$\begin{aligned}\Box \int_{\mathbb{R}^4} dt'd^3x'\, \rho(t', \boldsymbol{x}')\big(\mathsf{T}_{(t',\boldsymbol{x}')}G(t, \boldsymbol{x})\big) &= \int_{\mathbb{R}^4} dt'd^3x'\, \rho(t', \boldsymbol{x}')\Box_{(t,\boldsymbol{x})}\left(\big(\mathsf{T}_{(t',\boldsymbol{x}')}G(t, \boldsymbol{x})\big)\right)\\ &= \int_{\mathbb{R}^4} dt'd^3x'\, \rho(t', \boldsymbol{x}')\mathsf{T}_{(t',\boldsymbol{x}')}\delta(t, \boldsymbol{x})\\ &= \rho(t, \boldsymbol{x})\end{aligned}$$

which, formulated by convolutions, take the form of the, again purely formal expression:

$$\Box(G * \rho)(t, \boldsymbol{x}) = \big((\Box G) * \rho\big)(t, \boldsymbol{x}) = (\delta * \rho)(t, \boldsymbol{x}) = \rho(t, \boldsymbol{x}) \,.$$

Unfortunately, neither the support of G^{adv} nor of G^{ret} is compact such that the convolution $G * \rho$ is, up to this point, only defined in the case where ρ has compact support. We need another generalization of the convolution involving generalized functions to be able to construct solutions of the wave equations by linear combinations of translated retarded and advanced Green's functions for larger classes of sources.

Convolutions of Distributions of compatible Supports

A further generalization of the convolution by support properties can be achieved which is motivated by another purely formal manipulations. Let us falsely assume that G, ρ and $G * \rho$ are all representable by ordinary functions and the interchangeability of integrals appearing in the evaluation of $G * \rho$ on some test function f. Then this evaluation takes the following form:

$$\begin{aligned}(G * \rho)[f] &= \int_{\mathbb{R}^4} dtd^3x\, (G * \rho)(t, \boldsymbol{x})\, f(t, \boldsymbol{x})\\ &= \int_{\mathbb{R}^4} dtd^3x \left(\int_{\mathbb{R}^4} dt'd^3x'\, G(t - t', \boldsymbol{x} - \boldsymbol{x}')\rho(t', \boldsymbol{x}')\right) f(t, \boldsymbol{x})\\ &= \int_{\mathbb{R}^4} dt'd^3x' \int_{\mathbb{R}^4} dtd^3x\, G(t, \boldsymbol{x})\rho(t', \boldsymbol{x}')f(t + t', \boldsymbol{x} + \boldsymbol{x}') \,.\end{aligned}$$

This raises the speculation that, for the generalization of the convolution of G and ρ, only parts of the supports are relevant. More precisely, we only need to regard the preimage of the function $+|_{\text{sp}(G)\times\text{sp}(\rho)} : \text{sp}(G)\times\text{sp}(\rho) \to \mathbb{R}^N$, $\big((t, \boldsymbol{x}), (t', \boldsymbol{x}')\big) \mapsto (t+t', \boldsymbol{x}+\boldsymbol{x}')$ of $\text{sp}(f)$. If $(+|_{\text{sp}(G)\times\text{sp}(\rho)})^{-1}\big(\text{sp}(f)\big)$ is compact, we could restrict G and ρ to compact domains without expecting an influence on the value of $(G * \rho)[f]$. The evaluation of the convolution of two compactly supported distributions on a test function $f \in C_c^\infty(\mathbb{R}^4, \mathbb{C})$ is, however, already well defined by our preceding discussion. Hörmander showed in [HI83, Section 4.2] that this idea can indeed be executed in a rigorous manner to provide a generalization of the convolution on distributions of compatible supports[Def. 5.1.9]. We extend these results to the convolution of a finite number of distributions of compatible support and prov its sequential continuity (Definition/Theorem 5.1.10). This allows us to lift properties like associativity, commutativity and statements about the support of the convolution by restricting ourselves to the slightly stronger condition on the support of the involved distributions of strict compatibility (Def. 5.1.12):

Definition **Strictly Compatible Sets**

We call $m \in \mathbb{N}$ subsets $A_1, \dots, A_m$ of $\mathbb{R}^N$ strictly compatible if there exists $\varepsilon > 0$ such that $A_1^{+\varepsilon}, \dots, A_m^{+\varepsilon}$ are compatible where $A_j^{+\varepsilon} := A_j + \mathsf{B}_\varepsilon(\mathbf{0})$.

These statements are summarized in Definition/Theorem 5.1.13:

Definition **Theorem** **Conv. of Distr. on Strictly Compatible Supports**

Given $m \in \mathbb{N}_{\geq 2}$ strictly compatible subsets $A_1, \dots, A_m$ of $\mathbb{R}^N$ and distributions $u_j \in \mathscr{D}'(A_j)$ for all $j = 1, \dots, m$, there exists a unique way to define their convolution product $u_1 * \dots * u_m \in \mathscr{D}'$ such that

1. it is associative and commutative
2. $\operatorname{sp}(u_1 * \dots * u_m) \subset \operatorname{sp}(u_1) + \dots + \operatorname{sp}(u_m)$
3. $\partial(u_1 * \dots * u_m) = (\partial u_1) * u_2 * \dots * u_m = \dots = u_1 * \dots * u_{m-1} * (\partial u_m)$ for all $\partial \in \mathfrak{d}$
4. $(u_{j,i})_{i \in \mathbb{N}} \in \mathscr{D}'(A_j)$, with $u_{j,i} \xrightarrow{\mathscr{D}'} u_j$ for all $j = 1, \dots, m$ implies $u_{1,i} * \dots * u_{m,i} \xrightarrow{\mathscr{D}'} u_1 * \dots * u_m$ for $i \to \infty$.

Strict Compatibility of Cones

The condition of strict compatibility takes an explicit and easy-to-handle form in the special case of closed cones at compact sets (Def. 5.1.16):

Definition **Closed Cones**

We call a subset $\Gamma \in \mathbb{R}^N$ a closed cone at $\{0\} \in \mathbb{R}^N$ if Γ is closed and if $\forall \boldsymbol{x} \in \Gamma$ also $\lambda \boldsymbol{x} \in \Gamma$ for all $\lambda \in \mathbb{R}_{\geq 0}$. Thus, the set of closed cones at $\{0\}$ can be identified with the set of closed subsets of $\mathsf{S}_1(\mathbf{0}) := \{\boldsymbol{x} \in \mathbb{R}^N | |\boldsymbol{x}| = 1\}$ endowed with its subset topology.

Given compact $\mathsf{K} \subset \mathbb{R}^N$ and a closed cone Γ at $\{0\}$, we call the set $\Gamma_\mathsf{K} := \Gamma + \mathsf{K}$ the closed-Γ cone at K. If $\mathsf{K} = \{\boldsymbol{x}\}$ for $\boldsymbol{x} \in \mathbb{R}^N$ we abbreviate $\Gamma_{\boldsymbol{x}} := \Gamma_{\{\boldsymbol{x}\}}$.

We prov Proposition 5.1.18 on the characterization of the strict compatibility of closed cones:

Proposition **Strict Compatibility of Closed Cones**

Given two closed cones $\Gamma, \Gamma' \subset \mathbb{R}^N$ with $\Gamma \cap (-\Gamma') = \{\mathbf{0}\}$ and compact $\mathsf{K}, \mathsf{K}' \subset \mathbb{R}^N$, then Γ_K and $\Gamma'_{\mathsf{K}'}$ are strictly compatible.

Striclty Compatible Source with Respect to $G^{\mathrm{adv/ret}}$

The supports of G^{adv} and G^{ret} are, according to the preceding discussion (Item 5 of Theorem 4.4.3), given by $\Gamma^{\mathrm{light},-}_{(0,\mathbf{0})}$ respectively $\Gamma^{\mathrm{light},\mp}_{(0,\mathbf{0})}$, and thus, closed cones at $(0, \mathbf{0})$. There exist large classes of sources ρ with a strictly compatible and closed conical which are of particular interest in physics. This is due to the fact that in relativistic theories the constituents of charge distributions are usually assumed to move with velocities less then the speed of light, i.e. 1 in our unitless perspective. If there exists further a maximal velocity $0 \leq v_{\max}$ strictly less then 1 never exceeded by any of the constituents and some time interval in which the distribution is contained in a compact spatial region, then the support of such a source lies inside the closed cone $\Gamma_{\mathsf{K}}^{v_{\max}}$ with compact $\mathsf{K} \subset \mathbb{R}^4$ and $\Gamma^{v_{\max}} := \big\{(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^3 \big| |\boldsymbol{x}| \le v_{\max} \cdot t\big\}$. The maximal velocity $v_{\max}$ implies the strict compatibility of the advanced and retarded Green's functions and such a source such that $G^{\mathrm{adv/ret}} * \rho$ are solutions to Equation (2.1).

2.2.3 Particle Trajectories

An important special case of strictly compatible sources with respect to the advanced and retarded Green's function are pieces of smooth charge trajectories of either point-like or extended particles (Def. 5.2.1):

Definition Particle Source

Given $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ smooth such that $\exists v_{\max} < 1$ with $\|\dot{\mathbf{r}}(t)\| < v_{\max}$ forall $t \in \mathbb{R}$ and further some interval $I \subset \mathbb{R}$ we define

$$\mathbb{1}_I \rho_{\mathbf{r},\delta} : S(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, f \mapsto \int_{\mathbb{R}} dt\, q f(t, \mathbf{r}(t)) \mathbb{1}_I(t)$$

which we abbreviate by $\mathbb{1}_I(t)\rho_{\mathbf{r},\delta}(t,\boldsymbol{x}) = \mathbb{1}_I(t) q \delta\big(\boldsymbol{x} - \mathbf{r}(t)\big)$. For $I = \mathbb{R}$ we will define $\rho_{\mathbf{r},\delta} = \mathbb{1}_{\mathbb{R}}\rho_{\mathbf{r},\delta}$.

For a particle shape $\varrho \in \mathbb{C}_c^\infty(\mathbb{R}^3, \mathbb{C})$ we define $\mathbb{1}_I \rho_{\mathbf{r},\varrho} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} : (t, \boldsymbol{x}) \mapsto \mathbb{1}_I(t)\rho_{\mathbf{r},\varrho}(t, \boldsymbol{x}) := \mathbb{1}_I(t) q \varrho\big(\boldsymbol{x} - \mathbf{r}(t)\big)$.

We prove in Theorem 5.2.2 and Proposition 5.2.3 that they can indeed be understood as generalized functions in the setting of distributions and fullfil the desired support properties:

Theorem Properties of Particle Sources

Given a smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ ($\|\dot{\mathbf{r}}(t)\| \le v_{\max} < 1\, \forall t \in \mathbb{R}$), a particle shape $\varrho \in \mathbb{C}_c^\infty(\mathbb{R}^3, \mathbb{C})$ and an interval $I \subset \mathbb{R}$, then the following statements are true:

1. $\mathbb{1}_I \rho_{\mathbf{r},\delta}$ and $\mathbb{1}_I \rho_{\mathbf{r},\varrho}$ are in $\mathcal{S}'(\mathbb{R}^4)$.
2. $\mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta}) = \mathrm{graph}(\mathbf{r}|_I) \subset \Gamma^{v_{\max}}_{(t_0,\,\mathbf{r}(t_0))}$ for all $t_0 \in \mathbb{R}$.
3. $\mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\varrho}) = \mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta}) + \{0\} \times \mathrm{sp}(\varrho) \subset \Gamma^{v_{\max}}_{\{(t_0,\,\mathbf{r}(t_0))\} + \{0\} \times \mathrm{sp}(\varrho)}$ for all $t_0 \in \mathbb{R}$.
4. $\mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta})$ and $\mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\varrho})$ are strictly compatible with $\mathrm{sp}(G^{\mathrm{adv/ret}})$.

As a corollary of Definition/Theorem 5.1.13, we are able to construct solutions of the wave equation out of the advanced and retarded Green's functions and the preceding sources (Proposition 5.3.1):

Proposition Solutions as Distributions

Given a smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ ($\|\dot{\mathbf{r}}(t)\| \le v_{\max} < 1\, \forall t \in \mathbb{R}$) and an interval $I \subset \mathbb{R}$, the following statements hold true:

1. $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ is well-defined as a distribution on space-time.
2. $\mathrm{sp}(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}) \subset \Gamma^{\mathrm{light},\mp}_{\mathrm{graph}(\mathbf{r}|_I)}$.
3. $\square G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta} = \mathbb{1}_I \rho_{\mathbf{r},\delta}$.

Similarly, given additionally $\varrho \in \mathbb{C}_c^\infty(\mathbb{R}^3, \mathbb{C})$ the following statements hold true:

4. $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho}$ is well-defined as a distribution on space-time.
5. $\mathrm{sp}(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho}) \subset \Gamma^{\mathrm{light},\mp}_{\mathrm{graph}(\mathbf{r}|_I) + \{0\} \times \mathrm{sp}(\varrho)}$.

6. $\Box G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho} = \mathbb{1}_I \rho_{\mathbf{r},\varrho}$.

2.3 Regularity of Solutions

We turn now towards the discussion of the regularity of the potentials, i.e. solutions of the wave equation (Equation (2.1)) sourced by some charge distribution in $\mathscr{D}'(\mathbb{R}^4)$. As discussed in the introduction in Chapter 1, it is of particular importance for the discussion of initial value and asymptotic constraints that the regarded solutions possess a certain additional regularity with respect to time. The purposes of the next section is to present the subset of time foilable distributions of distributions on space-time and to lay out our results on their characterization. The detailed discussion including our proofs is outsourced to Chapter 6.

2.3.1 Time Foilable Distributions on Space-Time

As introduced, we discuss in the following in what sense and which cases our notion of generalized functions provided by distributions on space-time can be regarded as ordinary functions in the time direction but still possibly generalized functions with respect to space directions. We call this subset defined by our requirements for its elements time foilable distribution (Def. 6.1.6):

Definition **Time Foilabel Distribution on Space-Time**

A distribution $u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R}^4)$ is time foilable if there exists $u(\,\cdot\,)[\,\cdot\,]$ in the sense of the expressions

$$u(\,\cdot\,)[\,\cdot\,] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^3), t \mapsto u(t)[\,\cdot\,] \qquad \text{and} \qquad u(t)[\,\cdot\,] : C_c^\infty(\mathbb{R}^3, \mathbb{C}) \to \mathbb{C}, f \mapsto u(t)[f]\,,$$

such that for all $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, the function $t \mapsto u(t)\big[\boldsymbol{x} \mapsto f(t,\boldsymbol{x})\big]$ is Lebesgue integrable and the equation

$$u[f] = \int_{\mathbb{R}} dt\, u(t)\big[\boldsymbol{x} \mapsto f(t,\boldsymbol{x})\big] \tag{2.3}$$

holds true. The vector space $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3) := \big\{u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R}^4)\,\big|\, u[\,\cdot\,] \text{ is time foilabel}\big\}$ is called time foilable distributions.

Arbitrary functions with domain $\mathbb{R}$ and range $\mathscr{D}'(\mathbb{R}^3)$ will, in general, not lie within $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ even if the Lebesgue integrability condition in the definition of time foilable distributions is satisfied. This is due to the fact that according to the regularity theorem (Theorem 4.1.25) distributions possess some pointwise regularity in the time direction. We prove in Proposition 6.1.7 that this requirement is met within the setting of [Har18] of $\mathfrak{C}$-continuous differentiability generalized functions (Def. 6.1.3):

Definition **Proposition** **$\mathfrak{C}$-cont. Differentiability Generalized Functions**

A function $u(\,\cdot\,)[\,\cdot\,] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^3)$ is $\mathfrak{C}$-continuously differentiable of order $m \in \mathbb{N}_0$ if for all $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$ the evaluation $u(\,\cdot\,)[f] : \mathbb{R} \to \mathbb{C}, t \mapsto u(t)[f]$ is m times continuously differentiable. We define the vector space of m-times $\mathfrak{C}$-continuously differentiable distributions:

$$\mathfrak{C}^m\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^3)\big) := \big\{u : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^3)\big| u \text{ is } \mathfrak{C}\text{-continuously differentiable up to order } m \in \mathbb{N}_0\big\}\,.$$

$\mathfrak{C}^0\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^3)\big)$ is a subset of $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ via the canonical embedding in Equation (2.3).

It is remarkable that just a strong notion of continuity of $u(\,\cdot\,)[\,\cdot\,] \in \mathfrak{C}^0\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^3)\big)$ provides enough regularity in time direction to regard it as a distribution on space-time $u[\,\cdot\,]$ since, as such, $u[\,\cdot\,]$ is weak partial differentiable of arbitrary finite order in the time direction. However, the setting of $\mathfrak{C}^0\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^3)\big)$ will be too restrictive for our purposes.

Sufficient Conditions on the Foilablity of Distributions

We developed certain tools providing sufficient conditions on the foilablity of distributions. They depend on statements involving the interplay of convolutions and sequential continuity with the tensor product of distributions (Definition/Theorem 6.1.8). The reader is referred to Section 6.1.2 for the details and our proofs. The main idea can be sketched by considering a highly regular distribution $u[\,\cdot\,]$ representable by a smooth function $u(\,\cdot\,)$ and falsely assuming that the delta distribution is the canonical embedding of a symmetric, smooth and compactly supported function in $\mathscr{D}'(\mathbb{R}^4)$. Given $t \in \mathbb{R}$ and a spatial test function $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$ we follow along the subsequent calculation:

$$\begin{aligned} u(t)[f] &= \int_{\mathbb{R}^3} d^3x\, u(t, \boldsymbol{x})\, f(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^4} dt'd^3x\, u(t', \boldsymbol{x})\, \delta(t' - t) f(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^4} dt'd^3x\, u(t', \boldsymbol{x})\, \delta(t - t') \widetilde{f}(\boldsymbol{0} - \boldsymbol{x}) \\ &= \int_{\mathbb{R}^4} dt'd^3x\, u(t', \boldsymbol{x})\, (\delta \otimes \widetilde{f})(t - t', \boldsymbol{0} - \boldsymbol{x}) \\ &= \big(u * (\delta \otimes \widetilde{f})\big)(t, \boldsymbol{0})\,. \end{aligned}$$

Since we are able to assign a meaning to the convolution $u * (\delta \otimes \widetilde{f})$ for arbitrary distributions as an element of $\mathscr{D}'(\mathbb{R}^4)$, we can consider this expression as the candidate for a foliation of u. Since it needs to be evaluated at the points in $\mathbb{R} \times \{\boldsymbol{0}\}$ in space-time to regard it as a foliation, we call a candidate representable by functions in such a case (Def. 6.1.12):

Definition **Foliation Candidate and its Representation by Functions**

Given $u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R}^4)$, we call the family of distributions $u * (\delta \otimes [\,\widetilde{\cdot}\,])[\,\cdot\,] := \big\{u * (\delta \otimes \widetilde{f})[\,\cdot\,]\big\}_{f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})}$ the foliation candidate of $u[\,\cdot\,]$.

Further, we call a family of functions $\big(u * (\delta \otimes \widetilde{f})(\,\cdot\,,\,\cdot\,)\big)_{f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})}$ a representation of the foliation candidate by functions if for all $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$ the function $u * (\delta \otimes \widetilde{f})(\,\cdot\,,\,\cdot\,)$

- lies in $L^1_{loc}(\mathbb{R}^4, \mathbb{C})$ and
- is a representation of the distribution $u * (\delta \otimes f)[\,\cdot\,]$ in the sense of the embedding of L^1_{loc} in $\mathscr{D}'$.

If there exists such a family we call the foliation candidate representable by functions.

We were able to show that the equation $u(t)[f] = \big(u * (\delta \otimes \widetilde{f})\big)(t, \boldsymbol{0})$ indeed holds true for large classes of foilable distributions, any $t \in \mathbb{R}$ and any $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$ when picking a suitable representation of the foliation candidate (Theorem 6.1.13):

Theorem **Representation of the Foliation Candidate**

Let $u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ and $u(\,\cdot\,)[\,\cdot\,]$ be foliation of $u[\,\cdot\,]$. If for all $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$ the function

$$u(\,\cdot\,)[\mathsf{T}_{\cdot} f] : \mathbb{R}^4 \to \mathbb{C}, (t, \boldsymbol{x}) \mapsto u(t)[\mathsf{T}_{\boldsymbol{x}} f]$$

lies in $L^1_{loc}(\mathbb{R}^4, \mathbb{C})$ then $\big\{u(\,\cdot\,)[\mathsf{T}_{\cdot} f]\big\}_{f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})}$ is a representation of the foliation candidate by functions.

Given $u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R}^4)$, this raises the objective of finding sufficient sets of conditions on the foliation candidate which imply the foilablity of $u[\,\cdot\,]$. Despite the representability of $u * (\delta \otimes [\,\widetilde{\cdot}\,])[\,\cdot\,]$ by functions its representation $u * (\delta \otimes [\,\widetilde{\cdot}\,])(\,\cdot\,,\,\cdot\,)$ needs to fulfil additional regularity requirements in all three arguments. These are necessary to be able to define the function $u * (\delta \otimes [\,\widetilde{\cdot}\,])(\,\cdot\,, \boldsymbol{0})$ defined by

- $u * (\delta \otimes [\,\widetilde{\cdot}\,])(\,\cdot\,, \boldsymbol{0}) : \mathbb{R} \to \mathsf{LF}[C_c^\infty(\mathbb{R}^3, \mathbb{C}), \mathbb{C}], t \mapsto u * (\delta \otimes [\,\widetilde{\cdot}\,])(t, \boldsymbol{0})$ and

- $u * (\delta \otimes [\,\tilde{\cdot}\,])(t, \mathbf{0}) : C_c^\infty(\mathbb{R}^3) \to \mathbb{C}, f \mapsto u * (\delta \otimes \widetilde{f})(t, \mathbf{0})$,

and to be able to regard it as a foliation of $u[\,\cdot\,]$. We define the two notions of form-regularity and distribution-regularity of representations of the foliation candidate in Def. 6.1.14. Their exact definition is quite involved and we restrict ourselves at this point to the discussion of their implication. We summarize our findings (Theorems 6.1.15 and 6.1.17) and refer the reader for details to Section 6.1.2:

Theorem **Implications of Form- and Distribution-Regularity**

Given $u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R}^4)$ such that there exists a representation of its foliation candidate by functions which we denote by $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \bullet)$.

If $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \bullet)$ is form-regular, we are able to define the linear form $u * (\delta \otimes [\,\tilde{\cdot}\,])[\,\tilde{\cdot}\,, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ on the linear span of the tensor product of $C_c^\infty(\mathbb{R}, \mathbb{C})$ and $C_c^\infty(\mathbb{R}^3, \mathbb{C})$. Its value for functions $g \in C_c^\infty(\mathbb{R}, \mathbb{C})$ and $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$ is provided by

$$u * (\delta \otimes \widetilde{f})[\widetilde{g}, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)} := \int_{\mathbb{R}} dt\, \widetilde{g}(t) \big(u * (\delta \otimes \widetilde{f})\big)(t, \mathbf{0}) \,.$$

Further, form-regularity implies that $u * (\delta \otimes [\,\tilde{\cdot}\,])[\,\tilde{\cdot}\,, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ and $u[\,\cdot\,]$ agree in $\mathsf{LF}\big[\mathrm{span}\,\big(C_c^\infty(\mathbb{R}, \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^3, \mathbb{C})\big), \mathbb{C}\big]$.

If $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \bullet)$ is distribution-regular, there exists a canonical embedding of $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \mathbf{0})$ in $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ denoted by $u * (\delta \otimes [\,\tilde{\cdot}\,])[\,\tilde{\cdot}\,, \mathbf{0})$. The canonical foliation is denoted by $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \mathbf{0})$.

If $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \bullet)$ is both form- and distribution-regular, then $u[\,\cdot\,]$ lies in $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ and $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \mathbf{0})$ is a foliation of $u[\,\cdot\,]$.

A corollary of this theorem that, given $m \in \mathbb{N}$, $u[\,\cdot\,]$ lies in $\mathfrak{C}^m\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^3)\big)$ if and only if there exists a representation of the foliation candidate such that $u * (\delta \otimes [\,\tilde{\cdot}\,])(\,\cdot\,, \mathbf{0})$, can be defined an lies in $\mathfrak{C}^m\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^3)\big)$ (Propositions 6.1.7 and 6.1.18).

2.3.2 Regularity of Solutions of Particle Trajectories

We return to the by Proposition 5.3.1 provided solutions $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ sourced by a to an interval $I \subset \mathbb{R}$ corresponding parts of the smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ ($\|\dot{\mathbf{r}}(t)\| \le v_{\max} < 1\, \forall t \in \mathbb{R}$) of a point particle. The analysis of their regularity by the tools of the preceding section is quite advantageous since its foliation candidate take according to Theorem 6.2.1 the form

$$(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}) * (\delta \otimes [\,\tilde{\cdot}\,])[\,\cdot\,] := \big\{G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\widetilde{f}}[\,\cdot\,]\big\}_{f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})} ,$$

and, i.e., is related to the solutions of extended particles. This leads to not only the time foilablity of the point particle solutions but to the even stronger statement (Definition/Lemma 6.2.2 and Theorem 6.2.3):

Definition **Theorem** **Regularity of the Solutions**

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ be a smooth function such that there exists $0 \le v_{\max} < 1$ such that the inequality $\|\dot{\mathbf{r}}(t)\| \le v_{\max}$ holds true for all $t \in \mathbb{R}$. It follows that, given $(t, \boldsymbol{x}) \in \mathbb{R}^4$, the set $\big|\,\mathrm{graph}(\mathbf{r}) \cap \Gamma^{\pm}_{(t,\boldsymbol{x})}\big|$ contains a single point in both $\pm$ cases called the advanced respectively retarded space-time point and gets denoted by $\big(r_0^{\pm}(t, \boldsymbol{x}), \mathbf{r}^{\pm}(t, \boldsymbol{x})\big)$. The function allocating this space-time point to $(t, \boldsymbol{x}) \in \mathbb{R}^4$ is outside of the trajectory smooth, i.e. $(r_0^{\pm}, \mathbf{r}^{\pm})|_{\mathbb{R}^4 \setminus \mathrm{graph}(\mathbf{r})} \in C^\infty(\mathbb{R}^4 \setminus \mathrm{graph}(\mathbf{r}), \mathbb{R}^4)$. We define furthermore for the function $\boldsymbol{n}^{\pm}$ and $\dot{\mathbf{r}}^{\pm}$ by allocating to $(t, \boldsymbol{x}) \in \mathbb{R}^4$ the values

- $\boldsymbol{n}^{\pm}(t, \boldsymbol{x}) := \dfrac{\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{x})}{\|\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{x})\|}$ and
- $\dot{\mathbf{r}}^{\pm}(t, \boldsymbol{x}) := \dot{\mathbf{r}}\big(r_0^{\pm}(t, \boldsymbol{x})\big)$.

Given further an interval $I \subset \mathbb{R}$, then $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ lies in $L^1_{loc}(\mathbb{R}^4,\mathbb{R}) \cap C^\infty\big(\mathbb{R}^4 \setminus (\Gamma^{\text{light},\mp}_{\text{graph}(\mathbf{r}|_{\partial I})} \cup \text{graph}\,(\mathbf{r}|_I)), \mathbb{R}\big)$ and for $(t,\boldsymbol{x}) \in \mathbb{R}^4 \setminus \text{graph}(\mathbf{r}|_I)$ its pointwise value is

$$\big(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x}) = \mathbb{1}_{\Gamma^{\text{light},\mp}_{\text{graph}(\mathbf{r}|_I)}}(t,\boldsymbol{x}) \, \frac{1}{1 \pm \boldsymbol{n}^{\pm}(t,\boldsymbol{x}) \cdot \dot{\mathbf{r}}^{\pm}(t,\boldsymbol{x})} \, \frac{q}{4\pi \|\boldsymbol{x} - \mathbf{r}^{\pm}(t,\boldsymbol{x})\|} \,.$$

In particular lies $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ in $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$.

Let ϱ in $C^\infty_c(\mathbb{R}^3,\mathbb{R})$ be the particle shape, then its solution at $(t,\boldsymbol{x}) \in \mathbb{R}^4$ is related to the one of the point particle by the following equality:

$$\big(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho}\big)(t,\boldsymbol{x}) = \int_{\mathbb{R}^3} d^3y \, \big(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{y}) \varrho(\boldsymbol{x} - \boldsymbol{y}) \,.$$

2.4 Selection of the Potential

The solutions of the wave equation with partial particle trajectory source constructed in Proposition 5.3.1 provide the building blocks of important for potentials in Theory 5 of particle sourced scalar potential.

2.4.1 Initial Value Dynamics

So far, we are lacking of a full formulation of the massless wave function accompanied with initial values. Such a treatment is available in the case of Maxwell's fields by [Dec10, Theorem 4.14] for smeared particles, and partially, for special initial conditions, provided by [Har18, Theorem 4.2.1] for point-like particles. Still, we are able to discuss one, for the later application partially important, case.

Vanishing Initial Value

Given time $t_0 \in \mathbb{R}$, we want to construct a potential such that it and its partial derivative with respect to time vanish on $\{t_0\} \times \mathbb{R}^3$. It its due to the special support properties of the advanced and retarded Green's function in Theorem 4.4.3 that the following corollaries of Proposition 5.3.1 hold true:

- $G^{\mathrm{adv}} * \mathbb{1}_{(-\infty,t_0)} \rho_{\mathbf{r},\delta}$ is supported in $\mathring{\Gamma}^{\text{filled light},-}_{(t_0,\mathbf{r}(t_0))} := \{(t,\boldsymbol{x}) \in \mathbb{R}^4 |\, |t - t_0| - \|\boldsymbol{x} - \mathbf{r}(t_0)\| \geq 0 \text{ and } t \leq t_0\}$.
- $G^{\mathrm{ret}} * \mathbb{1}_{[t_0,+\infty)} \rho_{\mathbf{r},\delta}$ is supported in $\Gamma^{\text{filled light},+}_{(t_0,\mathbf{r}(t_0))} := \{(t,\boldsymbol{x}) \in \mathbb{R}^4 |\, |t - t_0| - \|\boldsymbol{x} - \mathbf{r}(t_0)\| \geq 0 \text{ and } t_0 \leq t\}$.

We define $\phi^{t_0}_{\mathbf{r},\delta} := G^{\mathrm{adv}} * \mathbb{1}_{(-\infty,t_0)} \rho_{\mathbf{r},\delta} + G^{\mathrm{ret}} * \mathbb{1}_{[t_0,+\infty)} \rho_{\mathbf{r},\delta}$ which takes according to Theorem 6.2.3 the following pointwise appearance:

$$\phi^{t_0}_{\mathbf{r},\delta}(t,\boldsymbol{x}) = \begin{cases} \dfrac{1}{1 - \boldsymbol{n}^{-}(t,\boldsymbol{x}) \cdot \dot{\mathbf{r}}^{-}(t,\boldsymbol{x})} \, \dfrac{q}{4\pi \|\boldsymbol{x} - \mathbf{r}^{-}(t,\boldsymbol{z})\|} & , \ (t,\boldsymbol{x}) \in \Gamma^{\text{filled light},+}_{(t_0,\mathbf{r}(t_0))} \setminus \text{graph}(\mathbf{r}) \\ 0 & , \ \text{elsewhere} \\ \dfrac{1}{1 + \boldsymbol{n}^{+}(t,\boldsymbol{x}) \cdot \dot{\mathbf{r}}^{+}(t,\boldsymbol{x})} \, \dfrac{q}{4\pi \|\boldsymbol{x} - \mathbf{r}^{+}(t,\boldsymbol{z})\|} & , \ (t,\boldsymbol{x}) \in \mathring{\Gamma}^{\text{filled light},-}_{(t_0,\mathbf{r}(t_0))} \setminus \text{graph}(\mathbf{r}) \,. \end{cases}$$

Since $\mathbb{1}_{(-\infty,t_0)} \rho_{\mathbf{r},\delta} + \mathbb{1}_{[t_0,+\infty)} \rho_{\mathbf{r},\delta} = \rho_{\mathbf{r},\delta}$ in the sense of distributions this is clearly a solution to the wave equation sourced by the full trajectory, i.e., the equation $\Box \phi^{t_0}_{\mathbf{r},\delta} = \rho_{\mathbf{r},\delta}$ of distributions holds true. Additionally, when regarding $\phi^{t_0}_{\mathbf{r},\delta}$ at $t = t_0$ as either a function or as a foliated distribution it and its (weak) time derivative equate to 0 outside $\mathbf{0} \in \mathbb{R}^3$. The point $\mathbf{0}$ must be considered separately which we will not pursue further in this thesis.

Shocks

Similar to the case of the Maxwell fields, this special solution of the initial value problem "$\phi^{t_0}_{\mathbf{r},\delta}(t,\boldsymbol{x})|_{t=t_0} = 0 = \partial_t \phi^{t_0}_{\mathbf{r},\delta}(t,\boldsymbol{x})|_{t=t_0}$" possess an irregularity "propagating" along the light-cones of $\big(t_0, \mathbf{r}(t_0)\big)$. However, these irregularities are just jumps in the case of $\phi^{t_0}_{\mathbf{r},\delta}$ in contrast to the δ-like shocks of $\mathbf{E}$ and $\mathbf{B}$. This is to be expected since the analogue of ϕ in Maxwell's theory are rather Φ and $\mathbf{A}$ while the force fields are sourced by derivatives of the charge distribution ρ and the current $\mathbf{j}$ leading. These additional derivative "convert" the jumps in the potentials to δ-like shocks in $\mathbf{E}$ and $\mathbf{B}$.

2.4.2 Dressing

We are now in a position to transfer the statements about the dressing in Theory 3 of by an extended particle sourced Maxwell fields to Theory 5 of by a point particle sourced scalar potentials.

Scalar Liénard-Wiechert Potentials

The solutions $\phi_{\mathbf{r}}^{\mathrm{adv/ret}} := \phi_{\mathbf{r}|_{I,\delta}}^{\mathrm{adv/ret}}$ for $I = \mathbb{R}$, i.e. sourced by the full trajectory, are the scalar versions of the advanced and retarded Liénard-Wiechert potential. These are smooth everywhere except on the trajectory graph($\mathbf{r}$), and thus, expected to be $\mathfrak{C}$-smooth as shown for the Maxwell fields in [Har18, Lemma 4.2.1].

Dressing

We expect in Theory 4, i.e. the fully coupled scalar Abraham model, an analogue of the radiational damping in Theory 2, i.e. the pseudo-relativistic Abraham model, which relaxes the movement of particles towards straight lines of constant velocity $\boldsymbol{v}$ when exiting regions of interaction. In such cases, the potential should relax in an appropriate sense to the analogue of a soliton field which gets canonically identified as "the potential accompanying the free charged particle" and turn out to be the by $\boldsymbol{v}_{\pm\infty}$ boosted Coulomb potentials attached to the charge. Similar to the discussion of Theory 3 of Maxwell's fields, we will analyze aspects of this process by considering a particle trajectory $\mathbf{r}$ which takes the form of straight lines of constant velocities outside a time interval $(T_{\min}, T_{\max})$, i.e., the following statements hold true:

- $\mathbf{r}(t) = \boldsymbol{x}_{-\infty} + \boldsymbol{v}_{-\infty}\, t$ for $t \leq T_{\min}$ with $\boldsymbol{x}_{-\infty}, \boldsymbol{v}_{-\infty} \in \mathbb{R}^3$ and $\|\boldsymbol{v}_{-\infty}\| < 1$.
- $\mathbf{r}(t) = \boldsymbol{x}_{+\infty} + \boldsymbol{v}_{+\infty}\, t$ for $t \geq T_{\max}$ with $\boldsymbol{x}_{+\infty}, \boldsymbol{v}_{+\infty} \in \mathbb{R}^3$ and $\|\boldsymbol{v}_{+\infty}\| < 1$.

The retarded and advanced scalar Liénard-Wiechert potentials play now an important role since

- for $t \leq T_{\min}$ the solution $\phi_{\mathbf{r},\delta}^{\mathrm{ret}}(t, \boldsymbol{x})$ is only sensible to the part of the trajectory before $T_{\min}$,
- for $t \geq T_{\max}$ the solution $\phi_{\mathbf{r},\delta}^{\mathrm{adv}}(t, \boldsymbol{x})$ is only sensible to the part of the trajectory after $T_{\max}$

resulting in

- the equality of $\phi_{\mathbf{r},\delta}^{\mathrm{ret}}$ and $\phi_{\mathbf{r}_{\mathrm{in}},\delta}^{\mathrm{ret}}$ with $\mathbf{r}_{\mathrm{in}} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}_{\mathrm{in}}(t) := \boldsymbol{x}_{-\infty} + \boldsymbol{v}_{-\infty}\, t$ on $(-\infty, T_{\min}] \times \mathbb{R}^3$ and
- the equality of $\phi_{\mathbf{r},\delta}^{\mathrm{adv}}$ and $\phi_{\mathbf{r}_{\mathrm{out}},\delta}^{\mathrm{adv}}$ with $\mathbf{r}_{\mathrm{out}} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}_{\mathrm{out}} := \boldsymbol{x}_{+\infty} + \boldsymbol{v}_{+\infty}\, t$ on $[T_{\max}, \infty) \times \mathbb{R}^3$.

Definition/Lemma 7.1.1 and Theorem 7.1.2 tell us that both $\phi_{\mathbf{r}_{\mathrm{in}},\delta}^{\mathrm{ret}}$ and $\phi_{\mathbf{r}_{\mathrm{out}},\delta}^{\mathrm{adv}}$ are by the in/out going velocity $v_{\mp\infty}$ boosted and to the particle attached Coulomb potentials:

Definition **Theorem** **Liénard-Wiechert Coulomb Potentials**

Given $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ in $\mathbb{R}^3$ with $\|\boldsymbol{v}_0\| < 1$, we define the following expressions:

- $\mathbf{r}_0 : \mathbb{R} \to \mathbb{R}^3, t \mapsto \boldsymbol{x}_0 + \boldsymbol{v}_0\, t$.
- $\gamma(\boldsymbol{v}_0) := (1 - \|\boldsymbol{v}_0\|^2)^{-1/2}$.
- $\mathsf{L}(\boldsymbol{v}_0) := \mathbb{1} + (\gamma(\boldsymbol{v}_0) - 1)/\|\boldsymbol{v}_0\|^2 \boldsymbol{v}_0 \otimes \boldsymbol{v}_0$.

The retarded and advanced scalar Liénard-Wiechert potentials of $\mathbf{r}_0$ take at $(t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \mathrm{graph}(\mathbf{r}_0)$ the value

$$\phi_{\mathbf{r}_0,\delta}^{\mathrm{adv/ret}}(t, \boldsymbol{x}) = \frac{q\, \gamma(\boldsymbol{v}_0)}{4\pi \|\mathsf{L}(\boldsymbol{v}_0)(\boldsymbol{x} - \boldsymbol{x}_0) - \gamma(\boldsymbol{v}_0)\boldsymbol{v}_0\, t\|}\, .$$

Combining the preceding statements we find for the pointwise evaluation of the advanced and retarded Liénard-Wiechert potentials of $\mathbf{r}$ the equalities

- $\phi_{\mathbf{r},\delta}^{\mathrm{ret}}(t, \boldsymbol{x}) = \dfrac{q\, \gamma(\boldsymbol{v}_{-\infty})}{4\pi \|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_{-\infty}) - \gamma(\boldsymbol{v}_{-\infty})\boldsymbol{v}_{-\infty} t\|}$ for all $t \leq T_{\min}$ and $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\mathbf{r}(t)\}$ and

- $\phi^{\text{adv}}_{\mathbf{r},\delta}(t,\boldsymbol{x}) = \dfrac{q\,\gamma(\boldsymbol{v}_{+\infty})}{4\pi\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_{+\infty}) - \gamma(\boldsymbol{v}_{+\infty})\boldsymbol{v}_{+\infty}t\|}$ for all $t \geq T_{\max}$ and $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\mathbf{r}(t)\}$.

Thus, these solutions play an exceptional role when substituting the law of dynamics with the condition that one of the asymptotics is supposed to represent the potential of a "freely moving charge":

- $\phi^{\text{ret}}_{\mathbf{r},\delta}$ corresponds to the situation of an ingoing charge accompanied with "its accompanying potential".
- $\phi^{\text{adv}}_{\mathbf{r},\delta}$ corresponds to the situation of an outgoing charge accompanied with "its accompanying potential".

This clearly emphasizes their role in scattering theory which precisely deals with these scenarios.

Germination of the Dressing

We pick an initial time $T_{\text{ini}} \in \mathbb{R}$ and regarding the partial solution $\phi^{\text{ret}}_{\mathbf{r}|_{[T_{\text{ini}},\infty)},\delta}$ generated by the parts of the trajectory of times larger than T_{ini}. Its value at $(t,\boldsymbol{x}) \setminus \text{graph}(\mathbf{r}|_{[T_{\text{ini}},\infty)})$ is due to Theorem 6.2.3 given by

$$\phi^{\text{ret}}_{\mathbf{r}_{[T,\infty)},\delta}(t,\boldsymbol{x}) = \begin{cases} \dfrac{1}{1-\boldsymbol{n}^-(t,\boldsymbol{x})\cdot\dot{\mathbf{r}}^-(t,\boldsymbol{x})}\,\dfrac{q}{4\pi\|\boldsymbol{x}-\mathbf{r}^-(t,\boldsymbol{z})\|} & t > T_{\text{ini}} \text{ and } \boldsymbol{x} \in \mathsf{B}_{(t-T_{\text{ini}})}\big(\mathbf{r}(T_{\text{ini}})\big) \\ 0 & \text{, elsewhere.} \end{cases}$$

Similar do we find when picking a final time $T_{\text{fin}} \in \mathbb{R}$ that $\phi^{\text{adv}}_{\mathbf{r}|_{(-\infty,T_{\text{fin}}]},\delta}$ takes the following value at $(t,\boldsymbol{x}) \setminus \text{graph}(\mathbf{r}|_{(-\infty,T_{\text{fin}}]})$:

$$\phi^{\text{adv}}_{\mathbf{r}|_{(-\infty,T_{\text{fin}}]},\delta}(t,\boldsymbol{x}) = \begin{cases} \dfrac{1}{1-\boldsymbol{n}^-(t,\boldsymbol{x})\cdot\dot{\mathbf{r}}^-(t,\boldsymbol{x})}\,\dfrac{q}{4\pi\|\boldsymbol{x}-\mathbf{r}^-(t,\boldsymbol{z})\|} & t < T_{\text{fin}} \text{ and } \boldsymbol{x} \in \mathsf{B}_{(T_{\text{fin}}-t)}\big(\mathbf{r}(T_{\text{fin}})\big) \\ 0 & \text{, elsewhere.} \end{cases}$$

We manage to prove Theorem 7.2.2, which stated in the terminology of this section takes the following form:

Theorem **Asymptotic Convergence to Liénard-Wiechert Potentials**

Let $\mathbf{r}:\mathbb{R}\to\mathbb{R}^3, t\mapsto\mathbf{r}(t)$ be a smooth trajectory such that there exists $0 \leq v_{\max} < 1$ with the property that $\|\dot{\mathbf{r}}(t)\| \leq v_{\max}$ for all $t \in \mathbb{R}$. Then, the following convergences hold true:

Convergence as Distributions on Space-Time:

- $\phi^{\text{ret}}_{\mathbf{r}_{[T_{\text{ini}},\infty)},\delta} \xrightarrow{\mathscr{D}'(\mathbb{R}^4)} \phi^{\text{ret}}_{\mathbf{r},\delta}$ for $T_{\text{ini}} \to -\infty$
- $\phi^{\text{adv}}_{\mathbf{r}|_{(-\infty,T_{\text{fin}}]},\delta} \xrightarrow{\mathscr{D}'(\mathbb{R}^4)} \phi^{\text{adv}}_{\mathbf{r},\delta}$ for $T_{\text{fin}} \to \infty$

Convergence of the Time Foliation in Distributions on Space: For all $t \in \mathbb{R}$

- $\phi^{\text{ret}}_{\mathbf{r}_{[T_{\text{ini}},\infty)},\delta}(t)[\,\cdot\,] \xrightarrow{\mathscr{D}'(\mathbb{R}^3)} \phi^{\text{ret}}_{\mathbf{r},\delta}(t)[\,\cdot\,]$ for $T_{\text{ini}} \to -\infty$
- $\phi^{\text{adv}}_{\mathbf{r}|_{(-\infty,T_{\text{fin}}]},\delta}(t)[\,\cdot\,] \xrightarrow{\mathscr{D}'(\mathbb{R}^3)} \phi^{\text{adv}}_{\mathbf{r},\delta}(t)[\,\cdot\,]$ for $T_{\text{fin}} \to \infty$

Pointwise Convergence: For all $(t,\boldsymbol{x}) \in \mathbb{R}^4 \setminus \text{graph}(\mathbf{r}|_I)$

- $\phi^{\text{ret}}_{\mathbf{r}_{[T_{\text{ini}},\infty)},\delta}(t,\boldsymbol{x}) \to \phi^{\text{ret}}_{\mathbf{r},\delta}(t,\boldsymbol{x})$ for $T_{\text{ini}} \to -\infty$
- $\phi^{\text{adv}}_{\mathbf{r}|_{(-\infty,T_{\text{fin}}]},\delta}(t,\boldsymbol{x}) \to \phi^{\text{adv}}_{\mathbf{r},\delta}(t,\boldsymbol{x})$ for $T_{\text{fin}} \to \infty$

Putting the statement of this theorem into words, the full retarded respectively advanced scalar Liénard-Wiechert potential "germinates" along the trajectory when "starting out of the vacuum" in the increasing past/future $T_{\text{ini}}/T_{\text{fin}}$.

Chapter 3

Outlook

The extension of the in this thesis developed tools within the setting of classical scalar potentials sourced by point particle trajectories and other theories can be expected to lead to a fruitful understanding of local regularity of potentials and the germination of scatting solutions.

Massless Scalar Potential

There are several different aspects of Theory 5 of the particle sourced scalar potential worth exploring:

Initial Value Dynamics: A systematic analysis of the initial value formulation of the theory would be desirable. We believe that the notion of time foilable distributions on space-time provides a fruitful ground. It is of interest to characterize subsets of tuples of distributions on space-time for which a generalization of Kirchhoff's formula ([Dec10, Corollary 4.13]) in $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ can be achieved, i.e. for which the so called "free evolution" exists in the sense of foilable distributions. This is directly related to the solutions of the massless wave equation (2.1) with vanishing sources in $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^3)$ which we will call "free potentials" in the following.

Free Potentials accompanying the Germination of the Dressing: Having the free evolution at hand allows one to add an additional free potential when germinating the dressing. This can be done by starting from the values of the free potential and its time derivative instead of out of the vacuum at the initial, and respectively, final time $T_{\text{ini}}/T_{\text{fin}}$. For trajectories which relax towards motion of constant velocities in the asymptotic regions these combined potentials will take the form of a boosted Coulomb potential plus the free potential. They can be regarded as scattering states of an in-/outgoing dressed particle and free potential.

Fourier Transformation of the Solutions: To understand the construction of scattering states via germination in their momentum representation, i.e., their spatial Fourier transformation, it would be desirable to replace the involved theorem of convolutions of distributions to one adapted to tempered distributions. Tools for this are provided within the context of generalizations of the classical wave-front set ([HI83, Definition 8.1.2]) which rely on the calculus of pseudo-differential operators. One such generalization is the C^∞ wave front set introduced in [Bon+91]. Coriasco and Maniccia later provided the slightly altered notion of the $\mathscr{S}$ wave front sets in [CM03]. Schulz uses these to provide a generalization of the multiplication on certain subsets of $\mathscr{S}'$ which is sequential continuous ([Sch14, Proposition 2.42]). This leads via the Fourier transformation to a sequential continuous convolution theorem based on the interplay of the behavior of involved tempered distributions on conical regions of space-time. We expect that the compatibility of the cone supports of the advanced, and respectively, retarded Green's functions and the particle trajectory sources suffice to apply this theorem.

Massive Scalar Potential

Replacing the dynamical law of the massless wave equation by its massive counterpart in Theory 5 leads to:

Theory 6 **Particle-Sourced Massive Scalar Potential**

The **ontological basis** in this model consists of

- one ϱ-shaped particle (either extended or point like) with charge q and a given smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ with some maximal velocity $v_{\max} < 1$ such that $\|\dot{\mathbf{r}}(t)\| \leq v_{\max}$
- and one field $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, (t, \boldsymbol{x}) \mapsto \phi(t, \boldsymbol{x})$ with a mass $m > 0$.

The **dynamics** of the particle is considered as externally fixed while the potential is governed by the **wave equation** $(\Box + m^2)\phi(t, \boldsymbol{x}) = \rho(t, \boldsymbol{x})$, which is sourced by $\rho(t, \boldsymbol{x}) = q\varrho\big(\boldsymbol{x} - \mathbf{r}(t)\big)$.

This theory is usually understood as an **initial value problem** which, by providing at some fixed time $t_0 \in \mathbb{R}$ initial

- particle positions and momenta $\big(\mathbf{r}_i(t_0), \mathbf{v}_i(t_0)\big) \in \mathbb{R}^3 \times \mathbb{R}^3$ for all particles $i \in [N]$
- and field configuration $\big(\phi(t_0, \cdot), \dot{\phi}(t_0, \cdot)\big) : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}, \boldsymbol{x} \mapsto \big(\phi(t_0, \boldsymbol{x}), \dot{\phi}(t_0, \boldsymbol{x})\big)$,

is expected to yield ϕ at all other time instances.

The relevance of a transfer of our insights form the massless to the massive potentials becomes clear when considering the quantized version of Theory 5. This seems, however, technically quite challenging since the support properties of the advanced and retarded Green's functions of the massive wave equation differ significantly from the massless ones. Our convolution theorem will not be applicable. This also prevents the use of the strong methods in the realm of pseudo-differential operators leading to the convolution theorem of tempered distributions.

Quantum Field Theory

There exist well-defined canonically bosonic quantized versions of Theories 2 to 6 with extended particles. They are formulated on some Hilbert space whose elements describe the state of the system. The degree of freedoms of the fields/potentials are encoded in a bosonic fock space. The dynamics is provided by the Schrödinger equation involving a self adjoint operator called the hamiltonian which is due to the external source time dependent in the quantum versions of Theories 5 and 6. These theories possess many familiarities with their classical counterparts:

Quantized Scalar Potential Sourced by Cl. Extended Charge Trajectory

We considered in another project the dynamics of the canonically bosonic quantized version of the massless and massive scalar potential with a fixed classical extended charge trajectory. We managed to find an explicit formulation of the time evolution which allows to mimic the germination of the dressing of the classical analogue in Section 2.4.2 for trajectories which relax to constant motion in finite time.

Massless Potential: The Möller operator, expected to be some limit $T_{\text{ini/fin}} \to \mp\infty$ of the germination process, does not exist in the massless case. This is called the infra-red catastrophe. Due to our explicit formula for the evolution, one can read of its reason: The operator representing the germination of the dressing in the quantum theory is an expression of the germinating classical dressing. At asymptotic times the classical dressing is the boosted and by the particle shape smeared Coulomb potential which lies due to its long tails not in $L^2(\mathbb{R}^3, \mathbb{C})$.

Massive Potential: If one however introduces a mass for the scalar potential, i.e., regards the quantization of Theory 6, then this problem will not occur. Still, the convergence of the Möller operator is quite weak. This can be traced back to the weak convergence of the classical dressing. We are able discuss several algorithms to construct a scattering theory for the massive model in which the limit $m \to 0$ can be performed.

Particle at Rest: When considering a charge at rest, the hamiltonian of this quantum theory does not depend on time. The resulting model is called the Van Hove-Miyatake model. The dressing of the stationary extended charge can instead of the dynamical germination out of the vacuum be obtain as the ground state of the hamiltonian. A detailed analysis is provided in [Ara18, Chapter 13].

Quantized Scalar Potential Sourced by Cl. Point Charge Trajectory

The description of the dynamics via a self-adjoint hamiltonian is not directly accessible. One could however regard the "operator" associated to the quantum scalar potential in the "Heisenberg picture" as an operator valued tempered distribution. It fulfills a, to this setting, generalized version of the wave equation including the particle trajectory source and an initial condition at some fixed time which is related to the free potential. The author is convinced that with the techniques of the convolution theorem of tempered distributions in place, the results of the classical massless Theory 5 of point particles achieved in this thesis are extendable to this quantum version of the model.

Learnings for Nelson's Model

The preceding quantum field theory can be viewed as an toy model of massless and massive Nelson's model, i.e. the canonical bosonic quantized versions of Theory 4 of the scalar Abraham model with either zero or non-zero mass of the potential.[1] Nelson shows in [Nel64] for the case of the massive potential that one can define the dynamics of the model with point charges as a limiting process of shrinking the extended charges to a point. The analysis of the dressing is an active area of research and many distinct methods are used. The extensions of our results on the classical scalar potential to its quantum version and the explicit formulas we found, provide a test ground of all these methods and grant, by their close resemblance to the classical theory, a clear picture of these constructions.

Gauge Potentials

As briefly discussed in Section 1.1.4, the Gauge potentials in Lorentz gauge fulfill the massless wave equations with similar sources to Theory 5 of the scalar potential sourced by particle trajectories. Since Lorentz gauge is just a partial gauge fix it is not directly clear whether the statements on the infra-red catastrophe in the quantized models take the same form for quantum gauge potentials. Ultimately, it would be desirable to generalize the results of this thesis to the theory of Maxwell's equations for the gauge potential without any Gauge fix in place.

[1] For massive case replace again $\square$ by $(\square + m^2)$ in the wave equation.

Chapter 4

Distributions and Green's Functions

In this chapter, we will construct a mathematical framework to describe the dynamics for our physical theories. In these fields are generated by fixed charges distributions of point particles. The dynamical laws are stated by linear differential equations and accompanied by either the implementation of certain initial conditions or the analysis of the asymptotics. In the first two section, will we give a didactic introduction into the formalism of distributions and Green's functions while providing some extensions. The last two sections are concerned with the application of these results to the massless wave equation.

Section Summaries:

4.1 Introduction to Distributions via Green's Function: This section will give a constructive introduction into the notion of distributions while following along the idea of the use of so called Green's functions. On this basis, we construct solutions for linear partial differential equations with highly regular inhomogeneities.

4.2 How to find Green's Functions: The techniques of generating solutions of the first section rest on the availability of Green's functions. We will, thus, discuss a line of thought, which leads to the generation of candidates. This results in the exploration of the Fourier-Laplace transformation, complex surface integrals and tempered distributions.

4.3 Relativistic Structure and Conventions: The notion of light cones is introduced which is connected to the relativistic structure. It also canonically suggests a from Section 4.2 deviating sign convention which we introduce here.

4.4 Advanced and Retarded Green's Functions: With the objective of constructing solutions to the massless wave equation in mind, we construct the advanced and retarded Green's functions (Def. 4.4.1). Their derived properties (Theorem 4.4.3) will emphasizes their special role. We will follow closely along the lines of Section 4.2.

4.1 Introduction to Distributions: Green's Function

As described in the introduction of this chapter we need to build a mathematical framework capable of expressing the idea of charge distributions of point particles ρ and their fields φ. To arrive there we will analyze the linear partial differential equation that these entities fulfill and as mentioned in the introduction of the chapter consider solutions in the context of very regular inhomogeneities first. We start our endeavour by reviewing the notion of differential equations in the setting of ordinary functions.

4.1.1 Differential Equation with Functions

In the classical setting a linear differential equation consists of

- a partial differential operator denoted by ∂ which is a finite linear combination of partial derivatives with $\deg(\partial)$ denoting the highest partial derivative compositing ∂
- an inhomogeneity ρ which is a function from $\mathbb{R}^N$ to $\mathbb{C}$

- a function ϕ which is $\deg(\partial)$-times differentiable

such that $\forall \boldsymbol{x} \in \mathbb{R}^N$ the equation $\partial\phi(\boldsymbol{x}) = \varrho(\boldsymbol{x})$ holds and ϕ will be called a solution to ∂ with inhomogeneity ρ. We will restrict ourselves to the case where the coefficients of the partial derivative in ∂ are constant, i.e. the translational invariant setting of the following definition.

Definition 4.1.1 Differential Equations of Functions

Given a finite set of multi-indices $\mathrm{A} \in \mathbb{N}_0^N$ and $\forall \boldsymbol{\alpha} \in \mathrm{A}$ functions $a_{\boldsymbol{\alpha}} : \mathbb{R}^N \to \mathbb{C}$, we define the linear partial differential operator (PDO)

$$\partial : C^{\mathrm{deq}(\partial)}(\mathbb{R}^N, \mathbb{C}) \to F(\mathbb{R}^N, \mathbb{C}), \phi \mapsto \sum_{\boldsymbol{\alpha} \in \mathrm{A}} a_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \phi$$

where $|\boldsymbol{\alpha}| = |(\alpha_1, \ldots, \alpha_N)| := \alpha_1 + \ldots + \alpha_N$, $\deg(\partial) := \max_{\boldsymbol{\alpha}\in\mathrm{A}} |\boldsymbol{\alpha}|$ and $\partial^{\boldsymbol{\alpha}} := \dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}}$.

The set of these linear PDOs with $a_{\boldsymbol{\alpha}}$ being constants will be denoted by $\boldsymbol{\eth}$ and $\partial \in \boldsymbol{\eth}$ is called translational invariant.

Thus, for given ∂ and $\rho \in F(\mathbb{R}^N, \mathbb{C}) := \mathbb{C}^{\mathbb{R}^N}$, we call $\phi \in C^{\mathrm{deq}(\partial)}(\mathbb{R}^N, \mathbb{C})$ a solution to (∂, ρ) if $\partial\phi = \rho$.

This setting will not be sufficient for our purposes since the fields we are interested in are those of point particles and their charge distribution ρ is of an irregularity which cannot be described by any function in $F(\mathbb{R}^N, \mathbb{C})$. Still the solution strategy of "elementary inhomogeneities" will lead our way.

4.1.2 The Idea of Green's Functions

We will see that a continuous inhomogeneity ϱ can be decomposed into linear combinations of translated elementary inhomogeneities. By the linearity and translational invariance of our PDOs of interest we expect that by finding a solution to the elementary inhomogeneity we can construct a solution for ρ by a linear combination of those.

The starting point is to fix a standardized family of "elementary" inhomogeneities $(\delta_\varepsilon)_{\varepsilon>0} \subset F(\mathbb{R}^N, \mathbb{C})$[1].

For an $\partial \in \boldsymbol{\eth}$ we assume that there exists an family $(G_\varepsilon^\partial)_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ of solutions to $\partial G_\varepsilon = \delta_\varepsilon$ for all ε which fulfills a specific convergence conditions and we will call it "approximating Green's functions". The convergence condition will be formulated with respect to the set $\mathcal{I}_{G^\partial} \subset F(\mathbb{R}^N, \mathbb{C})$ of inhomogeneities ρ on which $(G_\varepsilon^\partial)_{\varepsilon>0}$ allows to construct a solution $\phi \in C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ to $\partial\phi = \rho$.

Since the differential equation in general possess multiple solutions, there are multiple families $(G_\varepsilon^\partial)_{\varepsilon>0}$ which we will collect in the set $\mathcal{G}^\partial$. The convergence conditions for each family $(G_\varepsilon^\partial)_{\varepsilon>0} \in \mathcal{G}^\partial$ can be different resulting in different sets of inhomogeneities $\mathcal{I}_{G^\partial}$. Our priority will be to build for a framework enabling us to formulate this convergence condition in a convenient way. It should incorporate as many families $(G_\varepsilon^\partial)_{\varepsilon>0}$ as possible. This set is denote by:

$$\mathcal{G}_\varepsilon^{\boldsymbol{\eth}} := \{(G_\varepsilon^\partial)_{\varepsilon>0} \,|\, \partial \in \boldsymbol{\eth} \text{ and the family } (G_\varepsilon^\partial)_{\varepsilon>0} \text{ are approximating Green's function}\}$$

Thus, we will restrict to a common set $\mathcal{I}$ of inhomogeneities of which we expect that for any $(G_\varepsilon^\partial)_{\varepsilon>0} \in \mathcal{G}_\varepsilon^{\boldsymbol{\eth}}$ the convergence condition is satisfied. This guides us to the notion of distributions.

It comes at the price that having picked specific $(G_\varepsilon^\partial)_{\varepsilon>0}$ one could by similar techniques find a framework dedicated to this specific approximating Green's function which provides solutions to a larger set of inhomogeneities $\mathcal{I}_{G^\partial} \supsetneq \mathcal{I}$. Still the minimal condition of $\mathcal{I}_{G^\partial} \subset C(\mathbb{R}^N, \mathbb{N})$ is mandatory for this technique and thus this strategy is not capable of handling inhomogeneities pointwisely more irregular than being continuous. Thus, we need to extend our results in the aftermath anyway and accept this inconvenience.

[1]Actually we start with the family $(\chi_\varepsilon)_{\varepsilon>0}$ first but come soon afterwards to the notions discussed in this paragraph.

Elementary Inhomogeneity

Let us first introduce a symbol for the translation of a function:

Definition 4.1.2 Translation of a Function

Given $\boldsymbol{y} \in \mathbb{R}^N$ we define for $f \in F(\mathbb{R}^N, \mathbb{C})$ the function $\mathsf{T}_{\boldsymbol{y}} f : \mathbb{R}^N \to \mathbb{C}, \boldsymbol{x} \mapsto f(\boldsymbol{x} - \boldsymbol{y})$.

Fix, for all $\varepsilon > 0$, functions $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with the properties

- $\mathrm{sp}(\chi_\varepsilon) \subset [\text{-}\varepsilon, \varepsilon]^N$
- $\sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \chi_\varepsilon\big)(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \mathbb{R}^N$

which is, loosely speaking, to be understood as $\chi_\varepsilon \approx \mathbb{1}_{[-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon]^N}$.

Approximation of the Inhomogeneity

Restricting $\mathcal{I} \subset C(\mathbb{R}^N, \mathbb{C})$ the inhomogeneity $\rho \in \mathcal{I}$ can be pointwisely approximated by

$$\tilde{\rho}_\varepsilon(\boldsymbol{x}) := \sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \chi_\varepsilon\big)(\boldsymbol{x}) \rho(\boldsymbol{y})$$

or more precise for all $\boldsymbol{x} \in \mathbb{R}^N$, we have $\lim_{\varepsilon \to 0} \tilde{\rho}_\varepsilon(\boldsymbol{x}) = \rho(\boldsymbol{x})$.

Approximating Green's Function and Construction of other Solutions

Pick $\partial \in \mathfrak{d}$ and for an $\varepsilon > 0$ a function $\tilde{G}_\varepsilon^\partial \in C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ solving $\partial \tilde{G}_\varepsilon^\partial = \chi_\varepsilon$ which is called elementary solution. By the linearity and translational invariance of the PDOs in $\mathfrak{d}$ and restriction of the set of inhomogeneities $\mathcal{I}$ we can construct a candidate for an approximation of the solution

$$\tilde{\phi}_\varepsilon(\boldsymbol{x}) := \sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}_\varepsilon^\partial(\boldsymbol{x})\big) \rho(\boldsymbol{y})$$

out of $\tilde{G}_\varepsilon^\partial$. A sufficient restriction for the convergence of the sum is $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$, which we will assume from now on.

Different Approximating Green's Functions for Different Patches

In the approximation of the inhomogeneities for fixed ε we may divide $\varepsilon\mathbb{Z}^N$ into $i \in I$ regions $R_{i,\varepsilon} \subset \varepsilon\mathbb{Z}^N$ and use, for each region, a different solution $\tilde{G}_{i,\varepsilon}^\partial$. We circumvent this additional layer of complexity for now by restricting ρ to have support within the hull spanned by $R_{i,\varepsilon}$ for some $i \in I$. We will come back to this aspect in Section 5.1 after developing the tools for this simpler case.

We assume to have access to a family $(\tilde{G}_\varepsilon^\partial)_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ of these elementary solutions and repeat the construction for all $\varepsilon > 0$ producing a family $(\tilde{\phi}_\varepsilon)_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ of solutions to $\big(\partial, (\tilde{\rho}_\varepsilon)_{\varepsilon>0}\big)$.

Convergence of Solutions

We want to ensure that, similar to the convergence of $\tilde{\phi}_\varepsilon$ to ϕ, also the series $\tilde{\phi}_\varepsilon$ to converge to a function ϕ, which we expect to be a solution to (∂, ρ). But since, for general ∂, solutions are not unique, we need to imply conditions to ensure that the elementary solutions of the family $(\tilde{G}_\varepsilon^\partial)_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ "fit together". We approach this question in a very pragmatic way by referring to the reason of us constructing these families in the first place. We call such a family $(\tilde{G}_\varepsilon^\partial)_{\varepsilon>0}$ approximating Green's functions and say that it converges if we can find a sufficiently[2] large set of inhomogeneities $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$ such that

- $\lim_{\varepsilon \to 0} \sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}_\varepsilon^\partial(\boldsymbol{x})\big) \rho(\boldsymbol{y})$ exists for all $\boldsymbol{x} \in \mathbb{R}^N$,

[2] The disturbingly vague notion of "sufficiently large set $\mathcal{I}$" shall not bother us since our strategy is to start with the largest set $C(\mathbb{R}^N, \mathbb{C})$ possible and pick restrictions as sparse as possible.

- $\boldsymbol{x} \mapsto \lim_{\varepsilon\to 0} \sum_{\boldsymbol{y}\in\varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}^{\partial}_{\varepsilon}(\boldsymbol{x})\big)\rho(\boldsymbol{y})$ is $\deg(\partial)$-smooth
- and $\partial_{(\boldsymbol{x})} \lim_{\varepsilon\to 0} \sum_{\boldsymbol{y}\in\varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}^{\partial}_{\varepsilon}(\boldsymbol{x})\big)\rho(\boldsymbol{y}) = \rho(\boldsymbol{x})$

for all $\rho \in \mathcal{I}$ and indicate the set of them by:

$$\tilde{\mathcal{G}}^{\boldsymbol{\partial},\mathcal{I}}_{\varepsilon} := \{(\tilde{G}^{\partial}_{\varepsilon})_{\varepsilon>0} \subset C^{\deg(\partial)} \,|\, \partial \in \boldsymbol{\partial} \text{ and } (\tilde{G}^{\partial}_{\varepsilon})_{\varepsilon>0} \text{ are approximating Green's functions w.r.t } \mathcal{I}\}\,.$$

These conditions are economically sensible from the side of application of these approximating Green's functions to generate solutions to inhomogeneities of high regularity but not at all for finding them. Thus, we will try to replace them in various steps by conditions we hope to be not much more restrictive.

Summary

Elementary Inhomogeneity

Fix for all $\varepsilon > 0$ functions $\chi_\varepsilon \in C^\infty_c(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with the properties:

- $\mathrm{sp}(\chi_\varepsilon) \subset [\text{-}\varepsilon, \varepsilon]^N$.
- $\sum_{\boldsymbol{y}\in\varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}}\chi_\varepsilon\big)(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \mathbb{R}^N$.

Approximating Green's Function

A family $(\tilde{G}^{\partial}_{\varepsilon})_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ is called approximating Green's function with respect to $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$ and $\partial \in \boldsymbol{\partial}$ if

- $\partial \tilde{G}^{\partial}_{\varepsilon} = \chi_\varepsilon$ for all $\varepsilon > 0$,
- $\lim_{\varepsilon\to 0} \sum_{\boldsymbol{y}\in\varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}^{\partial}_{\varepsilon}(\boldsymbol{x})\big)\rho(\boldsymbol{y})$ exists for all $\boldsymbol{x} \in \mathbb{R}^N$,
- $\boldsymbol{x} \mapsto \lim_{\varepsilon\to 0} \sum_{\boldsymbol{y}\in\varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}^{\partial}_{\varepsilon}(\boldsymbol{x})\big)\rho(\boldsymbol{y})$ is $\deg(\partial)$-smooth,
- and $\partial_{(\boldsymbol{x})} \lim_{\varepsilon\to 0} \sum_{\boldsymbol{y}\in\varepsilon\mathbb{Z}^N} \big(\mathsf{T}_{\boldsymbol{y}} \tilde{G}^{\partial}_{\varepsilon}(\boldsymbol{x})\big)\rho(\boldsymbol{y}) = \rho(\boldsymbol{x})$

for any $\rho \in \mathcal{I}$.

Point Charge: Total Charge

Unfortunately, the point particle inhomogeneity varies very significantly on any scale at the particle position and cannot be expressed as a continuous function.

To move forward, we need a better understanding of the point particle charge distribution $\rho(t, \boldsymbol{x}) = q\delta(\boldsymbol{x} - \boldsymbol{r}(t))$. Lets suppose for simplicity a resting particle at position $\boldsymbol{r}(t) = \boldsymbol{0}$ of charge $q = 1$. In case of a continuous, static charge distribution represented by a smooth function $\varrho \in C^\infty(\mathbb{R}^3, \mathbb{R}_{\geq 0})$ the total charge Q contained in a open bounded spatial region $\mathsf{V} \in \mathbb{R}^3$ is given by

$$\mathsf{Q} = \int_{\mathsf{V}} d^N x\, \varrho(\boldsymbol{x})$$

while we get for the point particle

$$\mathsf{Q} = \begin{Bmatrix} q & \text{for } \boldsymbol{x} \in \mathsf{V} \\ 0 & \text{for } \boldsymbol{x} \notin \mathsf{V} \end{Bmatrix} \text{``}= q \int_{\mathsf{V}} d^N x\, \delta(\boldsymbol{x})\text{''}\,.$$

Te point particle charge distribution may, thus, be approximated by a sequence of smooth charge distributions of the form $\varrho_\varepsilon := q\,\delta_\varepsilon$ with δ_ε given by the following definition:

Definition 4.1.5 Smooth Approximating Unity

Given $\delta_1 \in C_c^\infty(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with $\int dx\, \delta_1(\boldsymbol{x}) = 1$ and $\varepsilon > 0$, we define the function:

$$\delta_\varepsilon := \left(\boldsymbol{x} \mapsto \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\tfrac{|\boldsymbol{x}|}{\varepsilon}\right)\right).$$

It possesses the property $\int d^N x\, \delta_\varepsilon(\boldsymbol{x}) = 1$ and we call the net of functions $\left(\delta_\varepsilon\right)_{\varepsilon \in \mathbb{R}_{>0}}$ a **smooth approximation of unity**[a].

[a] A net is a generalization of a sequence. The difference is not important here so in case you are not familiar with this just read it as sequence and replace $\mathbb{R}_{>0}$ with $\left\{1/n | n \in \mathbb{N}\right\}$

Point Charge: Dense Charge Cloud

Indeed, just concerning the total charge Q in some open space region, the point particle distribution can be described by

$$\varrho = q \lim_{\varepsilon \to 0} \delta_\varepsilon$$

as being understood in the context of the integration

$$\mathsf{Q} = q \lim_{\varepsilon \to 0} \int_{\mathsf{V}} d^N x\, \delta_\varepsilon(\boldsymbol{x})$$

in the case of $\mathbf{0} \in \mathring{\mathsf{V}}$.

Put differently, a point particle charge distribution can be expressed as the $\text{limit}_{\varepsilon \to 0}$ of smooth charge distributions ϱ_ε having support in an area of scale ε around $\mathbf{0}$ and values of the scale $\frac{1}{\varepsilon} q$ such that integrating along a space region V containing $\mathbf{0}$ will result in q.

By a simple redefinition, this looks exactly like the elementary inhomogeneity:

Redefinition of the Elementary Inhomogeneity

If we force our elementary inhomogeneity χ_ε to fulfil additionally

- $\int dx\, \chi_\varepsilon(\boldsymbol{x}) = \varepsilon^N$

the function $\delta_\varepsilon := \left(\frac{1}{\varepsilon}\right)^N \chi_\varepsilon$ will be an smooth approximation of unity.

Thus, by replacing $\chi_\varepsilon \to (\varepsilon)^N \delta_\varepsilon$ and further $\tilde{G}^\partial_\varepsilon \to \left(\frac{1}{\varepsilon}\right)^N G^\partial_\varepsilon$ with $G^\partial_\varepsilon := \varepsilon^N \tilde{G}^\partial_\varepsilon$, we get:

- $\tilde{\rho}_\varepsilon(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \varepsilon^N \left(\mathsf{T}_{\boldsymbol{y}} \delta_\varepsilon\right)(\boldsymbol{x}) \rho(\boldsymbol{y})$ as the approximation of the inhomogeneity.
- $\partial G^\partial_\varepsilon = \delta_\varepsilon$ for the approximating Green's function to be fulfilled.
- $\tilde{\phi}_\varepsilon(\boldsymbol{x}) = \left(\sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \varepsilon^N \left(\mathsf{T}_{\boldsymbol{y}} G^\partial_\varepsilon(\boldsymbol{x})\right) \rho(\boldsymbol{y})\right)$ as the approximation of a solution.

From Sums to Integrals

It will be more convenient later, instead of using the approximations by sums, to introduce a slightly adapted version using integrals:

- $\rho_\varepsilon(\boldsymbol{x}) := \int d^N y \left(\mathsf{T}_{\boldsymbol{y}} \delta_\varepsilon\right)(\boldsymbol{x}) \rho(\boldsymbol{y})$
- $\phi_\varepsilon(\boldsymbol{x}) := \int d^N y \left(\mathsf{T}_{\boldsymbol{y}} G_\varepsilon\right)(\boldsymbol{x}) \rho(\boldsymbol{y})$

The discussion from before can be repeated in the exact same way by just performing the replacements[3]:

- $\sum_{\boldsymbol{y} \in \varepsilon\mathbb{Z}^N} \varepsilon^N \to \int d^N y$

[3] This step looks like the limit of a Riemann sum which is not the case here since even though the grid size ε approaches 0 you need to keep in mind that the variation of δ_ε is even increasing relatively to the grid size in this limit.

- $\tilde{\rho}_\varepsilon \to \rho_\varepsilon$
- $\tilde{\phi}_\varepsilon \to \phi_\varepsilon$

In the setting of integrals, the properties of the elementary inhomogeneity can be weakened while still ensuring the pointwise approximation of the inhomogeneity[4].

The result is summarized here:

Summary

Elementary Inhomogeneity

Fix, for all $\varepsilon > 0$, functions $\delta_\varepsilon \in C_c^\infty(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with the properties:

- $\mathrm{sp}(\delta_\varepsilon) \subset [\text{-}\varepsilon, \varepsilon]^N$
- $\int d^N y\, \delta_\varepsilon(\boldsymbol{y}) = 1$

Approximating Green's Function

A family $(G^\partial_\varepsilon)_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ is called approximating Green's function with respect to $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$ and $\partial \in \boldsymbol{\partial}$ if

- $\partial G^\partial_\varepsilon = \delta_\varepsilon$ for all $\varepsilon > 0$,
- $\lim_{\varepsilon\to 0} \int d^N y \left(\mathsf{T}_{\boldsymbol{y}} G^\partial_\varepsilon\right)(\boldsymbol{x}) \rho(\boldsymbol{y})$ exists for all $\boldsymbol{x} \in \mathbb{R}^N$,
- $\boldsymbol{x} \mapsto \lim_{\varepsilon\to 0} \int d^N y \left(\mathsf{T}_{\boldsymbol{y}} G^\partial_\varepsilon\right)(\boldsymbol{x}) \rho(\boldsymbol{y})$ is $\deg(\partial)$-smooth,
- and $\partial_{(\boldsymbol{x})} \lim_{\varepsilon\to 0} \int d^N y \left(\mathsf{T}_{\boldsymbol{y}} G^\partial_\varepsilon\right)(\boldsymbol{x}) \rho(\boldsymbol{y}) = \rho(\boldsymbol{x})$

for any $\rho \in \mathcal{I}$.

Point Charge: Dense Charge Cloud

We argued, that the point particle charge distribution can be understood as a limit

$$\varrho = q \lim_{\varepsilon \to 0} \delta_\varepsilon$$

which cannot be given any sense in the setting of functions. This puts us in front of the task to find another framework. It naturally arises in the strategy of finding classical solutions to our linear, translation invariant differential equations.

4.1.3 Convenient Setting for Green's Functions: Distributions

The observation that the approximated objects mainly appear in linear pairings with the, by assumption, very regular (continuous and compactly supported) inhomogeneity, raises the expectation that their limit can be understood in the setting of linear forms on regular objects within a suitable topology. We will pursue this line of thought in the following sections.

Linear Forms

As mentioned, we do only need an understanding of G^∂_ε, δ_ε and $\partial G^\partial_\varepsilon = \delta_\varepsilon$ in the ε to 0 limit as linear forms mapping functions of specific regularities to complex numbers. Using a simple substitution, let us reshape

[4] It can actually be weakened even further than presented in the summary.

- the approximation of the inhomogeneity by

$$\rho_\varepsilon(\boldsymbol{x}) = \int d^N y \, (\mathsf{T}_{\boldsymbol{y}}\delta_\varepsilon)(\boldsymbol{x})\rho(\boldsymbol{y}) = \int d^N y \, \delta_\varepsilon(\boldsymbol{y})\rho(\boldsymbol{x}-\boldsymbol{y}) = \int d^N y \, \delta_\varepsilon(\boldsymbol{y})\widetilde{\rho}(\boldsymbol{y}-\boldsymbol{x}) = \int d^N y \, \delta_\varepsilon(\boldsymbol{y})(\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho})(\boldsymbol{y}) =: \delta_\varepsilon[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$$

with $\widetilde{\varrho} : \mathbb{R}^N \to \mathbb{C}, \boldsymbol{x} \mapsto \varrho(-\boldsymbol{x})$

- and the approximation of a solution by

$$\varphi_\varepsilon(\boldsymbol{x}) = \int d^N y \, (\mathsf{T}_{\boldsymbol{y}}G_\varepsilon)(\boldsymbol{x})\varrho(\boldsymbol{y}) = \int d^N y \, G_\varepsilon(\boldsymbol{y})(\mathsf{T}_{\boldsymbol{x}}\widetilde{\varrho})(\boldsymbol{y}) =: G_\varepsilon[\mathsf{T}_{\boldsymbol{x}}\widetilde{\varrho}] \tag{4.1}$$

and use this notation in our summary:

Summary

Elementary Inhomogeneity

Fix, for all $\varepsilon > 0$, functions $\delta_\varepsilon \in C_c^\infty(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with the properties:

- $\mathrm{sp}(\delta_\varepsilon) \subset [\text{-}\varepsilon, \varepsilon]^N$.
- $\int d^N y \, \delta_\varepsilon(\boldsymbol{y}) = 1$.

Approximating Green's Function

A family $(G_\varepsilon^\partial)_{\varepsilon>0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ is called approximating Green's function with respect to $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$ and $\partial \in \boldsymbol{\partial}$ if

- $\partial G_\varepsilon^\partial = \delta_\varepsilon$ for all $\varepsilon > 0$,
- $\lim_{\varepsilon\to 0} G_\varepsilon^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$ exists for all $\boldsymbol{x} \in \mathbb{R}^N$,
- $\boldsymbol{x} \mapsto \lim_{\varepsilon\to 0} G_\varepsilon^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$ is $\deg(\partial)$-smooth,
- and $\partial_{(\boldsymbol{x})} \lim_{\varepsilon\to 0} G_\varepsilon^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}] = \rho(\boldsymbol{x})$

for any $\rho \in \mathcal{I}$.

The procedure of linearly mapping a function to a complex number via integration with respect to another function will be part of the next definition:

Definition 4.1.8 Linear Forms

Given a subspace D of $F(\mathbb{R}^N, \mathbb{C})$ we define the **linear forms** by:

$$\mathsf{LF}[D, \mathbb{C}] := \{u \in F(D, \mathbb{C}) | \text{u is linear}\}$$

A locally bounded, integrable function $g \in L_{loc}^\infty(\mathbb{R}^N, \mathbb{C})$ canonical induces a linear form[a] on the domain of locally integrable functions $L_{loc}^1(\mathbb{R}^N, \mathbb{C})$ with decay properties depending on the decay of g:

- Having no information about the decay of g, we choose the functions in the domain to be compactly supported, i.e. in $L_{loc,c}^1$, i.e.,

$$g : L_{loc,c}^1 \to \mathbb{C}, f \mapsto g[f] := \int d^N x \, g(\boldsymbol{x}) f(\boldsymbol{x}) \, .$$

- If g is of compact support itself, we do not restrict the decay of the functions in the domain, i.e.,

$$g : L^1_{loc} \to \mathbb{C}, f \mapsto g[f] := \int d^N x\, g(\boldsymbol{x}) f(\boldsymbol{x})$$

We say that the function $g(\,\cdot\,)$ can be represented as a linear form $g[\,\cdot\,]$ (on the appropriate domain) and the other way around.

[a] We will denote the linear form with the same symbol. If we want to differentiate we call g as a function by $g(\,\cdot\,)$ and as a linear form by $g[\,\cdot\,]$.

Lemma 4.1.9 Properties of Linear Forms

$\mathsf{LF}[D, \mathbb{C}]$ is a $\mathbb{C}$-vector space by the definition of addition and scalar multiplication by

$$(u + c \cdot v) : D \mapsto \mathbb{C}, f \mapsto (u + c \cdot v)[f] := u[f] + c \cdot v[f]$$

for all $u, v \in \mathsf{LF}[D, \mathbb{C}]$ and $c \in \mathbb{C}$.

Topology on Linear Forms

In our definition of an elementary inhomogeneity and the properties of an approximating Green's function $(G^\partial_\varepsilon)_{\varepsilon>0}$ with respect to $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$, we demanded that, for all $\rho \in \mathcal{I}$ and $\boldsymbol{x} \in \mathbb{R}^N$,

- $\lim_{\varepsilon\to 0} \delta_\varepsilon[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$ exists and
- $\lim_{\varepsilon\to 0} G^\partial_\varepsilon[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$ exists

and the following expressions are well-defined[5]:

$$\delta : \mathcal{I} \to \mathbb{C}, f \mapsto \delta[f] := \lim_{\varepsilon\to 0} \delta_\varepsilon[f] \tag{4.2}$$

$$G^\partial : \mathcal{I} \to \mathbb{C}, f \mapsto G^\partial[f] := \lim_{\varepsilon\to 0} G^\partial_\varepsilon[f] \quad . \tag{4.3}$$

We call $G^\partial \in \mathsf{LF}[\mathcal{I}, \mathbb{C}]$ the Green's function corresponding to $(G^\partial_\varepsilon)_{\varepsilon>0}$. The, so called, delta form $\delta \in \mathsf{LF}[\mathcal{I}, \mathbb{C}]$ can explicitly determined by $\delta[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}] = \lim_{\varepsilon\to 0} \delta_\varepsilon[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}] = \rho(\boldsymbol{x})$ resulting in its given form in Definition/Proposition 4.1.11.

Limits of linear forms are thus to be understood in our context with respect to the following topology:

Definition 4.1.10 Topology on Linear Forms

Given linear forms $\mathsf{LF}[D, \mathbb{C}]$ we define its topology $\tau_{\mathsf{LF}[D]}$ to be the weak topology with respect to the domain. By $\mathsf{LF}^\tau[D, \mathbb{C}]$ we denote the topological vector space $\big(\mathsf{LF}[D, \mathbb{C}], \tau_{\mathsf{LF}[D]}\big)$.

Definition Proposition 4.1.11 Delta Form and its Approximations

We call the linear form $\delta : F(\mathbb{R}^N, \mathbb{C}) \to \mathbb{C}, f \mapsto \delta[f] := f(\mathbf{0})$ the δ form.

All approximating unities δ_ε converge to δ in $\mathsf{LF}^\tau[\{f \in L^\infty(\mathbb{R}^N, \mathbb{C}) | f \text{ is cointinous at } \mathbf{0}\}, \mathbb{C}]$.

Proof

Regarding an $f \in L^\infty(\mathbb{R}^N, \mathbb{C})$ being continuous at $\mathbf{0}$, we split the appearing integrals in the embedding of δ_ε in $\mathsf{LF}^\tau\big[\{f \in L^\infty(\mathbb{R}^N, \mathbb{C}) | f \text{ is cointinous at } \mathbf{0}\}, \mathbb{C}\big]$ in two regions, namely, $\mathsf{B}_c(\mathbf{0}) :=$

[5] We choose $\mathcal{I}$ such that its closed under translations and $f \mapsto \tilde{f}$.

$\{\boldsymbol{x} \in \mathbb{R}^N | |\boldsymbol{x}| < c\}$ and $\mathbb{R}^N \setminus \mathsf{B}_c(\mathbf{0})$ with $c > 0$. We estimate:

$$
\begin{aligned}
|\delta_\varepsilon[f] - \delta[f]| &= \left| \int_{\mathbb{R}^N} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) f(\boldsymbol{x}) - f(\mathbf{0}) \right| \\
&\leq \int_{\mathbb{R}^N} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) |f(\boldsymbol{x}) - f(\mathbf{0})| \\
&= \int_{\mathsf{B}_c(\mathbf{0})} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) |f(\boldsymbol{x}) - f(\mathbf{0})| + \int_{\mathbb{R}^N \setminus \mathsf{B}_c(\mathbf{0})} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) |f(\boldsymbol{x}) - f(\mathbf{0})| \\
&\leq \int_{\mathsf{B}_c(\mathbf{0})} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) |f(\boldsymbol{x}) - f(\mathbf{0})| + 2\|f\|_\infty \int_{\mathbb{R}^N \setminus \mathsf{B}_c(\mathbf{0})} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right)
\end{aligned}
$$

In this form, for given $\epsilon > 0$, may we estimate individually:

- There exists $c_\epsilon > 0$ s.t. $\forall |\boldsymbol{x}| < c_\epsilon$ we find $|f(\boldsymbol{x}) - f(\mathbf{0})| < \epsilon/2$ due to the continuity of f at $\mathbf{0}$.
- By substitution in step (i), we find

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus \mathsf{B}_{c_\epsilon}(\mathbf{0})} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} d^N x\, \mathbb{1}_{\mathbb{R}^N \setminus \mathsf{B}_{c_\epsilon}(\mathbf{0})}(\boldsymbol{x}) \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) \\
&\overset{(i)}{=} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} d^N y\, \mathbb{1}_{\mathbb{R}^N \setminus \mathsf{B}_{c_\epsilon}(\mathbf{0})}(\varepsilon \boldsymbol{y}) \delta_1(|\boldsymbol{x}|) \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} d^N y\, \mathbb{1}_{\mathbb{R}^N \setminus \mathsf{B}_{c_\epsilon/\varepsilon}(\mathbf{0})}(\boldsymbol{y}) \delta_1(|\boldsymbol{x}|) \\
&\overset{(ii)}{=} 0
\end{aligned}
$$

  and the application of dominated convergence in connection with the pointwise convergence of $\mathbb{1}_{\mathbb{R}^N \setminus \mathsf{B}_{c_\epsilon/\varepsilon}(\mathbf{0})}(\,\cdot\,)$ to 0. Furthermore, regarding (ii) we can find $\tilde{\varepsilon} > 0$, such that, for all $\varepsilon \leq \tilde{\varepsilon}$ we get

$$
\int_{\mathbb{R}^N \setminus \mathsf{B}_c(\mathbf{0})} d^N x \left(\tfrac{1}{\varepsilon}\right)^N \delta_1\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right) \leq \frac{\epsilon}{4\|f\|_\infty}.
$$

This results in the estimate

$$
|\delta_\varepsilon[f] - \delta[f]| \leq \frac{\epsilon}{2} + 2\|f\|_\infty \frac{\epsilon}{4\|f\|_\infty} = \epsilon, \quad \forall \varepsilon \leq \tilde{\varepsilon},
$$

i.e. the desired result.

With these notions in place we reformulate our summary:

Summary 4.1.12

Elementary Inhomogeneity

Fix, for all $\varepsilon > 0$, functions $\delta_\varepsilon \in C_c^\infty(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with the properties:

- $\mathrm{sp}(\delta_\varepsilon) \subset [\text{-}\varepsilon, \varepsilon]^N$
- $\int d^N y\, \delta_\varepsilon(\boldsymbol{y}) = 1$

Approximating Green's Function

A family $(G_\varepsilon^\partial)_{\varepsilon > 0} \subset C^{\deg(\partial)}(\mathbb{R}^N, \mathbb{C})$ is called approximating Green's function with respect to $\mathcal{I} \subset C_c(\mathbb{R}^N, \mathbb{C})$ and $\partial \in \boldsymbol{\mathfrak{d}}$ if

1. $\partial G_\varepsilon^\partial = \delta_\varepsilon$ for all $\varepsilon > 0$,

2. $G^\partial = \lim_{\varepsilon\to 0} G^\partial_\varepsilon$ exists in $\mathsf{LF}^\tau[\mathcal{I}, \mathbb{C}]$,
3. $\boldsymbol{x} \mapsto G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$ is $\deg(\partial)$-smooth,
4. and $\partial_{(\boldsymbol{x})} G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}] = \rho(\boldsymbol{x})$

for any $\rho \in \mathcal{I}$.

Point Charge: Dense Charge Cloud

The notion of linear forms on functions already enables us to take a step forward in the formalization of our discussion of the total charge Q in the spacetime region V.

- In the case of a smooth static charge distribution $\varrho \in C^\infty(\mathbb{R}^3, \mathbb{R}_{\geq 0})$ we get

$$\mathsf{Q} = \int_{\mathsf{V}} d^N x\, \varrho(\boldsymbol{x}) = \varrho[\mathbb{1}_{\mathsf{V}}]\,,$$

- while we find for the static point particle $\varrho = q\,\delta$

$$\mathsf{Q} = \begin{Bmatrix} q & \text{for } \boldsymbol{x} \in \mathsf{V} \\ 0 & \text{for } \boldsymbol{x} \notin \mathsf{V} \end{Bmatrix} = q\,\delta[\mathbb{1}_{\mathsf{V}}] = \varrho[\mathbb{1}_{\mathsf{V}}]\,.$$

Furthermore, the notion of convergence on the linear forms fits well with the approximation of a point charge by an increasingly dense charge cloud of fixed Q. We observe that, under the assumption $\mathbf{0} \notin \mathsf{V}$,

$$\lim_{\varepsilon\to 0} \varrho_\varepsilon[\mathbb{1}_{\mathsf{V}}] = q \lim_{\varepsilon\to 0} \delta_\varepsilon[\mathbb{1}_{\mathsf{V}}] \quad = q \lim_{\varepsilon\to 0} \delta_\varepsilon[\chi_{\mathsf{V}}] = q\delta[\chi_{\mathsf{V}}] \quad = q\delta[\mathbb{1}_{\mathsf{V}}] = \varrho[\mathbb{1}_{\mathsf{V}}]$$

where χ_{V} approximates the incontinuouty $\mathbb{1}_{\mathsf{V}}$ on a scale smaller than $\mathrm{dist}(\mathbf{0}, \partial\mathsf{V})$

Remark 4.1.13

Let us be aware that, so far, our treatment of stationary singular inhomogeneities **is not** on the same footing the inhomogeneities in our strategy. It is merely a byproduct of our solution strategy. Within "the construction phase" of our strategy in this section the set of inhomogeneities is very regular. It will be the main goal of Section 5.1 to apply our strategy to inhomogeneities of less regularity.

Weak Derivative on Linear Forms

We are able to understand the limits of δ_ε and of G^∂_ε in the sense of linear forms. Let us also analyze the limit of their relation, i.e. $\partial G^\partial_\varepsilon = \delta_\varepsilon$ for $\varepsilon \to 0$, in this setting. Since $G^\partial_\varepsilon[\,\cdot\,]$ is representable by a smooth function, we can define its derivatives by referring to the notion of partial derivatives in the regular setting. Its limit might not be of this form. This implies the need for another approach to define derivatives for linear forms of any kind which will be guided by the following calculation:

$$\begin{aligned}
\big(\partial^{\boldsymbol{\alpha}} G^\partial_\varepsilon\big)[f] &= \int d^N y\, \big(\partial^{\boldsymbol{\alpha}} G^\partial_\varepsilon\big)(\boldsymbol{y}) f(\boldsymbol{y}) \\
&= \int d^N y\, G^\partial_\varepsilon(\boldsymbol{y}) (\text{-}1)^{|\boldsymbol{\alpha}|} \big(\partial^{\boldsymbol{\alpha}} f\big)(\boldsymbol{y}) \\
&= G^\partial_\varepsilon\big[(\text{-}1)^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} f\big]
\end{aligned}$$

Under the condition, that the domain $\mathcal{I}$

- consists of differentiable functions $\big(\rho \in \mathcal{I} \implies \rho \in C^1(\mathbb{R}^N, \mathbb{C})\big)$
- and is closed under differentiation $\big(\rho \in \mathcal{I} \implies \partial_i \rho \in \mathcal{I}\big)$,

$\Big\} \implies \mathcal{I} \subset C^\infty(\mathbb{R}^N, \mathbb{C})$

we are able to define the following weak derivative of linear forms, which commutes with approximations in the topology $\tau_{\mathsf{LF}[D]}$:

Definition 4.1.14 Weak Derivative on Linear Forms with $D \subset C^\infty$

Given a subspace D of $F(\mathbb{R}^N, \mathbb{C})$ consisting of differentiable functions, which is also closed under differentiation, we define the weak derivative for $u \in \mathsf{LF}[D, \mathbb{C}]$ by

$$\partial^{\boldsymbol{\alpha}} u : D \to \mathbb{C}, f \mapsto (\partial^{\boldsymbol{\alpha}} u)[f] := u\big[(\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} f\big]$$

and observe $\partial^{\boldsymbol{\alpha}} u \in \mathsf{LF}[D, \mathbb{C}]$.

For any given $\partial = \sum_{\boldsymbol{\alpha}\in \mathrm{A}} a_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}$ with $\mathrm{A} \subset \mathbb{N}_0^N$ and $a_{\boldsymbol{\alpha}} \in \mathbb{C}$, we define $\partial^{\mathrm{t}} = \sum_{\boldsymbol{\alpha}\in \mathrm{A}} a_{\boldsymbol{\alpha}} (\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}}$ and

$$\partial u : D \to \mathbb{C}, f \mapsto (\partial u)[f] := u[\partial^{\mathrm{t}} f]\,.$$

Lemma 4.1.15 Weak Conitnuity of Partial Diff. Operators

Given a subspace D of $F(\mathbb{R}^N, \mathbb{C})$ consisting of differentiable functions, which is also closed under differentiation, and for all $\varepsilon > 0$ forms $u_\varepsilon, u \in \mathsf{LF}[D, \mathbb{C}]$, such that $\lim_{\varepsilon\to 0} u_\varepsilon = u$ in $\mathsf{LF}^\tau[D, \mathbb{C}]$, then

$$\lim_{\varepsilon\to 0} \partial^{\boldsymbol{\alpha}} u_\varepsilon = \partial^{\boldsymbol{\alpha}} u\,.$$

Proof

Given $f \in D$, then by assumption we have also $(\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} f \in D$, and thus,

$$\lim_{\varepsilon\to 0} \partial^{\boldsymbol{\alpha}} u_\varepsilon[f] = \lim_{\varepsilon\to 0} \partial^{\boldsymbol{\alpha}} u_\varepsilon[(\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} f] = u[(\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} f] = \partial^{\boldsymbol{\alpha}} u[f]\,,$$

i.e., $\lim_{\varepsilon\to 0} \partial^{\boldsymbol{\alpha}} u_\varepsilon = \partial^{\boldsymbol{\alpha}} u$ in $\mathsf{LF}^\tau[D, \mathbb{C}]$.

Thus, abbreviate in the language of weak derivatives on linear forms, ew have shown

$$\partial G^\partial = \partial \lim_{\varepsilon\to 0} G^\partial_\varepsilon = \lim_{\varepsilon\to 0} \partial G^\partial_\varepsilon = \lim_{\varepsilon\to 0} \delta_\varepsilon = \delta\,. \tag{4.4}$$

Point Charge: Solution

We are now able to formulate the differential equation for a field to be generated by an at the origin resting point particle. We understand for given $\varphi \in \mathsf{LF}[C_c^\infty, \mathbb{C}]$ the equation

$$\partial\varphi = \varrho$$

with $\varrho = q\,\delta$ as an equation of linear forms. ∂ acts on ϕ in the weak sense of Def. 4.1.14. Having a Green's function $G^\partial \in \mathsf{LF}[C_c^\infty, \mathbb{C}]$ available, we can even provide an explicit solution by $\varphi = q\,G^\partial$.

We will see now that the condition $\mathcal{G} \subset C^\infty(\mathbb{R}^N, \mathbb{C})$ and $\mathcal{G}$ being closed under differentiation is convenient also from another perspective.

Distributions

We turn now towards the condition that a Green's function, corresponding to approximating Green's functions $(G^\partial_\varepsilon)_{\varepsilon>0}$ with respect to $\mathcal{G}$, needs to have the property that for all $\rho \in \mathcal{G}$, the function $\boldsymbol{x} \mapsto G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]$ is $\deg(\partial)$-smooth. Our goal for this section is to analyze it and replace it with one of similar strength but easier accessibility.

We try to understand the limit of the differential quotient of $G^\partial[\mathsf{T}.\widetilde{\rho}]$, given by

$$\begin{aligned}\lim_{h\to 0}\tfrac{1}{h}\Big(G^\partial[\mathsf{T}_{\boldsymbol{x}+h\boldsymbol{e}_i}\widetilde{\rho}] - G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]\Big) &= \lim_{h\to 0}\tfrac{1}{h}\Big(G^\partial_{(\cdot)}[\rho(\boldsymbol{x}+h\boldsymbol{e}_i-\cdot)] - G^\partial_{(\cdot)}[\rho(\boldsymbol{x}-\cdot)]\Big)\\ &= \lim_{h\to 0} G^\partial_{(\cdot)}\Big[\tfrac{1}{h}\big(\rho(\boldsymbol{x}+h\boldsymbol{e}_i-\cdot)-\rho(\boldsymbol{x}-\cdot)\big)\Big]\\ &= \lim_{h\to 0} G^\partial_{(\cdot)}\Big[\text{-}\tfrac{1}{h}\Big(\rho(\boldsymbol{x}-\cdot)-\rho\big(\boldsymbol{x}-(\cdot-h\boldsymbol{e}_i)\big)\Big)\Big]\\ &= \lim_{h\to 0} G^\partial_{(\cdot)}\Big[\text{-}\mathsf{T}_{\boldsymbol{x}}\Big(\tfrac{1}{h}\big(\widetilde{\rho}(\cdot)-\widetilde{\rho}(\cdot-h\boldsymbol{e}_i)\big)\Big)\Big]\,.\end{aligned}$$

Suppose for a moment that we are allowed to pull $\lim_{h\to 0}$ through the evaluation of $G^\partial[\cdot]$. Then, under the condition that $\mathcal{I}$

- consists of differentiable functions $\big(\rho\in\mathcal{I} \implies \rho\in C^1(\mathbb{R}^N,\mathbb{C})\big)$
- and is closed under differentiation $\big(\rho\in\mathcal{I} \implies \partial_i\rho\in\mathcal{I}\big)$

$\Big\}\implies \mathcal{I}\subset C^\infty(\mathbb{R}^N,\mathbb{C})\,,$

we find that $G^\partial[\mathsf{T}.\widetilde{\rho}]$ is partially differentiable and $\partial_{\boldsymbol{x}_i}\big(G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]\big) = G^\partial_{(\cdot)}\big[\partial_{\cdot_i}\big(\text{-}\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}(\cdot)\big)\big]$.

By induction, we get for all $\boldsymbol{\alpha}\in\mathbb{N}_0^N$ the relation $\partial^{\boldsymbol{\alpha}}_{(\boldsymbol{x})}\big(G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}]\big) = G^\partial_{(\cdot)}\big[(\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}}_{(\cdot)}\big(\mathsf{T}_{\boldsymbol{x}}\widetilde{\rho}(\cdot)\big)\big]$, still under the same assumption that

$$\forall f\in\mathcal{I}\qquad \lim_{h\to 0} G^\partial_{(\cdot)}\big[\tfrac{1}{h}\big(f(\cdot)-f(\cdot-h\boldsymbol{e}_i)\big)\big] = G^\partial_{(\,\cdot\,)}\big[\partial_{\cdot_i}f(\cdot)\big]\,, \tag{4.5}$$

holds true. Further, this allows, given $\boldsymbol{x}\in\mathbb{R}^N$, the following manipulations

$$\begin{aligned}\partial\big(G^\partial[\mathsf{T}_{\boldsymbol{x}}\widetilde{\varrho}]\big) &= G^\partial_{(\cdot)}\big[\partial^{\mathsf{t}}_{\cdot}\,\mathsf{T}_{\boldsymbol{x}}\widetilde{\varrho}(\cdot)\big]\\ &\overset{(i)}{=} (\partial G^\partial)[\mathsf{T}_{\boldsymbol{x}}\widetilde{\varrho}] && (i)\text{ by Def. 4.1.14}\\ &\overset{(ii)}{=} \delta[\mathsf{T}_{\boldsymbol{x}}\widetilde{\varrho}] && (ii)\text{ by Equation (4.4)}\\ &\overset{(iii)}{=} \varrho(\boldsymbol{x}) && (iii)\text{ by Definition/Proposition 4.1.11}\end{aligned} \tag{4.6}$$

which implies that both Item 3 and Item 4 in Summary 4.1.12 can be replaced by this condition.

Following this line of argumentation, we may choose $C_c(\mathbb{R}^N,\mathbb{C})\cap C^\infty(\mathbb{R}^N,\mathbb{C}) = C_c^\infty(\mathbb{R}^N,\mathbb{C})$[6] as the set of inhomogeneities, i.e., $\mathcal{I} = C_c^\infty(\mathbb{R}^N,\mathbb{C})$.

Continuity: The condition in Equation (4.5) can be replaced by endowing C_c^∞ with a topology $\tau_{C_c^\infty}$, such that

$$f\in C_c^\infty(\mathbb{R}^N,\mathbb{C})\quad \tfrac{1}{h}\big(f(\cdot)-f(\cdot-h\boldsymbol{e}_i)\big)\xrightarrow{\tau_{C_c^\infty}}\partial_{\cdot_i}f(\cdot)\text{ for }h\to 0 \tag{4.7}$$

and forcing $G^\partial : (C_c^\infty,\tau_{C_c^\infty})\to(\mathbb{C},|\cdot|)$ to be continuous.

This replacement leads to the interesting question:

How much more restrictive is the new condition?

The new condition is sufficient but seems far from necessary to imply Equation (4.5). If one would like to try to extend the idea of Green's functions from the one that is constructed in the following setting, this seems to be the best starting point. Fortunately this possibly large step in the restriction of our framework is still sufficient to supply us with the tools relevant for our physical theories and the author is not aware of any Green's functions relying on an extension of this kind. Additionally, the continuity condition on the Green's function will later be the key feature to extend the formalism to irregular inhomogeneities.

In the following we will analyze Equation (4.5) to find minimal requirement for $\tau_{C_c^\infty}$ to fulfil.

[6] Remember that the condition $\mathcal{I}\subset C_c$ has been chosen in order to approximate the inhomogeneity (continuity) and for the well-definedness of the potential approximation of a solution (compact support).

The L^1_{loc} Case We want to draw guidance from our understanding of interchanges of limits in the context of functions. Let us suppose for a moment, that $G^\partial(\,\cdot\,)$ lies in $L^1_{loc}(\mathbb{R}^N,\mathbb{C})$, which is the most general case in which the embedding of functions into linear forms of domain $C^\infty_c(\mathbb{R}^N,\mathbb{C})$ by Def. 4.1.8 is applicable.

In this setting the condition in Equation (4.5) turns into of a more familiar form of an interchange of limit and integral:

$$\lim_{h\to 0}\int d^Ny\, G^\partial(\boldsymbol{y})\tfrac{1}{h}\big(f(\boldsymbol{y})-f(\boldsymbol{y}\text{-}h\boldsymbol{e}_i)\big)=\int d^Ny\, G^\partial(\boldsymbol{y})\,\partial_{\boldsymbol{y}_i}f(\boldsymbol{y})\,.$$

Now, since $f\in\mathcal{I}$ is of compact support, we define a set $\mathsf{K}:=\overline{\{\boldsymbol{x}+h\boldsymbol{e}_i\in\mathbb{R}^N\,|\,\boldsymbol{x}\in\mathrm{sp}(f)\text{ and }h\in[\text{-}1,1]\}}$, which is again compact and thus $\|G^\partial\|_{L^1(\mathsf{K},\mathbb{C})}<\infty$ enabling the following estimation for $|h|\le 1$:

$$\begin{aligned}&\left|\int d^Ny\, G(\boldsymbol{y})\Big(\tfrac{1}{h}\big(f(\boldsymbol{y})-f(\boldsymbol{y}\text{-}h\boldsymbol{e}_i)\big)-\partial_{\boldsymbol{y}_i}f(\boldsymbol{y})\Big)\right|\\ &\le\int d^Ny\,|G(\boldsymbol{y})|\left|\tfrac{1}{h}\big(f(\boldsymbol{y})-f(\boldsymbol{y}\text{-}h\boldsymbol{e}_i)\big)-\partial_{\boldsymbol{y}_i}f(\boldsymbol{y})\right|\\ &\le\|G^\partial\|_{L^1(\mathsf{K},\mathbb{C})}\sup_{\boldsymbol{y}\in\mathsf{K}}\left|\tfrac{1}{h}\big(f(\boldsymbol{y})-f(\boldsymbol{y}\text{-}h\boldsymbol{e}_i)\big)-\partial_{\boldsymbol{y}_i}f(\boldsymbol{y})\right|\\ &=:\|G^\partial\|_{L^1(\mathsf{K},\mathbb{C})}\underbrace{\left\|\Big(\tfrac{1}{h}\big(f(\,\cdot\,)-f(\,\cdot\,\text{-}h\boldsymbol{e}_i)\big)-\partial_{\cdot_i}f(\,\cdot\,)\Big)\right\|_{\mathsf{K},\infty}}_{\longrightarrow 0\text{ for }h\to 0}\,.\end{aligned}\tag{4.8}$$

Note that, since the inequality in Equation (4.8) needs to hold for any $f\in C^\infty_c$ as well as possibly many $G^\partial\in L^1_{loc}$ of very different shape, there is not really any hope to find more relaxed but still sufficient alternatives to our line of argumentation which is valid for all these cases.

The property of $G^\partial(\,\cdot\,)\in L^1_{loc}(\mathbb{R}^N,\mathbb{C})$ which we take advantage in Equation (4.8) in order to show Equation (4.5) is that

$$\forall\,\mathsf{K}\subset\mathbb{R}^N\text{ compact }\exists\,C_{\mathsf{K}}\ge 0\text{ such that }\forall f\in C^\infty_c(\mathsf{K},\mathbb{C})\text{ it follows }\big|G^\partial[f]\big|\le\|f\|_{\mathsf{K},\infty}\,,$$

which is a statement about the continuity of $G^\partial[\,\cdot\,]$ (see Theorem 4.1.18 below) as a map from $(C^\infty_c,\tau^{\infty,\mathbf{0}}_{C^\infty_c})$ to $(\mathbb{C},|\cdot|)$ with the topology $\tau^{\infty,\mathbf{0}}_{C^\infty_c}$ given by Def. 4.1.17.

Definition 4.1.17 Topology $\tau^{\infty,\mathbf{0}}_{C^\infty_c}$ on C^∞_c

We will build a locally convex space $\big(C^\infty_c(\mathbb{R}^N,\mathbb{C}),\tau^{\infty,\mathbf{0}}_{C^\infty_c}\big)$ as a strict inductive limit of the spaces $\big(C^\infty_c(\mathsf{K}_n,\mathbb{C}),\tau^{\infty,\mathbf{0}}_{C^\infty_c(\mathsf{K}_n)}\big)$ step by step:

- First we observe, that $C^\infty_c(\mathbb{R}^N,\mathbb{C})=\bigcup_{n\in\mathbb{N}}C^\infty_c(\mathsf{K}_n,\mathbb{C})$ with $\mathsf{K}_n:=[-n,n]^N$.
- On each $C^\infty_c(\mathsf{K}_n,\mathbb{C})$, we consider the norm
$$\|\cdot\|_{\mathsf{K}_n,\infty}:C^\infty_c(\mathsf{K}_n,\mathbb{C})\to\mathbb{R}_{\ge 0},f\mapsto\|f\|_{\mathsf{K}_n,\infty}:=\sup_{\boldsymbol{x}\in\mathsf{K}_n}|f(\boldsymbol{x})|\,.$$
- We endow $C^\infty_c(\mathsf{K}_n,\mathbb{C})$ with the topology $\tau^{\infty,\mathbf{0}}_{C^\infty_c(\mathsf{K}_n)}$ induced by the norm $\|\cdot\|_{\mathsf{K}_n,\infty}$.
- And finally, we consider the inclusion maps
$$\iota_n:\big(C^\infty_c(\mathsf{K}_n,\mathbb{C}),\tau^{\infty,\mathbf{0}}_{C^\infty_c(\mathsf{K}_n)}\big)\hookrightarrow\big(C^\infty_c(\mathbb{R}^N,\mathbb{C}),\tau^{\infty,\mathbf{0}}_{C^\infty_c}\big)$$
and choose the finest $\tau^{\infty,\mathbf{0}}_{C^\infty_c}$, such that for all $n\in\mathbb{N}$ ι_n is continuous.

Theorem 4.1.18 **Characterization of Continuity on** $\mathrm{LF}\Big[\big(C_c^\infty, \tau_{C_c^\infty}^{\infty,\mathbf{0}}\big), \big(\mathbb{C}, |\cdot|\big)\Big]$

For a map $u : \big(C_c^\infty, \tau_{C_c^\infty}^{\infty,\mathbf{0}}\big) \to \big(\mathbb{C}, |\cdot|\big)$ the following properties are equivalent:

- u is continuous.
- For all $n \in \mathbb{N}$, there exists $C_n \geq 0$, such that $\forall f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$

$$|u[f]| \leq C_n \|f\|_{\mathsf{K}_n,\infty}\,.$$

Proof

[RSI80, Theorem V.2]

If we want to include such cases into our framework, we get the requirement that $\tau_{C_c^\infty}$ is a refinement of $\tau_{C_c^\infty}^{\infty,\mathbf{0}}$.

Weak Derivatives of L^1_{loc} Comparing Green's functions of fixed $\partial \in \boldsymbol{\partial}$ with those, for given $\boldsymbol{\alpha} \in \mathbb{N}_0^N$, of $\partial\partial^{\boldsymbol{\alpha}}$, we should expect higher regularity of the later. If there exists, for fixed $\boldsymbol{\alpha} \in \mathbb{N}_0^N$, a family $(G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}})_{\varepsilon>0}$ of approximating Green's functions with respect to $\partial\partial^{\boldsymbol{\alpha}}$ and $\mathcal{I}$ whose limit $G^{\partial\partial^{\boldsymbol{\alpha}}}$ lies in $L^1_{loc}(\mathbb{R}^N, \mathbb{C})$, the we are able to construct $(G_\varepsilon^{\partial})_{\varepsilon>0}$ by defining $G_\varepsilon^{\partial} := \partial^{\boldsymbol{\alpha}} G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}}$ and observe:

- $$\begin{aligned} \partial G_\varepsilon^{\partial} &\overset{(i)}{=} \partial\partial^{\boldsymbol{\alpha}} G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}} && (i) \text{ by the definition } G_\varepsilon^{\partial} \\ &= \delta_\varepsilon \end{aligned}$$

  We conclude, that $(G_\varepsilon^{\partial})_{\varepsilon>0}$ fulfils Item 1 in Summary 4.1.12.

- $$\begin{aligned} G^{\partial}[f] &:= \lim_{\varepsilon\to 0} G_\varepsilon^{\partial}[f] \\ &\overset{(i)}{=} \lim_{\varepsilon\to 0} \big(\partial^{\boldsymbol{\alpha}} G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}}\big)[f] && (i) \text{ by the definition } G_\varepsilon^{\partial} \\ &= \lim_{\varepsilon\to 0} \int d^N y \, \big(\partial^{\boldsymbol{\alpha}} G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}}\big)(\boldsymbol{y}) f(\boldsymbol{y}) \\ &= \lim_{\varepsilon\to 0} \int d^N y \, G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}}(\boldsymbol{y}) \, (\text{-}1)^{|\boldsymbol{\alpha}|} \big(\partial^{\boldsymbol{\alpha}} f\big)(\boldsymbol{y}) \\ &= \lim_{\varepsilon\to 0} G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}}[(\text{-}1)^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} f] \\ &\overset{(ii)}{=} G^{\partial\partial^{\boldsymbol{\alpha}}}[(\text{-}1)^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} f] && (ii) \text{ by Item 2 in Summary 4.1.12 for } G^{\partial\partial^{\boldsymbol{\alpha}}} \\ &\overset{(iii)}{=} \big(\partial^{\boldsymbol{\alpha}} G^{\partial\partial^{\boldsymbol{\alpha}}}\big)[f] && (iii) \text{ by Def. 4.1.14} \end{aligned}$$

  I.e., the limit $\varepsilon \to 0$ of G_ε^{∂} exists leading to the fulfillment of Item 2 in Summary 4.1.12.

- $$\begin{aligned} \boldsymbol{x} &\mapsto G^{\partial}[\mathsf{T}_{\boldsymbol{x}} \widetilde{\rho}] \\ &= G_{(\cdot)}^{\partial\partial^{\boldsymbol{\alpha}}}[(\text{-}1)^{|\boldsymbol{\alpha}|} \partial_{(\cdot)}^{\boldsymbol{\alpha}} \mathsf{T}_{\boldsymbol{x}} \widetilde{\rho}_{(\cdot)}] \\ &= G^{\partial\partial^{\boldsymbol{\alpha}}}[\mathsf{T}_{\boldsymbol{x}} \widetilde{\rho}'] \end{aligned}$$

  Hereby, we define $\rho'(\boldsymbol{x}) := (\text{-}1)^{|\boldsymbol{\alpha}|} \partial_{(\boldsymbol{x})}^{\boldsymbol{\alpha}} \rho(\boldsymbol{x})$ and, thus, smooth, by Item 3 in Summary 4.1.12 for $(G_\varepsilon^{\partial\partial^{\boldsymbol{\alpha}}})_{\varepsilon>0}$,implying the same statement for $(G_\varepsilon^{\partial})_{\varepsilon>0}$.

- $$\begin{aligned} \partial_{(\boldsymbol{x})} G^{\partial}[\mathsf{T}_{\boldsymbol{x}} \widetilde{\rho}] &\overset{(i)}{=} \partial_{(\boldsymbol{x})} G_{(\cdot)}^{\partial\partial^{\boldsymbol{\alpha}}}[(\text{-}1)^{|\boldsymbol{\alpha}|} \partial_{(\cdot)}^{\boldsymbol{\alpha}} \mathsf{T}_{\boldsymbol{x}} \widetilde{\rho}_{(\cdot)}] && (i) \text{ by the preceding point} \\ &\overset{(ii)}{=} (\partial\partial^{\boldsymbol{\alpha}})_{(\boldsymbol{x})} G_{(\cdot)}^{\partial\partial^{\boldsymbol{\alpha}}}[\mathsf{T}_{\boldsymbol{x}} \widetilde{\rho}_{(\cdot)}] && (ii) \text{ since } G^{\partial\partial^{\boldsymbol{\alpha}}} \text{ is in } L^1_{loc}(\mathbb{R}^N, \mathbb{C}) \\ &\overset{(iii)}{=} \rho(\boldsymbol{x}) && (ii) \text{ by Item 4 in Summary 4.1.12 for } G^{\partial\partial^{\boldsymbol{\alpha}}} \end{aligned}$$

  This leads finally to the fulfillment of fulfils Item 4 in Summary 4.1.12.

This shows that $(G_\varepsilon^{\partial})_{\varepsilon>0}$ is an approximating Green's function with respect to ∂ and $\mathcal{I}$. The corresponding limit Green's function G^{∂} is the $\partial^{\boldsymbol{\alpha}}$-weak derivative of $G^{\partial\partial^{\boldsymbol{\alpha}}} \in L^1_{loc}$ and in general not representable as a function L^1_{loc} and, thus, not necessarily continuous as a linear form with domain $(C_c^\infty, \tau_{C_c^\infty}^{\infty;\mathbf{0}})$ mapping to $(\mathbb{C}, |\cdot|)$. Picking $f \in C_c^\infty$, and by defining $\mathsf{K} := \operatorname{sp} f$, we can mimic the estimate in Equation (4.8) resulting in the following calculation:

$$\begin{aligned} \left|G^\partial[f]\right| &= \left|G^{\partial\partial^{\boldsymbol{\alpha}}}[(\text{-}1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} f]\right| \\ &= \left|\int d^N y\, G^{\partial\partial^{\boldsymbol{\alpha}}}(\boldsymbol{y})\,(\text{-}1)^{|\boldsymbol{\alpha}|}(\partial^{\boldsymbol{\alpha}} f)(\boldsymbol{y})\right| \\ &\leq \int d^N y \left|G^{\partial\partial^{\boldsymbol{\alpha}}}(\boldsymbol{y})\right| |(\partial^{\boldsymbol{\alpha}} f)(\boldsymbol{y})| \\ &\leq \left\|G^{\partial\partial^{\boldsymbol{\alpha}}}\right\|_{L^1(\mathsf{K},\mathbb{C})} \left\|\partial^{\boldsymbol{\alpha}} f\right\|_{\mathsf{K},\infty} \end{aligned} \tag{4.9}$$

We find in the case where G^∂ is a $\partial^{\boldsymbol{\alpha}}$-weak derivative of an L^1_{loc}, that the condition in Equation (4.5) is implied by another continuity condition of $G^\partial[\cdot]$. However, now its domain needs to be endowed with the topology originating from the inductive limit of the semi-norms $\left(\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}}\right)_{n\in\mathbb{N}}$ with $\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} := \|\partial^{\boldsymbol{\alpha}}\cdot\|_{\mathsf{K}_n,\infty}$ denoted by $\tau_{C_c^\infty}^{\infty;\boldsymbol{\alpha}}$. We conclude that $\tau_{C_c^\infty}$ should be a refinement of it.

The Topology $\tau_{C_c^\infty}$ Combining these results we, require that $\tau_{C_c^\infty}$ is a common refinement of $\left(\tau_{C_c^\infty}^{\infty;\boldsymbol{\alpha}}\right)_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}$. Thus, we choose the finest one among such topologies:

Definition 4.1.19 $\mathcal{D}$ and its Topology

The topological space $\mathcal{D} := \left(C_c^\infty(\mathbb{R}^N, \mathbb{C}), \tau_{C_c^\infty}\right)$ is build as the strict inductive limit of $\left(C_c^\infty(\mathsf{K}_n, \mathbb{C}), \tau_{C_c^\infty(\mathsf{K}_n)}\right)$ step by step:

- First we observe, that $C_c^\infty(\mathbb{R}^N, \mathbb{C}) = \bigcup_{n\in\mathbb{N}} C_c^\infty(\mathsf{K}_n, \mathbb{C})$ with $\mathsf{K}_n := [-n, n]^N$.
- On each $C_c^\infty(\mathsf{K}_n, \mathbb{C})$, we consider, for any $\boldsymbol{\alpha} \in \mathbb{N}_0^N$, the semi-norm

$$\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} : C_c^\infty(\mathsf{K}_n, \mathbb{C}) \to \mathbb{R}_{\geq 0}, f \mapsto \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} := \sup_{\boldsymbol{x}\in\mathsf{K}_n} |\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x})|\,.$$

- We endow $C_c^\infty(\mathsf{K}_n, \mathbb{C})$ with the topology $\tau_{C_c^\infty(\mathsf{K}_n)}$ induced by the family of semi-norms $\left(\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}}\right)_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}$.
- And finally, we consider the inclusion maps

$$\iota_n : \left(C_c^\infty(\mathsf{K}_n, \mathbb{C}), \tau_{C_c^\infty(\mathsf{K}_n)}\right) \hookrightarrow \left(C_c^\infty(\mathbb{R}^N, \mathbb{C}), \tau_{C_c^\infty}\right)$$

  and choose the finest $\tau_{C_c^\infty}$, such that $\forall n \in \mathbb{N}\ \iota_n$ is continuous.

The following theorem confirms that this kind of continuity embodies Equation (4.9):

Theorem 4.1.20 Characterization of Continuity

A linear map $u : \mathcal{D} \to (\mathbb{C}, |\cdot|)$ is continuous if, and only if, the following condition holds true:

$$\forall n \in \mathbb{N}\ \exists C_c > 0 \text{ and } a_n \in \mathbb{N}_0 \text{ s.t. } |u[f]| \leq C_n \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N, |\boldsymbol{\alpha}|\leq a_n} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} \text{ holds } \forall f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})\,.$$

Proof

[RSI80, Theorem V.16]

We remind you that our endeavor is to find a topology $\tau_{C_c^\infty}$ on C_c^∞ such that Equation (4.7) and Equation (4.5) hold true when choosing G^∂ to be continuous with respect to this domain. I.e., for all $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ and $h \to 0$, the convergences

- $\frac{1}{h}\big(f(\,\cdot\,) - f(\,\cdot - h\boldsymbol{e}_i)\big) \to \partial_{\cdot_i} f(\,\cdot\,)$ in $\big(C_c^\infty, \tau_{C_c^\infty}\big)$ and
- $G^\partial_{(\cdot)}\big[\frac{1}{h}\big(f(\,\cdot\,) - f(\,\cdot - h\boldsymbol{e}_i)\big)\big] \to G^\partial_{(\,\cdot\,)}\big[\partial_{\cdot_i} f(\,\cdot\,)\big]$ in $\big(\mathbb{C}, |\cdot|\big)$

shall hold true in that case. Thus, it remains to be shown, that the differential quotient converges in $\tau_{C_c^\infty}$:

Proposition 4.1.21 Convergence of Differential Quotient in $\tau_{C_c^\infty}$

For all $f \in \mathcal{D}$, we find that, for $h \to 0$, the convergence $\frac{1}{h}\big(f(\,\cdot\,) - f(\,\cdot - h\boldsymbol{e}_i)\big) \xrightarrow{\tau_{C_c^\infty}} \partial_{\cdot_i} f(\,\cdot\,)$ holds true.

Proof

This property follows directly by an application of Taylor's theorem and the boundedness of all derivatives in $\|\cdot\|_\infty$.

It is convenient to introduce a shorthand notation:

Definition Lemma 4.1.22 Convolution of $\mathsf{LF}[D, \mathbb{C}]$ and D

For $u \in \mathsf{LF}[D, \mathbb{C}]$ and $f \in D$, we define their convolution by $u * f : \mathbb{R}^N \to \mathbb{C}, \boldsymbol{x} \mapsto \big(u * f\big)(\boldsymbol{x}) := u[\mathsf{T}_{\boldsymbol{x}} \widetilde{f}]$.

For $u \in L^1_{loc}$ and $f \in D$, the convolution takes the following form: $u * f(\boldsymbol{x}) = \int d^N y\, u(\boldsymbol{y}) f(\boldsymbol{x} - \boldsymbol{y})$.

Proof

This immediately follows from the Definition/Lemma 4.1.22 of the convolution and the embedding in Def. 4.1.8 of $L^1_{loc} \subset \mathsf{LF}[D, \mathbb{C}]$.

Proposition 4.1.23 Smoothness of Convolutions of $\mathcal{D}'$ and $\mathcal{D}$

Given $u \in \mathcal{D}'(\mathbb{R}^N)$ and $f \in \mathcal{D}(\mathbb{R}^N, \mathbb{C})$, then $u * f \in C^\infty(\mathbb{R}^N, \mathbb{C})$ and. for any $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$, the following equalities hold true:

$$\partial^{\boldsymbol{\alpha}} u * f(x) = (\partial^{\boldsymbol{\alpha}} u) * f(x) = u * (\partial^{\boldsymbol{\alpha}} f)(x)$$

Proof

[HI83, Theorem 4.1.1]

This line of thought thus leads us to the following definition:

Definition 4.1.24 Distributions

Given $\mathcal{D} := \big(C_c^\infty(\mathbb{R}^N, \mathbb{C}), \tau_{C_c^\infty}\big)$, we call the following the topological vector space the space of distributions:

$$\mathcal{D}'(\mathbb{R}^N) := \Big(\big\{u \in \mathsf{LF}[C_c^\infty, \mathbb{C}]\,\big|\, u : \mathcal{D} \to (\mathbb{C}, |\cdot|) \text{ is continuous}\big\}, \tau_{\mathsf{LF}[C_c^\infty]}\Big)$$

Before we summarize our results from this introductory section, let us answer the natural question of whether $\mathcal{D}'$ contains more abstract objects than the finite derivatives of L^1_{loc}-functions. Due to the restrictiveness of the continuity condition with respect to $\tau_{C_c^\infty}$, the answer is no and is given by the following theorem:

Theorem 4.1.25 Regularity Theorem for Distributions

Given $u \in \mathcal{D}'$, then there exists $(u_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}$ in $C(\mathbb{R}^N,\mathbb{C})$ such that, for all compact $\mathsf{K} \subset \mathbb{R}^N$, the set $\{\boldsymbol{\alpha} \in \mathbb{N}_0^N \mid \operatorname{sp} u_{\boldsymbol{\alpha}} \cap \mathsf{K} \neq \varnothing\}$ is finite and the following equality holds true:

$$u = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N} \partial^{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}$$

Proof

[HI83, Theorem 4.4.7]

Thus, on compact sets any distribution is a finite weak derivative of a continuous function.

4.1.4 Summary

At this stage of our strategy, we were able to replace the approximate series with linear forms that fulfill certain continuity requirements. This led us to the following notion of Green's functions:

Definition 4.1.26 Greens Function

A Green's function G with respect to a linear and translation invariant PDO $\partial \in \eth$ is a distribution $(G \in \mathcal{D}')$ which fulfills $\partial G = \delta$ in the weak sense.

The construction results in this theorem:

Theorem 4.1.27 Solutions out of Green's Functions

Given a Green's function G with respect to $\partial \in \eth$ and any $\rho \in \mathcal{D}$ then $\phi := G^{\partial} * \rho$ solves $\partial\phi = \rho$ in the strong sense[a].

[a] Meaning as a differential equation on differentiable functions.

4.2 How to find Green's Functions

Since most linear PDOs have a non vanishing set of homogeneous solutions, i.e. solutions to the equation $\partial\phi = 0$, we expect the existence of a multitude of Green's functions. A method of full characterization is not known to the author but we will give an instruction to a systematic framework on how to find a large subset. The idea is laid out in Section 4.2.1. This will lead us to a construction of candidates for Green's functions via complex surface integrals in Section 4.2.3. In Section 4.2.4, we will explore the context of tempered distributions which provides strong tools for the task but can be quite restrictive depending on the PDO of interest.

4.2.1 The Idea of Linear Representations via Trivials

Now we turn to the question of finding linear forms $G \in \mathsf{LF}[C_c^\infty, \mathbb{C}]$ solving for given $\partial \in \eth$ the equation $\partial G = \delta$ in the sense of the weak derivative in Def. 4.1.14 i.e. $\forall f \in C_c^\infty$:

$$G[\partial^{\mathsf{t}} f] = \delta[f] \tag{4.10}$$

Even though the elementary inhomogeneity δ is very useful when constructing solutions to regular inhomogeneities as summarized in Section 4.1.4, it is of advantage to again capitalize on the linearity of the differential equation to represent it by a linear decomposition with respect to objects on which the action of ∂ becomes simple. However it is much simpler to dissect objects in $\mathbb{C}_c^\infty$ instead of those in $\mathsf{LF}[C_c^\infty, \mathbb{C}]$ and we can still use the resulting dissection maps by duality. After finding candidates in

$\mathsf{LF}[C_c^\infty, \mathbb{C}]$, their continuity in the sense of Theorem 4.1.20 can be checked and, if affirmative, lead to Green's functions in the sense of Def. 4.1.26.

Trivial Inhomogeneities

Since we only consider partial differential operators which are not only linear but have also constant coefficients, there exists a large set of smooth functions on which uniformly for all $\partial \in \mathfrak{d}$ the action of ∂ reduces to the multiplication by a $\mathbb{C}$ number.

Definition Lemma 4.2.1 Trivials and the Characteristic Polynomial

The set $\mathcal{T} := \left\{ t_{\boldsymbol{k}}(\,\cdot\,) := e^{\mathrm{i}\langle \boldsymbol{k}|\cdot\rangle} \mid \boldsymbol{k} \in \mathbb{C} \right\} \subset C^\infty(\mathbb{R}^N, \mathbb{C})$ is called the set of **trivial inhomogeneities**, or in short, **trivials**. For the simplicity of the notation, we will naturally identify $\mathcal{T} \simeq \{\boldsymbol{k} \in \mathbb{C}^N\}$ and also call T the trivials having this identification in mind.

For a given $\partial = \sum_{\boldsymbol{\alpha} \in \mathrm{A}} a_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \in \mathfrak{d}$, we define its **Characteristic Polynomial** by:

$$m_\partial : \mathbb{C}^N \to \mathbb{C}, \boldsymbol{k} \mapsto m_\partial(\boldsymbol{k}) := \sum_{\boldsymbol{\alpha} \in \mathrm{A}} a_{\boldsymbol{\alpha}} (\mathrm{i}\boldsymbol{k})^{\boldsymbol{\alpha}} \,.$$

The action of ∂ on $t_{\boldsymbol{k}}$ reduces to $\partial t_{\boldsymbol{k}} = m_\partial(\boldsymbol{k}) t_{\boldsymbol{k}}$. We abbreviate $\{m_\partial = 0\} := \{\boldsymbol{k} \in \mathbb{C}^N | m_\partial(\boldsymbol{k}) = 0\}$.

Linear Representation of $\mathbb{C}_c^\infty$ by Trivials

We will try to find a way to express any $f \in C_c^\infty$ as linear combinations of trivials. If the summation in this linear mapping commutes with the PDO, its action will just reduce to a multiplication of the coefficients in front of the trivials by the characteristic polynomial. Lets forge these ideas into a precise definition:

Definition 4.2.2 Linear Representation via Trivials

We call the tuple $(\mathcal{C}oef, \mathcal{LC})$ a linear representation via the trivials $\mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})} \subset \mathcal{T}$ if:

- $\mathcal{C}oef : C_c^\infty(\mathbb{R}^N, \mathbb{C}) \to \operatorname{ran}(\mathcal{C}oef) \subset F(\mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})}, \mathbb{C})$ is linear.
- $\mathcal{LC} : \operatorname{ran}(\mathcal{C}oef) \to C_c^\infty(\mathbb{R}^N, \mathbb{C})$ is linear.
- $\mathcal{LC} = \mathcal{C}oef^{-1}$.
- $\mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})} = \{\boldsymbol{k} \in \mathcal{T} \mid \exists f_{\mathcal{C}oef} \in \operatorname{ran}(\mathcal{C}oef) \text{ such that } f_{\mathcal{C}oef}(\boldsymbol{k}) \neq 0\}$.
- Any $\partial \in \mathfrak{d}$ gets diagonalized by $\mathcal{LC}$ in the following way: $\partial\, \mathcal{LC} = \mathcal{LC}\, m_\partial$.

$\mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})}$ is called the support of $(\mathcal{C}oef, \mathcal{LC})$.

Putting Def. 4.2.2 into words:

- The map $\mathcal{LC}$ constructs (possibly "non finite") linear combinations of $e^{\mathrm{i}\langle \boldsymbol{k}|\cdot\rangle} \in \mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})}$ with coefficients $f_{\mathcal{C}oef}(\boldsymbol{k})$.
- For functions $f \in C_c^\infty$, the map $\mathcal{C}oef$ gives rise to coefficients which, when used in $\mathcal{LC}$ give back f.
- $\mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})}$ indicates the subset of the trivials used in the representation.
- Given coefficients $f_{\mathcal{C}oef} \in \operatorname{ran}(\mathcal{C}oef)$, then $\mathcal{LC}(f_{\mathcal{C}oef})$ is to be understood as some kind of summation of trivials $e^{\mathrm{i}\langle \boldsymbol{k}|\cdot\rangle}$ with the respective coefficients $f_{\mathcal{C}oef}(\boldsymbol{k})$. When applying a PDO ∂, we want to be able to "pull it through" this summation so that it acts directly on the trivial $e^{\mathrm{i}\langle \boldsymbol{k}|\cdot\rangle}$ in the by $\boldsymbol{k}$ indexed summand. By Definition/Lemma 4.2.1, this results in the multiplication of the respective coefficient $f_{\mathcal{C}oef}(\boldsymbol{k})$ by $m_\partial(\boldsymbol{k})$, i.e. we demand $\partial\, \mathcal{LC} = \mathcal{LC}\, m_\partial$.

If we regard for $\partial \in \mathfrak{d}$ and $f, g \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, the equation $\partial f = g$ is equivalent to $m_\partial \cdot \mathcal{C}oef(f) = \mathcal{C}oef(g)$ which puts forward the canonical candidate "$\mathcal{LC}\big((m_\partial)^{-1} \cdot \mathcal{C}oef(g)\big)$". The quotation marks are put in place because it is not clear whether such an object can be given a meaning. Before turning to this question, we realize that when searching for Green's functions δ takes the place of g which leads to the necessity of a generalization to $\mathsf{LF}[C_c^\infty, \mathbb{C}]$.

Linear Representation of $\mathsf{LF}[C_c^\infty, \mathbb{C}]$ by Trivials

We have already seen how linear maps on C_c^∞ can be lifted to $\mathsf{LF}[C_c^\infty, \mathbb{C}]$ in Def. 4.1.14, i.e., strong derivatives on C_c^∞ got lifted to weak derivatives on $\mathsf{LF}[C_c^\infty, \mathbb{C}]$ by duality. The same can be done for our linear representations. This method is called pullback:

Definition 4.2.3 **Pullback of Linear Forms**

Let $L : V_1 \to V_2$ be a linear map between the two vector spaces V_1 and V_2, then we define its pullback by

$$L^* : \mathsf{LF}[V_2, \mathbb{C}] \to \mathsf{LF}[V_1, \mathbb{C}], u[\,\cdot\,] \mapsto L^* u[\,\cdot\,] := u[L\,\cdot\,]$$

If $V_1 = V_2$ is a vector space over the field F, and L is just the multiplication by an element $m \in F$, we will denote the pullback by $m\cdot$.

Indeed the interplay of the weak derivative and the pullback of the linear representation on $\mathsf{LF}[C_c^\infty, \mathbb{C}]$ is the same as the one of derivatives and linear representations on C_c^∞:

Lemma 4.2.4 **Pullback Representation by Trivials**

Given a linear representation $(\mathcal{C}oef, \mathcal{LC})$ via trivials, we find

- $\mathcal{LC}^* : \mathsf{LF}[C_c^\infty(\mathbb{R}^N, \mathbb{C}), \mathbb{C}] \to \mathsf{LF}[\mathrm{ran}(\mathcal{C}oef), \mathbb{C}]$ is linear,
- $\mathcal{C}oef^* : \mathsf{LF}[\mathrm{ran}(\mathcal{C}oef), \mathbb{C}] \to \mathsf{LF}[C_c^\infty(\mathbb{R}^N, \mathbb{C}), \mathbb{C}]$ is linear,
- $\mathcal{C}oef^* = (\mathcal{LC}^*)^{-1}$,
- and any $\partial \in \mathfrak{D}$ gets diagonalized by $\mathcal{C}oef^*$ in the following way: $\partial\, \mathcal{C}oef^* = \mathcal{C}oef^*\, m_{\partial^t}$,

i.e., the roles of $\mathcal{C}oef$ and $\mathcal{LC}$ are interchanged by the pullback.

Proof

All but the last point follow directly from Def. 4.2.3 of the pullback. Let $\partial \in \mathfrak{D}$ and $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$

$$\begin{aligned} \partial \mathcal{C}oef^* u[f] &\overset{(i)}{=} \mathcal{C}oef * u[\partial^t f] && (i) \text{ by Def. 4.1.14 of weak deriv.} \\ &\overset{(ii)}{=} u[\mathcal{C}oef \partial^t f] && (ii) \text{ by the definition of } \mathcal{C}oef* \\ &\overset{(iii)}{=} u[m_{\partial^t} \cdot \mathcal{C}oef f] && (iii) \text{ by Def. 4.2.2} \\ &\overset{(iv)}{=} m_{\partial^t} \cdot u[\mathcal{C}oef f] && (iv) \text{ by Def. 4.2.3} \\ &\overset{(i)}{=} \mathcal{C}oef^* m_{\partial^t} \cdot u[f]\,. \end{aligned}$$

Similar to the C_c^∞, the equation $\partial G = \delta$ for $\partial \in \mathfrak{D}$ is equivalent to $m_{\partial^t} \cdot \mathcal{LC}^*(G) = \mathcal{LC}^*(\delta)$ leading to the candidate "$\mathcal{C}oef^*\big((m_{\partial^t})^{-1} \cdot \mathcal{LC}^*(\delta)\big)$".

Complications

If we want to generate a Green's function G by considering "$\mathcal{C}oef^*\big((m_{\partial^t})^{-1} \cdot \mathcal{LC}^*(\delta)\big)$" we need to answer the following two questions:

1. Does the multiplication of $\mathcal{LC}^*(\delta)$ by $(m_{\partial^t})^{-1}$ result in an object in $\mathsf{LF}[\mathrm{ran}(\mathcal{C}oef), \mathbb{C}]$?

2. If so, is the linear form $\mathcal{C}oef^*\big((m_{\partial^t})^{-1} \cdot \mathcal{LC}^*(\delta)\big) \in \mathsf{LF}[C_c^\infty(\mathbb{R}^N, \mathbb{C}), \mathbb{C}]$ continuous in the sense of a mapping $\mathcal{D}$ to $(\mathbb{C}, |\cdot|)$?

For the first question it is certainly of advantage to choose the representation such that its support $\mathcal{T}_{(\mathcal{C}oef, \mathcal{LC})}$ has no overlap with $\{m_{\partial^t} = 0\}$. One can try to answer the second question by considering representations whose image $\mathrm{ran}(\mathcal{C}oef)$ can be equipped with a topology $\tau_{\mathrm{ran}(\mathcal{C}oef)}$ such that $\mathcal{C}oef : \mathcal{D} \to \big(\mathrm{ran}(\mathcal{C}oef), \tau_{\mathrm{ran}(\mathcal{C}oef)}\big)$ is continuous. Then, the continuity of $(m_{\partial^t})^{-1}$ as a function from $\big(\mathrm{ran}(\mathcal{C}oef), \tau_{\mathrm{ran}(\mathcal{C}oef)}\big)$ to $(\mathbb{C}, |\cdot|)$ implies $\mathcal{C}oef^*\big((m_{\partial^t})^{-1} \cdot \mathcal{LC}^*(\delta)\big) \in \mathcal{D}'$. We will explore these ideas in the setting where the coefficients $\mathcal{C}oef$ in front of the trivials $e^{\mathrm{i}\langle \boldsymbol{k} | \cdot \rangle}$ are constructed by integration of the variable $\boldsymbol{k}$ with respect to the Lebesgue measure, i.e., the setting of Fourier transformations.

4.2.2 Coefficients: Fourier-Laplace Transformation

We will see in the following that the Fourier-Laplace transformation is a reliable source of coefficient functions in the context of linear representation via subsets of trivials:

Definition 4.2.5 **Fourier-Laplace Transformation on C_c^∞**

We define the Fourier-Laplace transformation by:

$$\mathcal{FL} : C_c^\infty(\mathbb{R}^N, \mathbb{C}) \to F(\mathcal{T} \simeq \mathbb{C}^N, \mathbb{C}), f(\cdot) \mapsto (\mathcal{FL} f)(\cdot) := \left(\boldsymbol{k} \mapsto (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x \, f(\mathbf{x}) \, e^{-\mathrm{i}\langle \boldsymbol{k} | \mathbf{x} \rangle} \right) .$$

Indeed the Fourier-Laplace transform fulfils the desired commutation relations with PDOs and their characteristic polynomials:

Lemma 4.2.6 **Diagonalization Property of $\mathcal{FL}$**

Given any $\partial \in \boldsymbol{\partial}$, we find $\mathcal{FL}\partial = m_\partial \mathcal{FL}$ on $C_c^\infty(\mathbb{R}^N, \mathbb{C})$.

Proof

Let $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ and $\boldsymbol{k} \in \mathbb{C}^N$, then

$$\begin{aligned} (\mathcal{FL}\partial f)(\boldsymbol{k}) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x \, (\partial f)(\mathbf{x}) \, e^{-\mathrm{i}\langle \boldsymbol{k} | \mathbf{x} \rangle} \\ &\overset{(i)}{=} (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x \, f(\mathbf{x}) \, \partial_{\boldsymbol{x}}^{\mathrm{t}} e^{-\mathrm{i}\langle \boldsymbol{k} | \mathbf{x} \rangle} && (i) \text{ by partial integration} \\ &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x \, f(\mathbf{x}) \, m_{\partial^t}(-\boldsymbol{k}) e^{-\mathrm{i}\langle \boldsymbol{k} | \mathbf{x} \rangle} \\ &= m_\partial(\boldsymbol{k})(\mathcal{FL} f) . \end{aligned}$$

Since $\mathrm{ran}(\mathcal{FL})$ consists of functions with support on all of $\mathcal{T}$, it seems unfortunate to use it in order to build Green's functions, considering the complication raised in the first question. The following proposition shows, however, that $\mathcal{FL}(f)$ contains redundant information about $f \in \mathbb{C}_c^\infty(\mathbb{R}^N, \mathbb{C})$:

Proposition 4.2.7 **Regularity of the Fourier-Laplace Transformation**

Let $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, then, its Fourier-Laplace transform $\mathcal{FL} f$ is holomorphic, i.e $\mathcal{FL} f \in H(\mathbb{C}^N, \mathbb{C}) := \{h : \mathbb{C}^N \to \mathbb{C} \,|\, h \text{ is holomorphic}\}$.

Proof

[HI83, Theorem 7.1.14]

Since holomorphic functions can be uniquely identified by specifying their value on certain true subsets $t \subset \mathcal{T}$, we can, given $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, restrict the result of the Fourier-Laplace transform $\mathcal{FL} f$ without any loss of information:

Definition 4.2.8 Restricted Fourier-Laplace Transformation

Given $t \subset \mathcal{T}$, we denote by $\mathcal{FL}_t$ the restriction of the Fourier-Laplace transformation in the following sense:

$$\mathcal{FL}_t : C_c^\infty(\mathbb{R}^N, \mathbb{C}) \to F(t, \mathbb{C}), f(\,\cdot\,) \mapsto \big(\mathcal{FL} f\big)|_t(\,\cdot\,)\,.$$

Since our objective is to build a linear representation of the trivials t by defining $(\mathcal{C}oef, \mathcal{LC}) := \big(\mathcal{FL}_t, (\mathcal{FL}_t)^{-1}\big)$, we need to make sure that after shrinking to t $\mathcal{FL}_t$ is still injective. We will discuss a mechanism to find these sets of trivials in the following.

4.2.3 Trivials: Sufficient Subsets

A particularly advantageous subset is $\mathbb{R}^N \subset \mathcal{T}$, since there exists a superset $S(\mathbb{R}^N, \mathbb{C})$ of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{FL}_{\mathbb{R}^N}$ is the restriction of an isomorphism on S, and thus, is invertible on its range:

Definition 4.2.9 Schwartz Functions

The set S called Schwartz functions is defined by

$$S(\mathbb{R}^N, \mathbb{C}) := \Big\{ f \in C^\infty(\mathbb{R}^N, \mathbb{C}) \,\Big|\, \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^N \|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}} < \infty \Big\}$$

with $\mathbf{x}^{\boldsymbol{\beta}} = (x_1, \ldots, x_N)^{(\beta_1, \ldots, \beta_N)} := x_1^{\beta_1} \cdot \ldots \cdot x_N^{\beta_N}$ in

$$\|\cdot\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}} : S(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}_{\geq 0}, f \mapsto \|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}} := \sup_{\mathbf{x} \in \mathbb{R}^N} \big|\boldsymbol{x}^{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} f(\boldsymbol{x})\big|$$

Definition Theorem 4.2.10 Fourier Transform on Schwartz Functions

The following map, called Fourier transformation

$$\mathcal{F} : S(\mathbb{R}^N, \mathbb{C}) \to S(\mathbb{R}^N, \mathbb{C}), f(\,\cdot\,) \mapsto (\mathcal{F} f)(\,\cdot\,) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x\, f(\boldsymbol{x})\, e^{-\mathrm{i}\langle\,\cdot\,|\boldsymbol{x}\rangle}\,,$$

is invertible and its inverse is given by:

$$_{inv}\mathcal{F} : S(\mathbb{R}^N, \mathbb{C}) \to S(\mathbb{R}^N, \mathbb{C}), f(\,\cdot\,) \mapsto (_{inv}\mathcal{F} f)(\,\cdot\,) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N k\, f(\boldsymbol{k})\, e^{\mathrm{i}\langle\boldsymbol{k}|\,\cdot\,\rangle}\,.$$

Proof

[HI83, Theorem 7.1.5]

This implies, as described before:

Corollary 4.2.11

$(\mathcal{C}oef_{\mathbb{R}^N}, \mathcal{LC}_{\mathbb{R}^N}) := \big(\mathcal{FL}_{\mathbb{R}^N}, (\mathcal{FL}_{\mathbb{R}^N})^{-1}\big) = \big(\mathcal{F}|_{C_c^\infty}, {}_{inv}\mathcal{F}_{\mathcal{F}(C_c^\infty)}\big)$ is a linear representation via $\mathbb{R}^N \subset \mathcal{T}$.

Further we have an explicit formula for the evaluation of linear combination of trivials with coefficients $f_{\mathcal{C}oef} \in \mathcal{F}(C_c^\infty)$ at point $\boldsymbol{x} \in \mathbb{R}^N$:

$$\mathcal{LC}_{\mathbb{R}^N}(f_{\mathcal{C}oef})(\boldsymbol{x}) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N k\, f_{\mathcal{C}oef}(\boldsymbol{k})\, e^{\mathrm{i}\langle\boldsymbol{k}|\,\boldsymbol{x}\,\rangle}\,. \tag{4.11}$$

As disused before, $f_{\mathcal{C}oef}$ is the restriction of a holomorphic function on $\mathbb{R}^N$. Equation (4.11) can be regarded as a surface integral of the holomorphic function $\left(\boldsymbol{k} \mapsto f_{\mathcal{C}oef}(\boldsymbol{k})\, e^{\mathrm{i}\langle \boldsymbol{k}|\, \boldsymbol{x}\,\rangle}\right)$ along the surface $\mathbb{R}^N$. There exist generalized versions of the Cauchy integral theorem, which allow to smoothly alter the surface integration on the domain $\mathbb{R}^N$ to other N real-dimensional surfaces in $t \subset \mathbb{C}^N$ without changing the value of the integral. This results in a new linear representation on $(\mathcal{LC}_{\mathrm{t}}, \mathcal{C}oef_{\mathrm{t}})$ by defining $\mathcal{C}oef_{\mathrm{t}} := \mathcal{FL}_{\mathrm{t}}$ and

$$\mathcal{LC}_{\mathrm{t}}(f_{\mathcal{C}oef})(\boldsymbol{x}) = (2\pi)^{-N/2} \int_t d^N k\, f_{\mathcal{C}oef}(\boldsymbol{k})\, e^{\mathrm{i}\langle \boldsymbol{k}|\, \boldsymbol{x}\,\rangle}$$

for $f_{\mathcal{C}oef} \in \mathcal{FL}(\mathbb{C}_c^\infty)$ and $\boldsymbol{x} \in \mathbb{R}^N$ where $\int_t d^N k$ is not understood as an Lebesgue integral but a complex surface integral. The Cauchy integral theorem allows one to alter $\mathbb{R}^N$ only within compact regions of $\mathbb{C}^N$. However, the following proposition shows that the Fourier-Laplace transformations of C_c^∞-functions fulfilling certain strong decay properties along $\mathbb{R}^N + i\boldsymbol{c}$ for $\boldsymbol{c} \in \mathbb{R}^N$ which can be used to alter $\mathbb{R}^N$ outside of compact regions:

Proposition 4.2.12 Regularity of the Fourier-Laplace Transformation

Given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^N$ and compact $\mathsf{K} \subset \mathbb{R}^N$, then, $\forall n \in \mathbb{N}$ there exists $C_{\mathsf{K},\boldsymbol{\alpha},\boldsymbol{\beta},n} \geq 0$ such that $\forall f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$

$$\sup_{\boldsymbol{k} \in \mathbb{R}^N + i\mathsf{K}} \left| \boldsymbol{k}^{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} \mathcal{FL} f(\boldsymbol{k}) \right| \leq C_{\mathsf{K},\boldsymbol{\alpha},\boldsymbol{\beta},n} \sum_{\boldsymbol{\delta} \in \mathbb{N}_0^N, |\boldsymbol{\delta}| \leq |\boldsymbol{\beta}|} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\delta}}\,.$$

which implies for given $\mathbf{c} \in \mathbb{R}^N$ that $\imath_{(\mathbb{R}^N + i\mathbf{c})} \mathcal{FL}(f)$ is a Schwartz function in the sense that

$$\left(\mathbb{R}^N \to \mathbb{C} : \mathbf{k} \mapsto \imath_{(\mathbb{R}^N + i\mathbf{c})} \mathcal{FL}(f)(\mathbf{k} + \mathrm{i}\mathbf{c})\right) \in S(\mathbb{R}^N, \mathbb{C})$$

Proof

We take advantage of the compact support of f in steps (i) when using the dominated convergence theorem and (ii) when performing a partial integration in the following calculation

$$\begin{aligned}
\boldsymbol{k}^{\boldsymbol{\beta}} \partial_{\boldsymbol{k}}^{\boldsymbol{\alpha}} \mathcal{FL} f(\boldsymbol{k}) &= \boldsymbol{k}^{\boldsymbol{\beta}} \partial_{\boldsymbol{k}}^{\boldsymbol{\alpha}} (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x\, f(\mathbf{x})\, e^{-\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \\
&\overset{(i)}{=} (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x\, f(\mathbf{x})\, \boldsymbol{k}^{\boldsymbol{\beta}} \partial_{\boldsymbol{k}}^{\boldsymbol{\alpha}} e^{-\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x\, f(\mathbf{x})\, \boldsymbol{k}^{\boldsymbol{\beta}} (-i\boldsymbol{x})^{\boldsymbol{\alpha}} e^{-\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x\, f(\mathbf{x})\, (-i\boldsymbol{x})^{\boldsymbol{\alpha}} (i\partial_{\boldsymbol{x}})^{\boldsymbol{\beta}} e^{-\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \\
&\overset{(ii)}{=} (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x\, \left((-i\partial_{\boldsymbol{x}})^{\boldsymbol{\beta}} (-i\boldsymbol{x})^{\boldsymbol{\alpha}} f(\mathbf{x})\right) e^{-\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} d^N x \sum_{|\boldsymbol{\alpha}'| \leq |\boldsymbol{\alpha}|, |\boldsymbol{\beta}'| \leq |\boldsymbol{\beta}|} c_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} \left((-i\boldsymbol{x})^{\boldsymbol{\alpha}'} (-i\partial_{\boldsymbol{x}})^{\boldsymbol{\beta}'} f(\mathbf{x})\right) e^{-\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}
\end{aligned}$$

which leads to:

$$\begin{aligned}
\sup_{z \in \mathbb{R}^N + iK} \left| \boldsymbol{k}^{\boldsymbol{\beta}} \partial_{\boldsymbol{k}}^{\boldsymbol{\alpha}} \mathcal{FL} f(\boldsymbol{k}) \right| &\leq (2\pi)^{-N/2} \sum_{|\boldsymbol{\alpha}'| \leq |\boldsymbol{\alpha}|, |\boldsymbol{\beta}'| \leq |\boldsymbol{\beta}|} c_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} \int_{\mathbb{R}^N} d^N x \left| \boldsymbol{x}^{\boldsymbol{\alpha}'} \partial_{\boldsymbol{x}}^{\boldsymbol{\beta}'} f(\mathbf{x}) \right| \sup_{z \in \mathbb{R}^N + iK} e^{\langle \mathrm{Im}(\boldsymbol{k})|\mathbf{x}\rangle} \\
&\leq (2\pi)^{-N/2} \sum_{|\boldsymbol{\alpha}'| \leq |\boldsymbol{\alpha}|, |\boldsymbol{\beta}'| \leq |\boldsymbol{\beta}|} \tilde{c}_{\mathsf{K}_n,\boldsymbol{\alpha}',\boldsymbol{\beta}'} \left(\int_{\mathsf{K}_n} d^N x \left| \partial_{\boldsymbol{x}}^{\boldsymbol{\beta}'} f(\boldsymbol{x}) \right| \right) \sup_{\boldsymbol{x} \in \mathsf{K}_n, \boldsymbol{y} \in K} e^{\langle \boldsymbol{y}|\boldsymbol{x}\rangle} \\
&\leq C_{\mathsf{K},\boldsymbol{\alpha},\boldsymbol{\beta},n} \sum_{\boldsymbol{\delta} \in \mathbb{N}_0^N, |\boldsymbol{\delta}| \leq |\boldsymbol{\beta}|} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\delta}}\,.
\end{aligned}$$

We will, however, restrict ourselves to only alter the surface on compact regions in just one dimension, which does not involve generalized versions of:

Definition **Theorem 4.2.13** **Cauchy's Integral Theorem**

Given a continuous function $f : \mathbb{C} \to \mathbb{C}$, and further, a piecewise continuously differentiable contour $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{C}$, then we define the following complex contour integral:

$$\int_\gamma f(z)dz := \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt$$

We call a function $\gamma : [a, b] \to \mathbb{C}$ a contour, if the following conditions are meet:

- γ is piecewise continuously differentiable.
- $\gamma(a) = \gamma(b)$.
- for $t_1, t_2 \in (a, b)$ with $t_1 < t_2$ the quality $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = a$ and $t_2 = b$.

Given $f \in H(\mathbb{C}, \mathbb{C})$ and a contour $\gamma : [a, b] \to \mathbb{C}$, then $\int_\gamma f(z)dz = 0$.

For fixed $\boldsymbol{k}' = (k_2, \dots, k_N) \in \mathbb{R}^{N-1}$, we want to alter the domain of the $(\omega = k_1)$-integration in Equation (4.11) by Cauchy's integral theorem. We define

Definition 4.2.14 **ω-Surface and ω-Surface Integration**

We call a family of functions $\gamma = \{\gamma_{\boldsymbol{k}'} : \mathbb{R} \to \mathbb{C}\}_{\boldsymbol{k}' \in \mathbb{R}^{N-1}}$ ω-Cauchy curves, if forall $\boldsymbol{k}' \in \mathbb{R}^{N-1}$ there exists $T^{\pm}_{\boldsymbol{k}'} \in \mathbb{R}$ such that

- $\gamma_{\boldsymbol{k}'}(t) = \gamma_{\boldsymbol{k}'}(T^-_{\boldsymbol{k}'}) + (t - T^-_{\boldsymbol{k}'})$ for all $t \in (\text{-}\infty, T^-_{\boldsymbol{k}'}]$,
- $\gamma_{\boldsymbol{k}'}(t) = \gamma_{\boldsymbol{k}'}(T^+_{\boldsymbol{k}'}) + (t - T^+_{\boldsymbol{k}'})$ for all $t \in [T^-_{\boldsymbol{k}'},\ \infty)$,
- $\gamma_{\boldsymbol{k}'}(t)|_{[T^-_{\boldsymbol{k}'}, T^+_{\boldsymbol{k}'}]}$ is piecewise continuously differentiable

and we denote, by a slight abuse of notation, $\text{graph}(\gamma) := \big\{\big(\gamma_{\boldsymbol{k}'}(t), \boldsymbol{k}'\big) \in \mathbb{R}^N \big| (t, \boldsymbol{k}') \in \mathbb{R}^N\big\}$ as the ω-Cauchy surface associated to γ.

Given γ and a function f which is holomorphic on a neighborhood of $\text{graph}(\gamma)$ fulfilling that

- $\forall \boldsymbol{k}' \in \mathbb{R}^{N-1}$ $\big(\omega \mapsto f(\omega, \boldsymbol{k}')\big)$ is Lebesgue integrable on $\mathbb{R} \setminus [\gamma_{\boldsymbol{k}'}(T^-_{\boldsymbol{k}'}), \gamma_{\boldsymbol{k}'}(T^+_{\boldsymbol{k}'})]$,
- the map $\boldsymbol{k}' \mapsto \int_{\mathbb{R}\setminus[\gamma_{\boldsymbol{k}'}(T^-_{.}), \gamma_{\boldsymbol{k}'}(T^+_{.})]} d\omega f(\omega, \boldsymbol{k}')$ is Lebesgue integrable on $\mathbb{R}^{N-1}$,
- the map $\boldsymbol{k}' \mapsto \int_{\gamma_{\boldsymbol{k}'}|_{[T^-_{.}, T^+_{.}]}} d\omega f(\omega, \boldsymbol{k}')$ is Lebesgue integrable on $\mathbb{R}^{N-1}$ where for fixed $\boldsymbol{k}' \in \mathbb{R}^{N-1}$ the object $\int_{\gamma_{\boldsymbol{k}'}|_{[T^-_{.}, T^+_{.}]}} d\omega f(\omega, \boldsymbol{k}')$ is understood as a complex path integral,

we define the integral of f over $\text{graph}(\gamma)$ by:

$$\int_{\text{graph}(\gamma)} d^N k\, f(\boldsymbol{k}) := \int_{\mathbb{R}^{N-1}} d^{N\text{-}1}k' \left(\int_{\mathbb{R}\setminus[\gamma_{\boldsymbol{k}'}(T^-_{.}), \gamma_{\boldsymbol{k}'}(T^+_{.})]} d\omega f(\omega, \boldsymbol{k}') + \int_{\gamma_{\boldsymbol{k}'}|_{[T^-_{.}, T^+_{.}]}} d\omega f(\omega, \boldsymbol{k}')\right)$$

As of expected by the construction, we find:

Definition **Corollary 4.2.15**

Given ω-Cauchy curves γ, then $(\mathcal{Coef}_{\mathrm{graph}(\gamma)}, \mathcal{LC}_{\mathrm{graph}(\gamma)})$ with $\mathcal{Coef}_{\mathrm{graph}(\gamma)} := \mathcal{FL}_{\mathrm{graph}(\gamma)}$ and

$$\mathcal{LC}_{\mathrm{graph}(\gamma)} f(\boldsymbol{x}) := (2\pi)^{-N/2} \int_{\mathrm{graph}(\gamma)} d^N k \, f(\boldsymbol{k}) \, e^{\mathrm{i}\langle \boldsymbol{k} | \boldsymbol{x} \rangle}$$

for $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ and $\boldsymbol{x} \in \mathbb{R}^N$ is a linear representation via $\mathrm{graph}(\gamma) \subset \mathcal{T}$.

Proof

By the argumentation of this section.

We determine now $\mathcal{LC}^*_{\mathrm{graph}(\gamma)}(\delta)$

Lemma 4.2.16

$\mathcal{LC}^*_{\mathrm{graph}(\gamma)}(\delta) = (2\pi)^{-N/2}$ in the sense of $\mathsf{LF}[\mathrm{ran}(\mathcal{Coef}_{\mathrm{graph}(\gamma)}), \mathbb{C}]$.

Proof

Let $f_{\mathcal{Coef}} \in \mathrm{ran}(\mathcal{Coef}_{\mathrm{graph}(\gamma)})$ then

$$\mathcal{LC}^*_{\mathrm{graph}(\gamma)}(\delta)[f_{\mathcal{Coef}}] = \delta[\mathcal{LC}_{\mathrm{graph}(\gamma)} f_{\mathcal{Coef}}] = (2\pi)^{-N/2} \int_{\mathrm{graph}(\gamma)} d^N k \, f(\boldsymbol{k}) = (2\pi)^{-N/2}[f]$$

The Green's function candidate becomes "$\mathcal{Coef}^*_{\mathrm{graph}(\gamma)}\big((m_{\partial^{\mathrm{t}}})^{-1} \cdot (2\pi)^{-N/2}\big)$", where we consider $(m_{\partial^{\mathrm{t}}})^{-1} \cdot (2\pi)^{-N/2}$ just on $\mathrm{graph}(\gamma)$ and we can try to chose $\mathrm{graph}(\gamma)$ such that it is disjoint of $\{m_{\partial^{\mathrm{t}}} = 0\}$. This does not fully answer if it lies in $\mathsf{LF}[\mathrm{ran}(\mathcal{Coef}_{\mathrm{graph}(\gamma)}), \mathbb{C}]$. Furthermore, this technique does not provide us with a mechanism to ensure the continuity of the possible $G \in \mathsf{LF}[C_c^\infty, \mathbb{C}]$.

4.2.4 Tempered Distributions

The strategy of searching for a linear representation, such that $\mathrm{ran}(\mathcal{Coef})$ can be equipped with some topology $\tau_{\mathrm{ran}(\mathcal{Coef})}$ which makes $\mathcal{Coef} : \mathcal{D} \to \big(\mathrm{ran}(\mathcal{Coef}), \tau_{\mathrm{ran}(\mathcal{Coef})}\big)$ a homeomorphism sketched to answer the second concern (Item 2), leads to the notion of tempered distributions and their Fourier transformations. A caveat however is that one needs to pass to an increased topological set of test functions to $(S(\mathbb{R}^N, \mathbb{C}), \tau_S)$ of which $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ is a true subset and its inclusion is continuous. This implies that the set of continuous linear forms on $(S(\mathbb{R}^N, \mathbb{C}), \tau_S)$ is only a subset of $\mathcal{D}'$, and we might not be able to find all possible Green's functions. We define this topological space:

Definition 4.2.17 **Schwartz Space**

The topological space $\mathcal{S} := \big(S(\mathbb{R}^N, \mathbb{C}), \tau_S\big)$ with its topology τ_S given by the family of semi-norms $(\|\cdot\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}})_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}_0^N}$ is called the Schwartz space.

Proposition 4.2.18 **Embedding**

The inclusion map $\iota_{\mathcal{D},\mathcal{S}} : \mathcal{D} \to \mathcal{S}, f \mapsto f$ is continuous.

Proof

$\tau_{C_c^\infty}$ is constructed according to Def. 4.1.19 as the final topology of the family of maps $\big(\iota_n : C_c^\infty(\mathsf{K}_n, \mathbb{C}) \to C_c^\infty(\mathbb{R}^N, \mathbb{C})\big)_{n\in\mathbb{N}}$ with $C_c^\infty(\mathsf{K}_n, \mathbb{C})$ topologized by the family $(\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}$ of semi-norms. By the universal property of the final topology, $\iota_{\mathscr{D},\mathscr{S}}$ is continuos if, and only if, for all $n \in \mathbb{N}$ $\iota_n \circ \iota_{\mathscr{D},\mathscr{S}} : \big(C_c^\infty(\mathsf{K}_n, \mathbb{C}), \tau_{C_c^\infty(\mathsf{K}_n)}\big) \to \mathscr{S}$ is continuos. Since $\iota_n \circ \iota_{\mathscr{D},\mathscr{S}}$ is linear and due to

$$\|\iota_n \circ \iota_{\mathscr{D},\mathscr{S}}(f)\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}} = \|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}} = \sup_{\boldsymbol{x}\in\mathbb{R}^N} \left|\boldsymbol{x}^{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} f(\boldsymbol{x})\right| \leq n^{|\boldsymbol{\beta}|} \sup_{\boldsymbol{x}\in\mathbb{R}^N} |\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x})| = C^{|\boldsymbol{\beta}|}\|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}}$$

for all $f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$ which according to [RSI80, Theorem V.2] implies the continuity of $\iota_n \circ \iota_{\mathscr{D},\mathscr{S}}$.

This thus implies that the linear continuous forms on $\mathscr{S}$ can be regarded as a subset of $\mathscr{D}'$:

Definition 4.2.19 Tempered Distributions

Given the topological vector space $\mathscr{S}$, we call the topological vector space

$$\mathscr{S}' := \Big(\big\{u \in \mathsf{LF}[S, \mathbb{C}] \,|\, u : \mathscr{S} \to (\mathbb{C}, |\cdot|) \text{ is continuous}\big\}, \tau_{\mathsf{LF}[S]}\Big)$$

the space of tempered distributions. Further $\mathscr{S}' \subset \mathscr{D}'$ by Proposition 4.2.18.

The reason for looking at this space of test functions lies now in the following theorem:

Theorem 4.2.20 Conitnuity of the Fourier Transformation on $\mathscr{S}$

The Fourier transform $\mathscr{F}$ and its inverse ${}_{inv}\mathscr{F}$ of Definition/Theorem 4.2.10 are isomorphisms of $\mathscr{S}$.

Thus, $\mathscr{F}^*$ and ${}_{inv}\mathscr{F}^*$ are isomorphisms of $\mathscr{S}'$. Since they agree with $\mathscr{F}$ on $S(\mathbb{R}^N, \mathbb{C}) \subset \mathscr{S}'$, we leave out the $*$.

Proof

[HI83, Lemma 7.1.3] and [HI83, Theorem 7.1.10]

Thus, this linear map gives rise to a linear representation of $S(\mathbb{R}^N, \mathbb{C})$[7] via $\mathbb{R}^N \subset \mathscr{T}$:

Proposition 4.2.21

$(coef^S_{\mathbb{R}^N}, \boldsymbol{\mathcal{LC}}^S_{\mathbb{R}^N}) := (\mathscr{F}, {}_{inv}\mathscr{F})$ is a linear representation of $S(\mathbb{R}^N, \mathbb{C})$ supported on $\mathbb{R}^N$ and $\boldsymbol{\mathcal{C}oef}^S_{\mathbb{R}^N}$ and $\boldsymbol{\mathcal{LC}}^S_{\mathbb{R}^N}$ are $\mathscr{S} \to \mathscr{S}$ continuos .

Proof

The only thing left to show is that $\mathscr{F}\partial = m_\partial \mathscr{F}$ holds true on $S \setminus C_c^\infty$ for any ∂. This is, however, proven in [HI83, Lemma 7.1.3].

With the same reasoning as in Lemma 4.2.16 we find:

Lemma 4.2.22

The equality $\big(\boldsymbol{\mathcal{LC}}^S_{\text{graph}(\gamma)}\big)^*(\delta) = (2\pi)^{-N/2}$ holds true in the sense of $\mathsf{LF}[\boldsymbol{\mathcal{C}oef}^S_{\mathbb{R}^N}, \mathbb{C}]$.

[7] A linear representation of S via trivials is to be understood by replacing in Def. 4.2.2 C_c^∞ by S.

Our candidate for a Green's function is therefore "$\big(\mathcal{C}oef^{S}_{\mathbb{R}^N}\big)^*\big((m_{\partial^t})^{-1}\cdot(2\pi)^{-N/2}\big)$". We can now approach the question of whether our candidate is continuous, i.e. lies in $\mathcal{S}'$. Since $\mathcal{C}oef^{S}_{\mathbb{R}^N}, \mathcal{LC}^{S}_{\mathbb{R}^N} = \mathcal{F}$ is an isomorphism onto $\mathcal{S}'$, the continuity of the candidate is equivalent to $(m_{\partial^t})^{-1}\cdot(2\pi)^{-N/2}\in\mathcal{S}'$. As for the case of the distribution in Theorem 4.1.25, there exists a theorem that describes the local regularity of the elements of $\mathcal{S}'$ and at the same time identifies their global behavior:

Theorem 4.2.23 Regularity Theorem for Tempered Distributions

$\mathcal{S}'(\mathbb{R}^N) = \big\{\partial^{\boldsymbol{\alpha}} g \,\big|\, g\in C(\mathbb{R}^N,\mathbb{C}) \text{ and polynomially bounded}, \boldsymbol{\alpha}\in\mathbb{N}_0^N\big\}$ with weak derivatives in the sense of Def. 4.1.14.

Proof

[RSI80, Theorem V.10]

Additionally, there exists an accessible characterization of the continuity of linear forms $\mathsf{LF}[\mathcal{S},(\mathbb{C},|\cdot|)]$, the analog of Theorem 4.1.18 for distributions:

Theorem 4.2.24 Characterization of Continuity

A map $u\in\mathsf{LF}[\mathcal{S},(\mathbb{C},|\cdot|)]$ is continuous if, and only if, there exists $a,b\in\mathbb{N}_0$ and $C>0$, such that, for all $f\in\mathcal{S}$ the following estimate holds true:

$$|u[f]|\leq C\sum_{|\boldsymbol{\alpha}|\leq a,|\boldsymbol{\beta}|\leq b}\|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}}\,.$$

Proof

[RSI80, Theorem V.2] and [RSI80, Theorem V.9]

One could use these tools to assess whether "$\mathbb{R}^N\to\mathbb{C}, \boldsymbol{k}\mapsto\frac{(2\pi)^{-N/2}}{m_{\partial^t}(\boldsymbol{k})}$" lies in $\mathcal{S}'(\mathbb{R}^N)$. This could be quite tricky and it could be advantageous to construct approximate series in $\mathcal{S}'$ by changing the characteristic polynomial $m^{\varepsilon}_{\partial^t}$ by a small parameter ε. Fortunately, the following theorem ensures that we obtain a tempered distribution if $(2\pi)^{-N/2}/m^{\varepsilon}_{\partial^t}$ converges to $\mathsf{LF}^{\tau}[S,\mathbb{C}]$:

Theorem 4.2.25

$\mathcal{S}'$ is a closed subset of the topological space $\mathsf{LF}^{\tau}[S,\mathbb{C}]$ (Def. 4.1.10).

Proof

Given a net $(u_i)_{i\in I}$ in $\mathcal{S}'$ such that it converges to $u\in\mathsf{LF}^{\tau}[S,\mathbb{C}]$, then, by Def. 4.1.10 of $\tau_{\mathsf{LF}[S]}$, it follows, for all $f\in S(\mathbb{R}^4,\mathbb{C})$, the convergence $\lim_{i\in I}u_i[f]=u[f]$. In particular is, for any given $f\in S(\mathbb{R}^4,\mathbb{C})$, set $\{|u_i[f]||i\in I\}$ bounded in $\mathbb{C}$. By the principle of uniform boundedness on the Fréchet space $\mathcal{S}$ (see [RSI80, Theorem V.7]), we find $C>0$ and $a,b\in\mathbb{N}_0$, such that, for all $i\in I$ and $f\in S(\mathbb{R}^4,\mathbb{C})$, the following estimation holds true:

$$|u_i[f]|\leq C\sum_{|\boldsymbol{\alpha}|\leq a,|\boldsymbol{\beta}|\leq b}\|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}}\,.$$

This implies, again for all $i\in I$ and $f\in S(\mathbb{R}^4,\mathbb{C})$

$$|u[f]|\leq|u[f]-u_i[f]|+|u_i[f]|\leq|(u-u_i)[f]|+C\sum_{|\boldsymbol{\alpha}|\leq a,|\boldsymbol{\beta}|\leq b}\|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}}$$

and thus, due to $u_i\xrightarrow{\tau_{\mathsf{LF}[S]}}u$, finally $|u[f]|\leq C\sum_{|\boldsymbol{\alpha}|\leq a,|\boldsymbol{\beta}|\leq b}\|f\|_{\infty,\boldsymbol{\alpha},\boldsymbol{\beta}}$. This implies by Theorem 4.2.24 the continuity of u on Schwartz space

4.3 Relativistic Structure and Conventions

When considering physical theories in the context of special relativity, the convention is to use the signature of the pseudo-Riemannian metric (+,-,-,-) in the Fourier transforms:

Definition 4.3.1 Relativistic Physics Convention

The set of trivials $\mathcal{T}_{ph} := \big\{ t_{(\omega,\boldsymbol{k})}(\,\cdot\,) := (t,\boldsymbol{x}) \mapsto e^{\mathrm{i}(\omega t - \langle \mathbf{k}|\boldsymbol{x}\rangle)} \,|\, (\omega,\boldsymbol{k}) \in \mathbb{C}^4 \big\} \subset C^\infty(\mathbb{R}^4,\mathbb{C})$.

Characteristic polynomial w.r.t. $\partial \in \mathfrak{D}$ $m_{ph,\partial} : \mathbb{C}\times\mathbb{C}^3 \to \mathbb{C}, (\omega,\boldsymbol{k}) \mapsto \sum_{\boldsymbol{\alpha}\in \mathrm{A}} a_{\boldsymbol{\alpha}} \big(\mathrm{i}(-\omega,\boldsymbol{k})\big)^{\boldsymbol{\alpha}}$.

The set of 0-points of $m_{ph,\partial}$ w.r.t. $\partial \in \mathfrak{D}$ $Z(m_{ph,\partial}) := \big\{ (\omega,\boldsymbol{k}) \in \mathbb{C}^4 \,\big|\, m_{ph,\partial}(\omega,\boldsymbol{k}) = 0 \big\}$.

The Fourier-Laplace transformation:

$$\mathcal{FL}_{ph} : C_c^\infty(\mathbb{R}^4,\mathbb{C}) \to H(\mathbb{C}^4,\mathbb{C}), f(\,\cdot\,) \mapsto \left((\omega,\boldsymbol{k}) \mapsto (2\pi)^{-4/2} \int_{\mathbb{R}^4} dt d^3x\, f(t,\boldsymbol{x})\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\boldsymbol{x}\rangle)} \right)$$

The Fourier transformation and its inverse:

$$\mathcal{F}_{ph} : S(\mathbb{R}^4,\mathbb{C}) \to S(\mathbb{R}^4,\mathbb{C}), f(\,\cdot\,) \mapsto \big(\mathcal{F}_{ph} f\big)(\,\cdot\,) := \left((\omega,\boldsymbol{k}) \mapsto (2\pi)^{-4/2} \int_{\mathbb{R}^4} dt d^3x\, f(t,\boldsymbol{x})\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \right)$$

$${}_{inv}\mathcal{F}_{ph} : S(\mathbb{R}^4,\mathbb{C}) \to S(\mathbb{R}^4,\mathbb{C}), f(\,\cdot\,) \mapsto \big({}_{inv}\mathcal{F}_{ph} f\big)(\,\cdot\,) := \left((t,\boldsymbol{x}) \mapsto (2\pi)^{-4/2} \int_{\mathbb{R}^4} d\omega d^3k\, f(\omega,\boldsymbol{k})\, e^{\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \right)$$

This metric provides the structure of the light cones, which we will summarize in the following definition:

Definition 4.3.2 Light Cones

Given a spacetime point $(t,\boldsymbol{x}) \in \mathbb{R}\times\mathbb{R}^3$, we define its

- past light cone $\Gamma^{\text{light},-}_{(t,\boldsymbol{x})} := \{(t',\boldsymbol{x}') \in \mathbb{R}^4 |\, |t-t'| - \|\boldsymbol{x}-\boldsymbol{x}'\| = 0 \text{ and } t' \le t\}$,
- future light cone $\Gamma^{\text{light},+}_{(t,\boldsymbol{x})} := \{(\tilde{t},\tilde{\boldsymbol{x}}) \in \mathbb{R}^4 |\, |t-t'| - \|\boldsymbol{x}-\boldsymbol{x}'\| = 0 \text{ and } t' \ge t\}$

and further, for any $A \subset \mathbb{R}\times\mathbb{R}^3$, the future, and respectively, past light cone $\Gamma^{\text{light},\pm}_{A} := \bigcup_{(t,\boldsymbol{x})\in A} \Gamma^{\text{light},\pm}_{(t,\boldsymbol{x})}$ of A.

4.4 Advanced and Retarded Green's Functions

In this section, we will discuss two Green's functions of the wave operator that are of particular interest. We first summarize their definitions and properties in Section 4.4.1 and move the proof to Section 4.4.5. In Sections 4.4.2 to 4.4.4 we follow the train of thought that was examined in Section 4.2.

4.4.1 Definition and Properties

Definition 4.4.1 Advanced and Retarded Green's Functions

We will call the following linear forms the advanced and retarded Green's function[a]

$$G^{\mathrm{adv/ret}} : \mathcal{S}(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, f \mapsto \int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\Theta(\mp t)\frac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big)$$

which are abbreviated by writing:

$$G^{\mathrm{adv/ret}}(t, \boldsymbol{x}) = \Theta(\mp t)\frac{1}{4\pi|\boldsymbol{x}|}\delta(|\boldsymbol{x}| \pm t)\,.$$

However, this is just a formal expression not to be understood as the evaluation of a function or the composition of its ingredients.

[a] The justification of this labeling is part of the following theorem.

To determine their properties, we need to define certain paths in the complex plane that are used in one of the representations of $G^{\mathrm{adv/ret}}$:

Definition 4.4.2

Given $\boldsymbol{k} \in \mathbb{R}^3$ we define two ω-Cauchy curves (Def. 4.2.14) $\gamma^{\pm} = \big\{\gamma^{\pm}_{\boldsymbol{k}} : \mathbb{R} \to \mathbb{C}\big\}_{\boldsymbol{k}\in\mathbb{R}^3}$ by

$$\gamma^{\pm}_{\boldsymbol{k}}(t) := \begin{cases} -(1+|\boldsymbol{k}|) + (t+1+(1+||\boldsymbol{k}|)) & , t \in \big(-\infty, \text{-}1\text{-}(1+||\boldsymbol{k}|)\big] \\ -(1+|\boldsymbol{k}|) \qquad \pm \mathrm{i}\big(t+1+(1+|\boldsymbol{k}|)\big) & , t \in \big[\text{-}1\text{-}(1+||\boldsymbol{k}|), \text{-}(1+|\boldsymbol{k}|)\big] \\ -(1+|\boldsymbol{k}|) + (t+(1+|\boldsymbol{k}|)) \quad \pm \mathrm{i} & , t \in \big[\text{-}(1+|\boldsymbol{k}|), (1+|\boldsymbol{k}|)\big] \\ +(1+|\boldsymbol{k}|) \qquad \pm \mathrm{i}\big(1-t-(1+|\boldsymbol{k}|)\big) & , t \in \big[(1+|\boldsymbol{k}|), 1+(1+|\boldsymbol{k}|)\big] \\ +(1+|\boldsymbol{k}|) + (t-1-(1+||\boldsymbol{k}|)) & , t \in \big[1+(1+|\boldsymbol{k}|), \infty\big) \end{cases}$$

and call $\gamma^{\pm}$ by the symbols .

The following theorem summarizes the statements derived below. Its proof is provided at the end of this section after some preparation.

Theorem 4.4.3 Properties of the Green's Functions

The following properties all hold true for $G^{\mathrm{adv/ret}}$:

1. $G^{\mathrm{adv/ret}} \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^3)$
2. $\Box G^{\mathrm{adv/ret}} = \delta$.
3. When restricting $G^{\mathrm{adv/ret}}$ to $C^{\infty}_c(\mathbb{R} \times \mathbb{R}^3, \mathbb{C})$, they take the form:

$$f \mapsto G^{\mathrm{adv/ret}}[f] = (2\pi)^{\text{-}4/2} \int d\omega d^3k\, \frac{(\mathcal{F}\mathcal{L}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2}\,.$$

4. The Fourier transform of $G^{\mathrm{adv/ret}}$ is given by the limit of the tempered distributions

$$\mathcal{F}_{ph} G^{\mathrm{adv/ret}}_{\varepsilon} : \mathbb{R}^4 \to \mathbb{C}, (\omega, \boldsymbol{k}) \mapsto \frac{(2\pi)^{\text{-}4/2}}{-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2}\,,$$

i.e., the limit $\mathcal{F}_{ph} G^{\mathrm{adv/ret}} = \lim_{\varepsilon\to 0} \mathcal{F}_{ph} G^{\mathrm{adv/ret}}_{\varepsilon}$ exists in $\mathcal{S}'$.

5. $\mathrm{sp}(G^{\mathrm{adv/ret}}) = \Gamma^{\mathrm{light},\mp}_{(0,\mathbf{0})}$.

4.4.2 Preparations

The derivation of Green's functions of $\Box$ begins with the analysis of their characteristic polynomial:

$$m_{ph,\Box^{\mathrm{t}}} : \mathbb{C}^N \to \mathbb{C}, (\omega, \boldsymbol{k}) \mapsto m_{ph,\Box^{\mathrm{t}}}(\omega, \boldsymbol{k}) := (-i\omega)^2 - (-i\boldsymbol{k})^2 = -\omega^2 + \boldsymbol{k}^2 \,.$$

Since $\mathbb{R}^4 \cap \{m_{ph,\Box^{\mathrm{t}}} = 0\} = \{(\omega, \boldsymbol{k}) \in \mathbb{R} \times \mathbb{R}^3 \,|\, \omega = \pm|\boldsymbol{k}|\}$ is a rather large subset of $\mathbb{R}^4$, it is difficult to judge directly whether $\frac{(\sqrt{2\pi})^2}{-\omega^2+\boldsymbol{k}^2}$ leads to Green's functions. Therefore, we will follow two routes:

- We introduce an approximation series of $\frac{(\sqrt{2\pi})^2}{-\omega^2+\boldsymbol{k}^2}$ with parameter ε and show its convergence in $\mathcal{S}'$. To derive certain explicit formulas for their Fourier transforms, a second approximation (parameter σ) is used.
- We also use ▭ from Def. 4.4.2, which are chosen to bypass the set of zeros of $m_{ph,\Box^{\mathrm{t}}}$, to construct a candidate from the linear representation via ▭ := graph(▭) according to Section 4.2.1. To show the equivalence to $G^{\mathrm{adv/ret}}$, we introduce an approximation (parameter σ).

To maintain an overview, we list all relevant objects in the following definition. The well-definedness and their relationship to each other will be shown later.

Definition 4.4.4 Building blocks

Let $\varepsilon, \sigma > 0$ then we regard

- $G^{\pm}_{\mathcal{HL},em} : C_c^\infty(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, \qquad f \mapsto (2\pi)^{-4/2} \int_{▭} d\omega d^3k \, \dfrac{(\mathcal{HL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2}$
- $G^{\pm}_{\mathcal{HL},\sigma,em} : C_c^\infty(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, \qquad f \mapsto (2\pi)^{-4/2} \int_{▭} d\omega d^3k \, \dfrac{(\mathcal{HL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}}$
- $G^{\pm}_{\mathcal{HL},\sigma,tp} : \mathbb{R}^4 \to \mathbb{C}, \qquad (t, \boldsymbol{x}) \mapsto \pm\Theta(\pm t) \dfrac{1}{4\pi|\boldsymbol{x}|} \big(\delta_\sigma(|\boldsymbol{x}| - t) - \delta_\sigma(|\boldsymbol{x}| + t)\big)$
- $G^{\pm}_{\mathcal{HL},tp} : C_c^\infty(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, \qquad f \mapsto \int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\boldsymbol{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(\pm t) \dfrac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big)$
- $G^{\pm}_{\varepsilon,em} : \mathbb{R}^4 \to \mathbb{C}, \qquad (\omega, \boldsymbol{k}) \mapsto \dfrac{(2\pi)^{-4/2}}{-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2}$
- $G^{\pm}_{\varepsilon,\sigma,em} : \mathbb{R}^4 \to \mathbb{C}, \qquad (\omega, \boldsymbol{k}) \mapsto \dfrac{(2\pi)^{-4/2}}{-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}}$
- $G^{\pm}_{\varepsilon,\sigma,tp} : \mathbb{R}^4 \to \mathbb{C}, \qquad (t, \boldsymbol{x}) \mapsto \pm\Theta(\pm t) e^{-\varepsilon|t|} \dfrac{1}{4\pi|\boldsymbol{x}|} \big(\delta_\sigma(|\boldsymbol{x}| - t) - \delta_\sigma(|\boldsymbol{x}| + t)\big)$
- $G^{\pm}_{\varepsilon,tp} : S(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, \qquad f \mapsto \int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\boldsymbol{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(\pm t) e^{-\varepsilon|t|} \dfrac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big)$
- $G^{\pm}_{tp} : S(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, \qquad f \mapsto \int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\boldsymbol{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(\pm t) \dfrac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big)$

where $\delta_\sigma : \mathbb{R} \to \mathbb{R}, s \mapsto \delta_\sigma(s) := \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}s^2}$, $\mathsf{S}_1(\boldsymbol{0}) := \{\boldsymbol{x} \in \mathbb{R}^N \,|\, |\boldsymbol{x}| = 1\}$ and $d\Omega(\hat{\boldsymbol{x}})$ indicates the surface integral with integration variable $\hat{\boldsymbol{x}}$.

4.4.3 Via Cauchy Surface

The following lemma confirms that ▭ bypasses the zeros of $m_{ph,\Box^{\mathrm{t}}}$ such that the candidate of the construction with $(\mathcal{C}oef_{▭}, \mathcal{LC}_{▭})$ can indeed be regarded as a distribution:

Lemma 4.4.5

The following properties hold true for $G^{\pm}_{\mathcal{HL},em}$:

1. They are well-defined and distributions.

2. They indeed solve $\Box G^{\pm}_{\mathscr{FL}} = \delta$.

Proof

Item 1 can be shown, according to Theorem 4.1.20, by checking if, for all $n \in \mathbb{N}$, there exists $C_n \geq 0$ and $k_n \in \mathbb{N}$, such that, for all $f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$, the estimate

$$\left|G^{\pm}_{\mathscr{FL}}[f]\right| \leq C_n \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^N, |\boldsymbol{\alpha}| \leq k_n} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}}$$

with $\|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} := \sup_{\mathbf{x}\in\mathsf{K}_n} |\partial^{\boldsymbol{\alpha}} f(\mathbf{x})|$ holds true.

Since $\boxdot := \bigcup_{\boldsymbol{k}\in\mathbb{R}^3} \gamma^{\pm}_{\boldsymbol{k}}(\mathbb{R}) \times \{\boldsymbol{k}\} \subset (\mathbb{R} + i[-1,1]) \times \mathbb{R}^3$ we know, by Proposition 4.2.12, that on the domain of integration there exists $C > 0$, such that

$$\begin{aligned}
(\mathscr{FL}_{ph} f)(\omega, \boldsymbol{k}) &\leq \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \sup_{(\omega,\boldsymbol{k})\in(\mathbb{R}+i[-1,1])\times\mathbb{R}^3} \left|(1+\omega^2)(1+|\boldsymbol{k}|^5)\mathscr{FL} f(\omega,\boldsymbol{k})\right| \\
&\leq \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \sum_{\beta\in\mathbb{N}_0,|\beta|\leq 7} \sup_{(\omega,\boldsymbol{k})\in(\mathbb{R}+i[-1,1])\times\mathbb{R}^3} \left|(\omega,\boldsymbol{k})^{\beta}\mathscr{FL} f(\omega,\boldsymbol{k})\right| \\
&\leq \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \sum_{\beta\in\mathbb{N}_0,|\beta|\leq 7} \sup_{(\omega,\boldsymbol{k})\in(\mathbb{R}+i[-1,1])\times\mathbb{R}^3} \left|(\omega,\boldsymbol{k})^{\beta}\mathscr{FL} f(\omega,\boldsymbol{k})\right| \\
&\leq C_n \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \sum_{\boldsymbol{\delta}\in\mathbb{N}_0^N,|\boldsymbol{\delta}|\leq 7} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\delta}} \,.
\end{aligned}$$

Furthermore, by the construction of $\boxdot$, we find $\left|(-\omega^2+\boldsymbol{k}^2)^{-1}\right| \leq 1$. Thus, we resume

$$\begin{aligned}
\left|G^{\pm}_{\mathscr{FL}}[f]\right| &\leq (2\pi)^{-4/2} \int_{\boxdot} d\omega d^3k \left|\frac{(\mathscr{FL}_{ph} f)(\omega,\boldsymbol{k})}{-\omega^2+\boldsymbol{k}^2}\right| \\
&\leq (2\pi)^{-4/2} \int_{\boxdot} d\omega d^3k \left|(\mathscr{FL}_{ph} f)(\omega,\boldsymbol{k})\right| \\
&\leq C_n \sum_{\boldsymbol{\delta}\in\mathbb{N}_0^N,|\boldsymbol{\delta}|\leq 7} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\delta}} \int_{\boxdot} d\omega d^3k \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \\
&\leq C_n \sum_{\boldsymbol{\delta}\in\mathbb{N}_0^N,|\boldsymbol{\delta}|\leq 7} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\delta}} \int_{\mathbb{R}^3} d^3k \left(\int_{\mathbb{R}} d\omega + \int_{\boxdot_{\boldsymbol{k}}} d\omega\right) \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \\
&\overset{(i)}{=} C_n \sum_{\boldsymbol{\delta}\in\mathbb{N}_0^N,|\boldsymbol{\delta}|\leq 7} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\delta}} \,,
\end{aligned}$$

where $\boxdot_{\boldsymbol{k}}$ represents $\gamma^{\pm}_{\boldsymbol{k}}|_{[-1-(1+||\boldsymbol{k}|),1+(1+||\boldsymbol{k}|)]}$ and step (i) is due to:

- $\displaystyle\int_{\mathbb{R}^3} d^3k \int_{\mathbb{R}} d\omega \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} < \infty$ and
- $\displaystyle\int_{\mathbb{R}^3} d^3k \int_{\boxdot_{\boldsymbol{k}}} d\omega \frac{1}{(1+\omega^2)(1+|\boldsymbol{k}|^5)} \leq \int_{\mathbb{R}^3} d^3k \frac{2+2(1+|\boldsymbol{k}|)}{(1+|\boldsymbol{k}|^5)} < \infty$, since $|\operatorname{dom}(\gamma^{\pm}_{\boldsymbol{k}})| = 2+2(1+|\boldsymbol{k}|)$ and $|(\gamma^{\pm}_{\boldsymbol{k}})'| = 1$.

Item 2: We consider $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ and calculate:

$$\begin{aligned}
\Box G^{\pm}_{\mathscr{FL}}[f] &= G^{\pm}_{\mathscr{FL}}[\Box f] \\
&= (2\pi)^{-4/2} \int_{\boxdot} d\omega d^3k \frac{(\mathscr{FL}_{ph}\Box f)(\omega,\boldsymbol{k})}{-\omega^2+\boldsymbol{k}^2} \\
&\overset{(i)}{=} (2\pi)^{-4/2} \int_{\boxdot} d\omega d^3k \frac{-\omega^2+\boldsymbol{k}^2}{-\omega^2+\boldsymbol{k}^2} (\mathscr{FL}_{ph} f)(\omega,\boldsymbol{k}) \qquad (i) \text{ by Lemma 4.2.6}
\end{aligned}$$

$$\overset{(ii)}{=} (2\pi)^{-4/2} \int_{\mathbb{R}^3} d^3k \int_{\mathbb{R}} d\omega\, (\mathcal{FL}_{ph} f)(\omega, \boldsymbol{k}) \qquad (ii) \text{ by Definition/Theorem 4.2.13}$$

$$\overset{(iii)}{=} f(0) \qquad (iii) \text{ by Definition/Theorem 4.2.10}$$

$$= \delta[f]\,.$$

To get in touch with the formulation of $G^{\mathrm{adv/ret}}$ in Def. 4.4.1, we first show that indeed $G^{\pm}_{\mathcal{FL},\sigma,em}$ approximates $G^{\pm}_{\mathcal{FL},em}$ for $\sigma \to 0$:

Lemma 4.4.6

$G^{\pm}_{\mathcal{FL},\sigma,em}$ converges for $\sigma \to 0$ to $G^{\pm}_{\mathcal{FL},em}$ in the sense of distributions as topologized in Def. 4.1.24.

Proof

Given $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, we consider the estimate

$$\left|G^{\pm}_{\mathcal{FL},\sigma,em}[f] - G^{\pm}_{\mathcal{FL},em}[f]\right| \leq (2\pi)^{-4/2} \int d\omega d^3k \left|\frac{(\mathcal{FL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2}\right| \left|e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} - 1\right|$$

and find that the integrant is dominated by

$$(\omega, \boldsymbol{k}) \mapsto 2\left|\frac{(\mathcal{FL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2}\right|$$

which is integrable as can be seen in the proof of Lemma 4.4.5. Since the integrant converges pointwisely to zero and also the contour integral is defined as an Lebesgue integral, we find by the dominated convergence theorem

$$\lim_{\sigma \to 0} \left|G^{\pm}_{\mathcal{FL},\sigma,em}[f] - G^{\pm}_{\mathcal{FL},em}[f]\right| = 0$$

i.e., the statement of this lemma.

With the regularization factor $e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}}$ we can now show the equivalence of the representations of the distributions at the level of approximations:

Proposition 4.4.7

$G^{\pm}_{\mathcal{FL},\sigma,em} = G^{\pm}_{\mathcal{FL},\sigma,tp} \in L^1_{loc,c}(\mathbb{R} \times \mathbb{R}^3, \mathbb{C})$ in the sense of the embedding in Def. 4.1.8.

Proof

Given $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, we consider

$$G^{\pm}_{\mathcal{FL},\sigma,em}[f] = (2\pi)^{-4/2} \int d\omega d^3k\, \frac{(\mathcal{FL}_{ph} f)(\omega, \boldsymbol{k})}{-\omega^2 + \boldsymbol{k}^2} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}}$$

$$= (2\pi)^{-4/2} \int d\omega d^3k\, \frac{1}{-\omega^2 + \boldsymbol{k}^2} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} (2\pi)^{-4/2} \int_{\mathbb{R}^4} dt d^3x\, f(t, \boldsymbol{x})\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k} | \mathbf{x} \rangle)}$$

and notice that, due to the complex contour, the integrand is smooth in both sets of variables, i.e., $(\omega, \boldsymbol{k})$ and $(t, \boldsymbol{x})$. Additionally, when regarding the complex contour integral in the sense of Lebesgue integrals, the regularizing factor $\exp\left\{\left(-\frac{\boldsymbol{k}^2}{2\sigma^2}\right)\right\}$ makes it integrable with respect to both sets of variables. Thus, Fubini's theorem can be allied leading to

$$G^{\pm}_{\mathcal{HL},\sigma,em}[f] = \int_{\mathbb{R}^4} dt d^3x \left(\int_{\mathbb{R}^3} d^3k\, (2\pi)^{-3} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \int_{\mathcal{C}_{\boldsymbol{k}}} d\omega\, (2\pi)^{-1} \frac{1}{-\omega^2 + \boldsymbol{k}^2}\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \right) f(t,\boldsymbol{x})$$

and with further manipulations the expression in the bracket becomes:

$$\int_{\mathbb{R}^3} d^3k\, (2\pi)^{-3} \frac{1}{2|\boldsymbol{k}|} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \left(\int_{\mathcal{C}_{\boldsymbol{k}}} d\omega\, (2\pi)^{-1} \left(\frac{1}{\omega - (-|\boldsymbol{k}|)} - \frac{1}{\omega - |\boldsymbol{k}|} \right) e^{-\mathrm{i}\omega t} \right) e^{\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} .$$

The time dependent contour integral in ω can now be computed by the residue theorem when closing the integration curve

- along the upper half-plane for $t < 0$, and respectively,
- along the lower half-plane for $t > 0$,

which resulting in:

$$\int_{\mathcal{C}_{\boldsymbol{k}}} d\omega\, (2\pi)^{-1} \left(\frac{1}{\omega - (-|\boldsymbol{k}|)} - \frac{1}{\omega - |\boldsymbol{k}|} \right) e^{-\mathrm{i}\omega t} = \mp\Theta(\pm t) i \left(e^{\mathrm{i}|\boldsymbol{k}|t} - e^{-\mathrm{i}|\boldsymbol{k}|t} \right) .$$

Evaluating the angular and radial parts of the $\boldsymbol{k}$-integration successively leads to the following calculation:

$$\begin{aligned}
&\int_{\mathbb{R}^3} d^3k\, (2\pi)^{-3} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \int_{\mathcal{C}_{\boldsymbol{k}}} d\omega\, (2\pi)^{-1} \frac{1}{-\omega^2 + \boldsymbol{k}^2}\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi} \int_{\mathbb{R}_{\geq 0}} d|\boldsymbol{k}| \int_{[0,2\pi]} d\phi \int_{[-1,1]} d\cos\theta\, |\boldsymbol{k}|^2 (2\pi)^{-2} \frac{1}{|\boldsymbol{k}|} e^{-\frac{|\boldsymbol{k}|^2}{2\sigma^2}} \left(e^{-\mathrm{i}|\boldsymbol{k}|t} - e^{\mathrm{i}|\boldsymbol{k}|t} \right) \mathrm{i}\, e^{\mathrm{i}|\boldsymbol{k}||\boldsymbol{x}|\cos\theta} \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi} \int_{\mathbb{R}_{\geq 0}} d|\boldsymbol{k}|\, (2\pi)^{-1} e^{-\frac{|\boldsymbol{k}|^2}{2\sigma^2}} \left(e^{-\mathrm{i}|\boldsymbol{k}|t} - e^{\mathrm{i}|\boldsymbol{k}|t} \right) \left(\int_{[-1,1]} d\cos\theta\, \mathrm{i}|\boldsymbol{k}|\, e^{\mathrm{i}|\boldsymbol{k}||\boldsymbol{x}|\cos\theta} \right) \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi|\boldsymbol{x}|} \int_{\mathbb{R}_{\geq 0}} d|\boldsymbol{k}|\, (2\pi)^{-1} e^{-\frac{|\boldsymbol{k}|^2}{2\sigma^2}} \left(e^{-\mathrm{i}|\boldsymbol{k}|t} - e^{\mathrm{i}|\boldsymbol{k}|t} \right) \left(e^{\mathrm{i}|\boldsymbol{k}||\boldsymbol{x}|} - e^{-\mathrm{i}|\boldsymbol{k}||\boldsymbol{x}|} \right) \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi|\boldsymbol{x}|} \int_{\mathbb{R}_{\geq 0}} d|\boldsymbol{k}|\, (2\pi)^{-1} e^{-\frac{|\boldsymbol{k}|^2}{2\sigma^2}} \left(e^{-\mathrm{i}|\boldsymbol{k}|(t-|\boldsymbol{x}|)} + e^{\mathrm{i}|\boldsymbol{k}|(t-|\boldsymbol{x}|)} - e^{-\mathrm{i}|\boldsymbol{k}|(t+|\boldsymbol{x}|)} - e^{\mathrm{i}|\boldsymbol{k}|(t+|\boldsymbol{x}|)} \right) \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi|\boldsymbol{x}|} \int_{\mathbb{R}} d|\boldsymbol{k}|\, (2\pi)^{-1} e^{-\frac{|\boldsymbol{k}|^2}{2\sigma^2}} \left(e^{\mathrm{i}|\boldsymbol{k}|(t-|\boldsymbol{x}|)} - e^{\mathrm{i}|\boldsymbol{k}|(t+|\boldsymbol{x}|)} \right) \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi|\boldsymbol{x}|} \left(\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}(t-|\boldsymbol{x}|)^2} - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}(t+|\boldsymbol{x}|)^2} \right) \\
&\quad = \pm\Theta(\pm t) \frac{1}{4\pi|\boldsymbol{x}|} \left(\delta_\sigma(|\boldsymbol{x}| - t) - \delta_\sigma(|\boldsymbol{x}| + t) \right) \\
&\quad = G^{\pm}_{\mathcal{HL},\sigma,tp}(t,\boldsymbol{x}) .
\end{aligned}$$

This leaves us with with the expression

$$G^{\pm}_{\mathcal{HL},\sigma,em}[f] = \int_{\mathbb{R}^4} dt d^3x\, G^{\pm}_{\mathcal{HL},\sigma,tp}(t,\boldsymbol{x}) f(t,\boldsymbol{x})$$

which implies the statement of the proposition.

It remains to be shown, that the approximation also converges on the other side of the picture when removing the regularization, i.e., for $\sigma \to 0$, the convergence $G_{\mathcal{HL},\sigma,tp} \to G^{\pm}_{\mathcal{HL},tp}$ holds true in distributions.

Theorem 4.4.8

The equality $G^{\pm}_{\mathcal{HL},em} = G^{\pm}_{\mathcal{HL},pt}$ holds true.

The following lemma will be helpful to prove the preceding theorem:

Lemma 4.4.9

Given a function $\delta_1 \in L^1(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ with $\int_{\mathbb{R}^N} d^N x\, \delta_1(\boldsymbol{x}) = 1$ and $\varepsilon > 0$ we define $\delta_\varepsilon := \big(\boldsymbol{x} \mapsto (1/\varepsilon)^N \delta_1(\boldsymbol{x}/\varepsilon)\big)$. Furthermore, let $h \in L^\infty(\mathbb{R}^N, \mathbb{C})$ and $\boldsymbol{x} \in \mathbb{R}^N$ such that h is continuous at $\boldsymbol{x}$. Then, we find

$$\lim_{\varepsilon \to 0} h * \delta_\varepsilon(\boldsymbol{x}) = h(\boldsymbol{x})\,.$$

Proof

Let $\epsilon > 0$, then we put the following preparations into place:

- By the continuity of h at $\boldsymbol{x}$, there exists $\delta > 0$ such that, for all $\boldsymbol{y} \in \mathsf{B}_\delta(\mathbf{0})$, the estimate $|h(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})| < \epsilon/2$ holds true.
- Since, given δ as in the step before, the function $\boldsymbol{y} \mapsto \mathbb{1}_{\mathbb{R}^N \setminus \mathsf{B}_{\delta/\varepsilon}(\mathbf{0})}(\boldsymbol{y})|\delta_1(\boldsymbol{y})|$ convergences pointwisely to zero and is dominated by the integrable function $|\delta_1|$, we find due to the dominated convergence theorem

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus \mathsf{B}_\delta(\mathbf{0})} d^N y\, \delta_\varepsilon(\boldsymbol{y}) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus \mathsf{B}_\delta(\mathbf{0})} d^N y\, \frac{1}{\varepsilon^N}\delta_1\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus \mathsf{B}_{\delta/\varepsilon}(\mathbf{0})} d^N z\, \delta_1(\boldsymbol{z})\\ &= 0\,.\end{aligned}$$

  Thus, there exists $\bar{\varepsilon} > 0$, such that, for all $\varepsilon < \bar{\varepsilon}$, we find $\int_{\mathbb{R}^N\setminus \mathsf{B}_\delta(\mathbf{0})} d^N y\, \delta_\varepsilon(\boldsymbol{y}) < \epsilon/(4\|h\|_\infty)$.

These estimates lead, for $\varepsilon < \bar{\varepsilon}$ to the calculation

$$\begin{aligned}&|h * \delta_\varepsilon(\boldsymbol{x}) - h(\boldsymbol{x})|\\ &\quad= \left|\int_{\mathbb{R}^N} d^N y\, h(\boldsymbol{x}-\boldsymbol{y})\delta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})\right|\\ &\quad\leq \int_{\mathbb{R}^N} d^N y\, |h(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})|\delta_\varepsilon(\boldsymbol{y})\\ &\quad\leq \int_{\mathsf{B}_\delta(\mathbf{0})} d^N y\, |h(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})|\delta_\varepsilon(\boldsymbol{y}) + \int_{\mathbb{R}^N\setminus \mathsf{B}_\delta(\mathbf{0})} d^N y\, |h(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})|\delta_\varepsilon(\boldsymbol{y})\\ &\quad\leq \delta\int_{\mathsf{B}_\delta(\mathbf{0})} d^N y\, \delta_\varepsilon(\boldsymbol{y}) + 2\|h\|_\infty \int_{\mathbb{R}^N\setminus \mathsf{B}_\delta(\mathbf{0})} d^N y\, \delta_\varepsilon(\boldsymbol{y})\\ &\quad= \epsilon/2 + \epsilon/2\end{aligned}$$

and, thus, $\lim_{\varepsilon\to 0} h * \delta_\varepsilon(\boldsymbol{x}) = h(\boldsymbol{x})$.

Proof Theorem 4.4.8

Since the equality $G^\pm_{\mathscr{H},em} \overset{(i)}{=} \lim_{\sigma\to 0} G^\pm_{\mathscr{H},\sigma,em} \overset{(ii)}{=} \lim_{\sigma\to 0} G^\pm_{\mathscr{H},\sigma,tp}$ holds true, with

(*i*) due to Lemma 4.4.6 and

(*ii*) by Proposition 4.4.7,

it remains to shown, that $G^\pm_{\mathscr{H},\sigma,tp} \xrightarrow{\sigma\to 0} G^\pm_{\mathscr{H},tp}$ in $\mathscr{D}'$. We will restrict our proof for clarity to the case $G^+_{\mathscr{H}}$. The other case plays out analogous. We need to show that, given $f \in C_c^\infty(\mathbb{R}^4, \mathbb{C})$ and in the limit $\sigma \to 0$, the following entity approaches 0:

$$\left|\int_{\mathbb{R}^4} dt d^3x\, \Theta(t)\frac{1}{4\pi|\boldsymbol{x}|}\big(\delta_\sigma(|\boldsymbol{x}|-t) - \delta_\sigma(|\boldsymbol{x}|+t)\big)f(t,\boldsymbol{x}) - \int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\Theta(t)\frac{1}{4\pi|t|}f\big(t,|t|\hat{\boldsymbol{x}}\big)\right|. \tag{4.12}$$

We start by picking $c > 0$ rewriting the fist term in the following way

$$
\begin{aligned}
&\int_{\mathbb{R}^4} dt d^3x\, \Theta(t) \frac{1}{4\pi|\boldsymbol{x}|} \big(\delta_\sigma(|\boldsymbol{x}| - t) - \delta_\sigma(|\boldsymbol{x}| + t)\big) f(t, \boldsymbol{x}) \\
&\qquad = \int_{\mathbb{R}_{\geq 0}} dt \underbrace{\int_{\mathbb{R}_{\geq 0}} dr\, \frac{r}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, \big(\delta_\sigma(r - t) - \delta_\sigma(r + t)\big) f(t, r\hat{\boldsymbol{x}})}_{=: F_\sigma(t)} \\
&\qquad \overset{(i)}{=} \int_c^\infty dt \int_0^\infty dr\, \frac{r}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, \big(\delta_\sigma(r - t) - \delta_\sigma(r + t)\big) f(t, r\hat{\boldsymbol{x}}) + \int_0^c dt F_\sigma(t) \\
&\qquad \overset{(ii)}{=} \int_c^\infty dt \int_{-t}^\infty du\, \frac{u + t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, \delta_\sigma(u) f\big(t, (u + t)\hat{\boldsymbol{x}}\big) \\
&\qquad\quad - \int_c^\infty dt \int_t^\infty dv\, \frac{v - t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, \delta_\sigma(v) f\big(t, (v - t)\hat{\boldsymbol{x}}\big) \\
&\qquad\quad + \int_0^c dt F_\sigma(t) \\
&\qquad = \int_c^\infty dt \int_{-\infty}^\infty du\, \frac{u + t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, \delta_\sigma(u) f\big(t, (u + t)\hat{\boldsymbol{x}}\big) \Theta(u + t) \\
&\qquad\quad - \int_c^\infty dt \int_{-\infty}^\infty dv\, \frac{v - t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, \delta_\sigma(v) f\big(t, (v - t)\hat{\boldsymbol{x}}\big) \Theta(v - t) \\
&\qquad\quad + \int_0^c dt F_\sigma(t) \\
&\qquad \overset{(iii)}{=} \int_{-\infty}^\infty du\, \delta_\sigma(u) \left(\int_c^\infty dt\, \frac{u + t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, f\big(t, (u + t)\hat{\boldsymbol{x}}\big) \Theta(u + t) \right) \\
&\qquad\quad - \int_{-\infty}^\infty dv\, \delta_\sigma(v) \left(\int_c^\infty dt\, \frac{v - t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, f\big(t, (v - t)\hat{\boldsymbol{x}}\big) \Theta(v - t) \right) \\
&\qquad\quad + \int_0^c dt F_\sigma(t) \\
&\qquad \overset{(iv)}{=} \delta_\sigma * H(0) - \delta_\sigma * J(0) + \int_0^c dt F_\sigma(t)\,,
\end{aligned}
$$

whereby the following arguments justify the preceding equations:

(i) due to $c > 0$.

(ii) by substitution with $u(r) := r - t$ and $v(r) := r + t$.

(iii) by Fubini.

(iv) by defining the following functions:

$$
\begin{aligned}
H_c(-u) &:= \int_c^\infty dt\, h(t, -u) \quad \text{with } h(t, -u) := \frac{u + t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, f\big(t, (u + t)\hat{\boldsymbol{x}}\big) \Theta(u + t) \text{ and} \\
J_c(-u) &:= \int_c^\infty dt\, j(t, -v) \quad \text{with } j(t, -v) := \frac{v - t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, f\big(t, (v - t)\hat{\boldsymbol{x}}\big) \Theta(v - t)\,.
\end{aligned}
$$

Further, $*$ is the convolution of functions.

Additionally, we rewrite also the second term:

$$
\begin{aligned}
&\int_{\mathbb{R}} dt\, t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(t) \frac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big) \\
&\qquad = \int_c^\infty dt\, t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(t) \frac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big) + \int_0^c dt\, t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(t) \frac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big) \\
&\qquad = H_c(\mathbf{0}) + \int_0^c dt\, \underbrace{t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(t) \frac{1}{4\pi|t|} f\big(t, |t|\hat{t}\big)}_{=: F(t)}
\end{aligned}
$$

$$= H_c(0) + \int_0^c dt F(t)\,.$$

Combining these two calculation leads to the following upper bound of (4.12)

$$\underbrace{|\delta_\sigma * H_c(0) - H_c(0)| + |\delta_\sigma * J_c(0)|}_{\text{Term II}} + \underbrace{\left|\int_0^c dt F_\sigma(t)\right| + \left|\int_0^c dt F(t)\right|}_{\text{Term I}}$$

and attack these terms successively. Let $\epsilon > 0$ be given.

Term I will be treated by finding some c_ϵ such that the sum of the integral terms become less then $\epsilon/2$. First, we find some dominating function for F_σ, by considering for $t \in \mathbb{R}$

$$\begin{aligned}
|F_\sigma(t)| &\leq \int_{\mathbb{R}_{\geq 0}} dr\, \frac{r}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}}) \left(\delta_\sigma(r-t) + \delta_\sigma(r+t)\right) |f(t, r\hat{\boldsymbol{x}})| \\
&\leq C \int_{\mathbb{R}} dr \big(\delta_\sigma(r-t) + \delta_\sigma(r+t)\big)(1+t^2)^{-1} \sup_{(\tilde{t},\tilde{\boldsymbol{x}})\in\mathbb{R}\times\mathbb{R}^3} \left|(1+\tilde{t}^2)(1+\tilde{\boldsymbol{x}}^2) f(\tilde{t},\tilde{\boldsymbol{x}})\right| \\
&\leq C \left(\sum_{\boldsymbol{\beta}\in\mathbb{N}_0^4, |\boldsymbol{\beta}|\leq 4} \|f\|_{\infty,\mathbf{0},\boldsymbol{\beta}} \right) (1+t^2)^{-1}
\end{aligned}$$

which tells us, that the convergence

$$\lim_{c\to 0} \left|\int_0^c dt F_\sigma(t)\right| \leq \lim_{c\to 0} C \left(\sum_{\boldsymbol{\beta}\in\mathbb{N}_0^4, |\boldsymbol{\beta}|\leq 4} \|f\|_{\infty,\mathbf{0},\boldsymbol{\beta}} \right) \int_0^c dt (1+t^2)^{-1} = 0$$

holds true. This is due to $f \in \mathbb{C}_c^\infty(\mathbb{R}^4, \mathbb{C}) \subset S(\mathbb{R}^4, \mathbb{C})$. Moreover, by

$$\begin{aligned}
|F(t)| &\leq t^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}}) \Theta(t) \frac{1}{4\pi\, t} \left| f\big(t, |t|\hat{\boldsymbol{x}}\big)\right| \\
&\leq C(1+t^2)^{-1} \sup_{(\tilde{t},\tilde{\boldsymbol{x}})\in\mathbb{R}\times\mathbb{R}^3} \left|(1+\tilde{t}^2)\tilde{t} f(\tilde{t},\tilde{\boldsymbol{x}})\right| \\
&\leq C \left(\sum_{\boldsymbol{\beta}\in\mathbb{N}_0^4, |\boldsymbol{\beta}|\leq 3} \|f\|_{\infty,\mathbf{0},\boldsymbol{\beta}} \right) (1+t^2)^{-1}
\end{aligned}$$

and with the same argument as before, we deduce $\lim_{c\to 0} \left|\int_0^c dt F(t)\right| = 0$. This lets us choose $c_\epsilon > 0$ such that the following expression holds true:

$$\left|\int_0^{c_\epsilon} dt F_\sigma(t)\right| + \left|\int_0^{c_\epsilon} dt F(t)\right| \leq \epsilon/2\,.$$

Term II can, with c_ϵ in place, be estimated by Lemma 4.4.9. We just need to proof that both H_{c_ϵ} and J_{c_ϵ} lie in $L^\infty(\mathbb{R}, \mathbb{C})$ and continuos at 0.

- **Continuity of** H_{c_ϵ} can be seen by first observing that, for $t \in [c_\epsilon, \infty)$, the integrant

$$u \mapsto h(t,u) = \frac{-u+t}{4\pi} \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, f\big(t, (-u+t)\hat{\boldsymbol{x}}\big)\Theta(-u+t)$$

is continuous in the neighborhood $(-1, c_\epsilon)$ of 0. This follows by dominated convergence due to the compactness of $\mathsf{S}_1(\mathbf{0})$, the smoothness of f and the continuity of $u \mapsto \Theta(-u+t)$ on $(-1, c_\epsilon)$. A u-independent, integrable dominating function is achieved via the estimate

$$|h(t,u)| \leq (1+t^2)^{-1} \sup_{(\tilde{t},\tilde{\boldsymbol{x}})\in\mathbb{R}\times\mathbb{R}^3} (1+\tilde{t}^2)(1+\tilde{\boldsymbol{x}}^2) f(\tilde{t},\tilde{\boldsymbol{x}}) \quad \leq C\Bigg(\sum_{\boldsymbol{\beta}\in\mathbb{N}_0^4, |\boldsymbol{\beta}|\leq 4} \|f\|_{\infty,\mathbf{0},\boldsymbol{\beta}} \Bigg)(1+t^2)^{-1}\,.$$

and another application of dominated convergence implies the continuity of H_{c_ϵ} at 0.

- **Boundedness of** H_{c_ϵ} is also directly implied by the dominating function.
- **Continuity and Boundedness of** J_{c_ϵ} follows immediately since, for $t \in [c_\epsilon, \infty)$ and $|v| < c_\epsilon$, the integrant

$$v \mapsto j(t,v) = \frac{-v-t}{4\pi} \int_{S_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\, f\big(t, (-v-t)\hat{\boldsymbol{x}}\big)\Theta(-v-t)$$

evaluates to 0.

Thus by referring to Definition/Proposition 4.1.11 there exists an $\tilde{\sigma} > 0$ such that for all $\sigma \leq \tilde{\sigma}$:

$$|\delta_\sigma * H_{c_\epsilon}(0) - H_{c_\epsilon}(0)| + |\delta_\sigma * J_{c_\epsilon}(0)| \leq \epsilon/2$$

Finally, we find, for all $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ and $\epsilon > 0$, that there exists $c_\epsilon > 0$ and $\tilde{\sigma} > 0$ such that, for all $\sigma \leq \tilde{\sigma}$ the inequality

$$\begin{aligned}\left| G^{\pm}_{\mathscr{HL},\sigma}[f] - \int_{\mathbb{R}} dt\, t^2 \int_{S_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\Theta(t)\frac{1}{4\pi|t|} f\big(t, |t|\hat{\boldsymbol{x}}\big)\right| & \\ \leq |\delta_\sigma * H_{c_\epsilon}(0) - H_{c_\epsilon}(0)| + |\delta_\sigma * J_{c_\epsilon}(0)| + \left|\int_0^{c_\epsilon} dt F_\sigma(t)\right| + \left|\int_0^{c_\epsilon} dt F(t)\right| & \\ \leq \epsilon &\end{aligned}$$

holds true and thus the statement of the theorem.

4.4.4 Via Tempered Distributions

Will will prove, for $\varepsilon \to 0$, the convergence of $G^{\pm}_{\varepsilon,em}$ in $\mathscr{S}'$ after passing to the space-time picture. To do so, we put, similar to the case before, the regularizing factor $\exp\big(-\frac{\boldsymbol{k}^2}{2\sigma^2}\big)$ in place, show that both $G^{\pm}_{\varepsilon,em}$ and its approximation are tempered distributions, prove the convergence of the approximation in energy-momentum space and finally establish that the approximation lies in L^1 enabling us to explicitly evaluate its Fourier transform:

Lemma 4.4.10

The following statements hold true:

1. $G^{\pm}_{\varepsilon,em}, G^{\pm}_{\varepsilon,\sigma,em} \in \mathscr{S}'(\mathbb{R} \times \mathbb{R}^3)$.
2. $G^{\pm}_{\varepsilon,\sigma,em} \to G^{\pm}_{\varepsilon,em}$ for $\sigma \to 0$ in $\mathscr{S}'(\mathbb{R}^4)$.
3. $G^{\pm}_{\varepsilon,em} \notin L^1(\mathbb{R}^4, \mathbb{C})$ but $G^{\pm}_{\varepsilon,\sigma,em} \in L^1(\mathbb{R}^4, \mathbb{C})$.

Proof

Item 1 is a consequence of the boundedness of both functions. We consider, for given $\varepsilon > 0$ and any $(\omega, \boldsymbol{k}) \in \mathbb{R} \times \mathbb{R}^3$, the estimate

$$\left|G^{\pm}_{\varepsilon,em}(\omega,\boldsymbol{k})\right| = \frac{(2\pi)^{-4/2}}{\left|-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2\right|} = \frac{(2\pi)^{-4/2}}{\sqrt{(\varepsilon^2 + \boldsymbol{k}^2 - \omega^2)^2 + 4\omega^2\varepsilon^2}} \overset{(i)}{\leq} \frac{(2\pi)^{-4/2}}{\varepsilon}$$

and justify the step (i) in the following. The denominator on the left-hand side of (i) possesses has the same position of minima as $(\omega, \boldsymbol{k}) \mapsto (\varepsilon^2 + \boldsymbol{k}^2 - \omega^2) + 4\omega^2\varepsilon^2$. This can be found by solving

- $0 = \partial_\omega(\varepsilon^2 + \boldsymbol{k}^2 - \omega^2)^2 + 4\omega^2\varepsilon^2 = 2\omega\big(-2(\varepsilon^2 + \boldsymbol{k}^2 - \omega^2) + 4\varepsilon^2\big) = -2\omega\, 2(-\varepsilon^2 + \boldsymbol{k}^2 - \omega^2)$
- $0 = \partial_{|\boldsymbol{k}|}(\varepsilon^2 + \boldsymbol{k}^2 - \omega^2)^2 + 4\omega^2\varepsilon^2 = 2|k|\, 2(\varepsilon^2 + \boldsymbol{k}^2 - \omega^2)$

for $(\omega, \boldsymbol{k})$. This implies $(\omega, \boldsymbol{k}) = (0, \boldsymbol{0})$. Additionally, we observe $\left|G^{\pm}_{\varepsilon,\sigma,em}(\omega, \boldsymbol{k})\right| \leq \left|G^{\pm}_{\varepsilon,em}(\omega, \boldsymbol{k})\right|$. Thus, both function are bounded and smooth and thus by Theorem 4.2.23 in $\mathcal{S}'(\mathbb{R}^4)$.

Item 2 is a direct consequence of the dominated convergence theorem since, for $f \in S(\mathbb{R}^4, \mathbb{C})$, we find

$$\begin{aligned}
\lim_{\sigma\to 0}\left|G^{\pm}_{\varepsilon,\sigma,em}[f] - G^{\pm}_{\varepsilon,em}[f]\right| &\leq \lim_{\sigma\to 0}\int_{\mathbb{R}^4} d\omega d^3k \left|\left(e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} - 1\right)\frac{(2\pi)^{-4/2}}{-(\omega\pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2} f(t,\boldsymbol{x})\right| \\
&\leq \lim_{\sigma\to 0}\int_{\mathbb{R}^4} d\omega d^3k \left|e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} - 1\right|\frac{(2\pi)^{-4/2}}{\varepsilon}|f(t,\boldsymbol{x})| \\
&= 0\,.
\end{aligned}$$

Its application is justified since $2\frac{(2\pi)^{-4/2}}{\varepsilon}|f(t,\boldsymbol{x})|$ is integrable and dominates the pointwisely to 0 convergent integrant.

Item 3 can be seen via considering again, for $\varepsilon, \sigma > 0$, the following statements:

- Considering,for fixed $\omega \in \mathbb{R}$, the function $\boldsymbol{k} \mapsto \left|G^{\pm}_{\varepsilon,em}(\omega, \boldsymbol{k})\right| = \frac{(2\pi)^{-4/2}}{\sqrt{(\varepsilon^2+\boldsymbol{k}^2-\omega^2)^2+4\omega^2\varepsilon^2}}$ cannot be integrated due to the insufficient $\frac{1}{\boldsymbol{k}^2}$ decay behavior. This leads to $G^{\pm}_{\varepsilon,em} \notin L^1(\mathbb{R}^4, \mathbb{C})$.
- Given $(\omega, \boldsymbol{k}) \in \mathbb{R}\times\mathbb{R}^3$, we regard the estimate

$$\begin{aligned}
\left|G^{\pm}_{\varepsilon,\sigma,em}(\omega, \boldsymbol{k})\right| &= \frac{(2\pi)^{-4/2}}{\sqrt{(\varepsilon^2+\boldsymbol{k}^2-\omega^2)^2+4\omega^2\varepsilon^2}}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \\
&= \Theta\big(\omega^2 - 2(\boldsymbol{k}^2+\varepsilon^2+1)\big)\frac{(2\pi)^{-4/2}}{\sqrt{(\varepsilon^2+\boldsymbol{k}^2-\omega^2)^2+4\omega^2\varepsilon^2}}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \\
&\quad + \Theta\big(2(\boldsymbol{k}^2+\varepsilon^2+1) - \omega^2\big)\frac{(2\pi)^{-4/2}}{\sqrt{(\varepsilon^2+\boldsymbol{k}^2-\omega^2)^2+4\omega^2\varepsilon^2}}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \\
&\overset{(i)}{\leq} \frac{(2\pi)^{-4/2}}{\sqrt{\left(\frac{\omega^2}{2}+1\right)^2+4\omega^2\varepsilon^2}}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} + \Theta\big(2(\boldsymbol{k}^2+\varepsilon^2+1)-\omega^2\big)\frac{(2\pi)^{-4/2}}{\varepsilon}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}}
\end{aligned}$$

  where we applied in (i) the implication $\omega^2 \geq 2(\boldsymbol{k}^2+\varepsilon^2+1) \implies \omega^2 - \boldsymbol{k}^2 - \varepsilon^2 \geq \frac{\omega^2}{2}+1 \implies (\varepsilon^2+\boldsymbol{k}^2-\omega^2)^2 \geq \left(\frac{\omega^2}{2}+1\right)^2$. The first expression clearly lies in $L^1(\mathbb{R}^4, \mathbb{C})$ when regarded as a function of $(\omega, \boldsymbol{k})$. Integrating the second expression, we find

$$\int_{\mathbb{R}^3} d^3k \int_{\mathbb{R}} dt \Theta\big(2(\boldsymbol{k}^2+\varepsilon^2+1)-\omega^2\big)\frac{(2\pi)^{-4/2}}{\varepsilon}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} = \int_{\mathbb{R}^3} d^3k 2\sqrt{2(\boldsymbol{k}^2+\varepsilon^2+1)}\frac{(2\pi)^{-4/2}}{\varepsilon}e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} < \infty\,.$$

  Thus we conclude $G^{\pm}_{\varepsilon,\sigma,em}$ lies $L^1(\mathbb{R}^4, \mathbb{C})$.

With the regularization, we can move from the energy-momentum image to the space-time image via the Fourier transform. We can then remove the regularization, i.e. set $\sigma \to 0$, and finally show the convergence of our approximation in the space-time image when performing the limit $\varepsilon \to 0$:

Proposition 4.4.11

The following statements hold true:

1. $G^{\pm}_{\varepsilon,\sigma,pt} = \mathcal{F}G^{\pm}_{\varepsilon,\sigma,em}$ in the sense of Theorem 4.2.20.
2. $G^{\pm}_{\varepsilon,\sigma,pt} \to G^{\pm}_{\varepsilon,pt}$ for $\sigma \to 0$ in $\mathcal{S}'(\mathbb{R}^4)$.
3. $G^{\pm}_{\varepsilon,pt} = \mathcal{F}G^{\pm}_{\varepsilon,em}$ in the sense of Theorem 4.2.20.
4. $G^{\pm}_{\varepsilon,pt} \to G^{\pm}_{pt}$ for $\sigma \to 0$ in $\mathcal{S}'(\mathbb{R}^4)$.

Proof

Item 1 can be calculated by considering $f \in S(\mathbb{R}^4, \mathbb{C})$. We consider the following calculations:

$$\begin{aligned}(\mathcal{F} G^{\pm}_{\varepsilon,\sigma,em})[f] &= G^{\pm}_{\varepsilon,\sigma,em}[\mathcal{F} f] \\ &= \int_{\mathbb{R}^4} d\omega d^3k\, G^{\pm}_{\varepsilon,\sigma,em}(\omega, \boldsymbol{k})(\mathcal{F} f)(\omega, \boldsymbol{k}) \\ &= \int_{\mathbb{R}^4} d\omega d^3k\, G^{\pm}_{\varepsilon,\sigma,em}(\omega, \boldsymbol{k})(2\pi)^{-4/2} \int_{\mathbb{R}^4} dt d^3x\, f(t, \boldsymbol{x})\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \\ &\overset{(i)}{=} \int_{\mathbb{R}^4} dt d^3x \left((2\pi)^{-4/2} \int_{\mathbb{R}^4} d\omega d^3k\, G^{\pm}_{\varepsilon,\sigma,em}(\omega, \boldsymbol{k})\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \right) f(t, \boldsymbol{x})\,.\end{aligned}$$

Step (i) is justified by Fubini's theorem, since, by Item 3 of Lemma 4.4.10, $G^{\pm}_{\varepsilon,\sigma,em}$ lies in L^1 and further $f \in S \subset L^1$. Furthermore, we follow along the manipulations:

$$\begin{aligned}&(2\pi)^{-4/2} \int_{\mathbb{R}^4} d\omega d^3k\, G^{\pm}_{\varepsilon,\sigma,em}(\omega, \boldsymbol{k})\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \\ &\quad= (2\pi)^{-4/2} \int_{\mathbb{R}^4} d\omega d^3k\, \frac{(2\pi)^{-4/2}}{-(\omega \pm \mathrm{i}\varepsilon)^2 + \boldsymbol{k}^2} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}}\, e^{-\mathrm{i}(\omega t - \langle \boldsymbol{k}|\mathbf{x}\rangle)} \\ &\quad= \int_{\mathbb{R}^3} d^3k\, (2\pi)^{-3} \frac{1}{2|\boldsymbol{k}|} e^{-\frac{\boldsymbol{k}^2}{2\sigma^2}} \left(\int_{\mathbb{R}} d\omega\, (2\pi)^{-1} \left(\frac{1}{\omega - (-|\boldsymbol{k}| \mp \mathrm{i}\varepsilon)} - \frac{1}{\omega - (|\boldsymbol{k}| \mp \mathrm{i}\varepsilon)} \right) e^{-\mathrm{i}\omega t} \right) e^{\mathrm{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}\,.\end{aligned}$$

The time dependent integral in ω in the preceding expression can be computed by the residue theorem when closing the integration curve,

- for $t < 0$, along the upper half-plane, and respectively,
- for $t > 0$, along the lower half-plane,

which results in:

$$\int_{\circlearrowleft} d\omega\, (2\pi)^{-1} \left(\frac{1}{\omega - (-|\boldsymbol{k}| \mp \mathrm{i}\varepsilon)} - \frac{1}{\omega - (|\boldsymbol{k}| \mp \mathrm{i}\varepsilon)} \right) e^{-\mathrm{i}\omega t} = \mp\Theta(\pm t) i \left(e^{\mathrm{i}|\boldsymbol{k}|t} - e^{-\mathrm{i}|\boldsymbol{k}|t} \right) e^{\mp\varepsilon t}\,.$$

From here on we just reproduce the calculations in the proof of Proposition 4.4.7 while carrying along the additional factor $e^{\mp\varepsilon t} = e^{\varepsilon|t|}$ leading us to:

$$\begin{aligned}(\mathcal{F} G^{\pm}_{\varepsilon,\sigma,em})[f] &= \int_{\mathbb{R}^4} dt d^3x \left(\pm\Theta(\pm t) e^{\mp\varepsilon t} \frac{1}{4\pi|\boldsymbol{x}|} \left(\delta_\sigma(|\boldsymbol{x}| - t) - \delta_\sigma(|\boldsymbol{x}| + t)\right) \right) f(t, \boldsymbol{x}) \\ &= \int_{\mathbb{R}^4} dt d^3x \left(\pm\Theta(\pm t) e^{-\varepsilon|t|} \frac{1}{4\pi|\boldsymbol{x}|} \left(\delta_\sigma(|\boldsymbol{x}| - t) - \delta_\sigma(|\boldsymbol{x}| + t)\right) \right) f(t, \boldsymbol{x})\,.\end{aligned}$$

Item 2 is a consequence of the proof of Theorem 4.4.8. This holds true without change when regarding f as a function in $S(\mathbb{R}^4, \mathbb{C})$ instead of $C^\infty_c(\mathbb{R}^4, \mathbb{C})$. The additional $e^{-\varepsilon|t|}$ does not obstruct the argumentation.

Item 3 follows from the calculation:

$$\begin{aligned}\mathcal{F} G^{\pm}_{\varepsilon,em} &\overset{(i)}{=} \mathcal{F} \lim_{\sigma\to 0} G^{\pm}_{\varepsilon,\sigma,em} && (i) \text{ by Item 2 of Lemma 4.4.10} \\ &\overset{(ii)}{=} \lim_{\sigma\to 0} \mathcal{F} G^{\pm}_{\varepsilon,\sigma,em} && (ii) \text{ by the definition of } \mathcal{F} \text{ via Def. 4.2.3} \\ &\overset{(iii)}{=} \lim_{\sigma\to 0} G^{\pm}_{\varepsilon,\sigma,pt} && (iii) \text{ by Item 1} \\ &\overset{(iv)}{=} G^{\pm}_{\varepsilon,pt}\,. && (iv) \text{ by Item 2}\,.\end{aligned}$$

Item 4 is a consequence of the dominated convergence theorem. Given $f \in S(\mathbb{R}^4, \mathbb{C})$, we find

$$\lim_{\varepsilon\to 0} \left|G^{\pm}_{\varepsilon,pt}[f] - G^{\pm}_{pt}[f]\right| \le \lim_{\varepsilon\to 0} \int_{\mathbb{R}} dt \int_{S_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{x}})\Theta(\pm t)\left|e^{-\varepsilon|t|} - 1\right| \frac{|t|}{4\pi} \left|f(t, |t|\hat{\boldsymbol{x}})\right|$$

$$\leq \lim_{\varepsilon\to 0}\int_{\mathbb{R}} dt\left|e^{-\varepsilon|t|}-1\right|4\pi\frac{|t|}{4\pi}(1+|t|^3)^{-1}\sup_{(\tilde{t},\tilde{\boldsymbol{x}})\in\mathbb{R}^4}\left|(1+|\tilde{t}|^3)f(\tilde{t},\tilde{\boldsymbol{x}})\right|$$
$$\leq C\left(\sum_{\boldsymbol{\beta}\in\mathbb{N}_0^4,|\boldsymbol{\beta}|\leq 3}\|f\|_{\infty,\mathbf{0},\boldsymbol{\beta}}\right)\lim_{\varepsilon\to 0}\int_{\mathbb{R}} dt\left|e^{-\varepsilon|t|}-1\right|\frac{|t|}{1+|t|^3}$$
$$\overset{(i)}{=} 0$$

with equality (i) since $t\mapsto 2\frac{|t|}{1+|t|^3}$ is integrable and $e^{-\varepsilon|t|}-1$ converges, for $\varepsilon\to 0$, pointwisely to 0.

Finally, it remains to show that $G^{\pm}_{pt}$ fulfills the property of the Green's function on $\mathcal{S}'$:

Corollary 4.4.12

The equation $\Box G^{\pm}_{pt}=\delta$ holds true on $\mathcal{S}'$.

Proof

Given $f\in\mathcal{S}$, we calculate

$$\begin{aligned}
\Box G^{\pm}_{pt}[f] &= G^{\pm}_{pt}[\Box f]\\
&\overset{(i)}{=} \lim_{\varepsilon\to 0} G^{\pm}_{\varepsilon,pt}[\Box f] && (i)\text{ by Proposition 4.4.11 Item 4}\\
&\overset{(ii)}{=} \lim_{\varepsilon\to 0} \mathcal{F}G^{\pm}_{\varepsilon,em}[\Box f] && (ii)\text{ by Proposition 4.4.11 Item 3}\\
&= \lim_{\varepsilon\to 0} G^{\pm}_{\varepsilon,em}[\mathcal{F}\Box f]\\
&\overset{(iii)}{=} \lim_{\varepsilon\to 0} G^{\pm}_{\varepsilon,em}[m_{ph,\Box}\mathcal{F}f] && (iii)\text{ by Proposition 4.2.21}\\
&\overset{(iv)}{=} \lim_{\varepsilon\to 0}\int_{\mathbb{R}^4} d\omega d^3k\,\frac{(2\pi)^{-4/2}}{-(\omega\pm \mathrm{i}\varepsilon)^2+\boldsymbol{k}^2}m_{ph,\Box}(\omega,\boldsymbol{k})(\mathcal{F}f)(\omega,\boldsymbol{k}) && (iv)\text{ by Lemma 4.4.10 Item 2}\\
&= \lim_{\varepsilon\to 0}(2\pi)^{-4/2}\int_{\mathbb{R}^4} d\omega d^3k\,\frac{-\omega^2+\boldsymbol{k}^2}{-(\omega\pm \mathrm{i}\varepsilon)^2+\boldsymbol{k}^2}(\mathcal{F}f)(\omega,\boldsymbol{k})\\
&\overset{(v)}{=} (2\pi)^{-4/2}\int_{\mathbb{R}^4} d\omega d^3k\,(\mathcal{F}f)(\omega,\boldsymbol{k})\\
&= {}_{inv}\mathcal{F}(\mathcal{F}f)(0,\mathbf{0})\\
&= f(0,\mathbf{0})\\
&= \delta[f]\,.
\end{aligned}$$

Step (v) is justified by the dominated convergence theorem, since $(\omega,\boldsymbol{k})\mapsto\frac{-\omega^2+\boldsymbol{k}^2}{-(\omega\pm\mathrm{i}\varepsilon)^2+\boldsymbol{k}^2}$ is bounded and $\mathcal{F}f$ lies inside $S(\mathbb{R}^4,\mathbb{C})$.

4.4.5 Proof of Properties

With all these preparations, we are in a position to prove Theorem 4.4.3:

Proof Theorem 4.4.3

Item 1 follows from Item 4 of Proposition 4.4.11, since by definition $G^{\text{adv/ret}}=G^{\mp}_{tp}$.

Item 2 is directly a consequence of Corollary 4.4.12, i.e., $\Box G^{\text{adv/ret}}=\Box G^{\mp}_{tp}=\delta$.

Item 3 holds true by Theorem 4.4.8 since the restriction of $G^{\text{adv/ret}}$ to C_c^∞ is equal to $G^{\mp}_{\mathcal{HL},em}$ by definition.

Item 4 follows by the following line of argumentation:

$$\begin{aligned}
\mathcal{F}G^{\text{adv/ret}} &= \mathcal{F}G^{\mp}_{tp} \\
&\overset{(i)}{=} \mathcal{F}\lim_{\varepsilon\to 0} G^{\mp}_{\varepsilon,tp} && (i) \text{ by Item 4 in Proposition 4.4.11} \\
&\overset{(ii)}{=} \lim_{\varepsilon\to 0} \mathcal{F}G^{\mp}_{\varepsilon,tp} && (ii) \text{ by Theorem 4.2.20} \\
&\overset{(iii)}{=} \lim_{\varepsilon\to 0} \mathcal{F}\mathcal{F}G^{\mp}_{\varepsilon,mp}\,. && (iii) \text{ by Item 3 in Proposition 4.4.11}
\end{aligned}$$

Furthermore, for $f \in S(\mathbb{R}^4, \mathbb{C})$, we find:

$$\begin{aligned}
\mathcal{F}\mathcal{F}G^{\mp}_{\varepsilon,mp}[f] &= G^{\mp}_{\varepsilon,mp}[(\omega,\boldsymbol{k}) \mapsto (\mathcal{F}\mathcal{F}f)(\omega,\boldsymbol{k})] \\
&\overset{(iv)}{=} G^{\mp}_{\varepsilon,mp}[(\omega,\boldsymbol{k}) \mapsto ({}_{inv}\mathcal{F}\mathcal{F}f)(-\omega,-\boldsymbol{k})] \\
&= G^{\mp}_{\varepsilon,mp}[(\omega,\boldsymbol{k}) \mapsto (f)(-\omega,-\boldsymbol{k})] \\
&= G^{\pm}_{\varepsilon,mp}[(\omega,\boldsymbol{k}) \mapsto (f)(\omega,\boldsymbol{k})] \\
&\overset{(v)}{=} G^{\pm}_{\varepsilon,mp}[f]\,.
\end{aligned}$$

Step (iv) is due to Definition/Theorem 4.2.10 of the Fourier transform. Additionally, (v) holds by the definition of $G^{\pm}_{\varepsilon,mp}$.

Item 5 can be read directly form Def. 4.4.1.

Chapter 5

Sources and Solutions

We generalize the notion of convolution of distributions and test functions to a larger setting. By formalizing the source of the massless wave equation as a distribution and using the advanced and retarded Green's functions from the previous chapter, we construct solutions along the generalization of the ideas of Section 4.1.

Section Summaries:

5.1 Irregular Sources and their Solutions: With respect to the singularities in the sources, it is necessary for our interest to extend the framework of Section 4.1. In Section 2.2 we have motivated the generalization of convolution to the product of a distribution with a compact support or the product of a finite set of distributions with strictly compatible support and will provide the rigorous justification for the statements.

5.2 Particle Source: This section deals with the proofs for the statements in Section 2.2.3. These state that the sources in Theory 5 of potentials originating from particle trajectories can be understood as generalized functions in the context of distributions on spacetime. Moreover, we will confirm properties such as the compatibility of their supports with those of advanced and retarded Green's functions.

5.3 Particle sourced Solutions: We substantiate the assertions made in Section 2.2.3 about the convolution of Green's functions and particle sources.

5.1 Irregular Sources and their Solutions

In section Section 4.1 we have established a solution theory of the linear partial differential equation $\partial\phi = \rho$ with $\partial \in \mathfrak{D}$ and an inhomogeneity of high local and global regularity, i.e. $\rho \in \mathbb{C}_c^\infty$. This is not sufficient in the case of our application:

Point Charge: Solution

In view of the previous discussions, we would like to describe the charge distribution of a point particle on a trajectory in the light of linear forms. We assume that its trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^N, t \mapsto \mathbf{r}(t)$ is smooth, i.e., $\mathbf{r}$ lies in $C^\infty(\mathbb{R}, \mathbb{R}^3)$, and set

$$\rho(t, \mathbf{x}) = q\,\delta\big(\mathbf{x} - \mathbf{r}(t)\big)$$

understood in terms of $\rho \in \mathsf{LF}[C_c^\infty(\mathbb{R} \times \mathbb{R}^3, \mathbb{C}), \mathbb{C}]$. In our theory, the potential is assumed to be a solution of a PDO $\partial \in \mathfrak{D}$ and we assume that we have access to a corresponding Green's function G.

Therefore, we want to extend these techniques to the situation where ρ is less regular. Since a solution of the PDE, if equipped with a Green's function $G \in \mathscr{D}'$, is given by $G * \rho$ if $\rho \in C_c^\infty$, two questions arise:

1. In which cases can the convolution be generalized so that the product of G and ρ is well-defined?
2. Can we still justify the manipulations that show that $G * \rho$ is actually a solution?

The answer to these two questions is complicated, and we will restrict ourselves to techniques that depend on ρ being within a subset $\mathcal{D}'$ dependent on G, characterized by considering the support of the two distributions:

Definition 5.1.1 Support of Distributions

Given $u \in \mathcal{D}'(\mathbb{R}^N)$, its support is the closed set which is defined by:

$$\mathrm{sp}(u) := \left\{ \boldsymbol{x} \in \mathbb{R}^N \middle| \exists \epsilon > 0 \text{ such that } \forall f \in C_c^\infty \text{ with } \mathrm{sp}(f) \subset \mathsf{B}_\epsilon(\boldsymbol{x}) \text{ one gets } u(f) = 0 \right\}^{\complement} .$$

Furthermore, given a closed subset $A \subset \mathbb{R}^N$, we denote set of on A supported distributions by $\mathcal{D}'(A)$.

These generalizations are all based on extensions of the following lemma:

Lemma 5.1.2 Support of Convolution Products

Given $u \in \mathcal{D}'$ and $f \in \mathcal{D}$, then $\mathrm{sp}(u * f) \subset \mathrm{sp}(u) + \mathrm{sp}(f)$

Proof

[HI83, Theorem 4.1.1]

Moreover, these methods lead to a notion of convolution that is continuous in a certain sense and allows us to discuss the dependence of solutions on the change of inhomogeneity. The methods are extensions of the ideas presented in [HI83, Section 4.2], which are complemented by the proof of sequential continuity of the convolutions. This leads to Definition/Theorem 5.1.13, which depends on a condition for the support of the distributions involved in the convolution, which we call strict compatibility. For compact and conic sets, the handling of this notion becomes particularly simple.

5.1.1 Convolution via Compact Support

Definition 5.1.3 Distributions of Compact Support

We denote with $\mathcal{E}'(\mathbb{R}^N) := \{u \in \mathcal{D}' | \, \mathrm{sp}(u) \text{ is compact}\} \subset \mathcal{D}'(\mathbb{R}^N)$ the set of compactly supported distributions.

Similar to the tempered distributions, $\mathcal{E}'$ can be considered as linear forms on the larger set $C^\infty(\mathbb{R}^N, \mathbb{C})$ of functions that are continuous with respect to a topology on $C^\infty(\mathbb{R}^N, \mathbb{C})$ so that the inclusion of $\mathcal{D}$ is continuous.

Definition Proposition 5.1.4

We define the topological space $\mathcal{E}(\mathbb{R}^N) := \left(C^\infty(\mathbb{R}^N, \mathbb{C}), \tau_{C^\infty} \right)$ via building its topology τ_C^∞ out of the family of semi-norms $(\| \cdot \|_{\mathsf{K}_n, \infty, \boldsymbol{\alpha}})_{n \in \mathbb{N}, \boldsymbol{\alpha} \in \mathbb{N}_0^N}$ defined in Def. 4.1.19. This is a Fréchet space.

$\mathcal{E}'(\mathbb{R}^N)$ is, as an topological space, equal $\left\{ u \in \mathsf{LF}\left[\mathcal{E}, (\mathbb{C}, |\cdot|)\right] \middle| u \text{ continous} \right\}$ covered with with its weak topology. Further, the continuity of $u \in \mathsf{LF}\left[\mathcal{E}, (\mathbb{C}, |\cdot|]\right]$ is equivalent to the existence of $C > 0$, $a \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that, for all $f \in \mathcal{E}$, the following bound holds true:

$$|u[f]| \leq C \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq a} \|f\|_{\mathsf{K}_n, \infty, \boldsymbol{\alpha}} .$$

Proof

[HI83, Theorem 2.3.1] and [RSI80, Theorem V.2]

Similar properties apply as for the convolutions of $\mathscr{D}'$ and $\mathscr{D}$ in Proposition 4.1.23 and Lemma 5.1.2:

Definition Lemma 5.1.5 Convolution of $\mathscr{E}'$ and Testfunction

Given $u \in \mathscr{E}'$, $f \in \mathscr{D}$ and $g \in \mathscr{E}$, then the following statements are true:

1. $u * f \in \mathscr{D}$.
2. $u * g := (\boldsymbol{x} \mapsto u[T_{\boldsymbol{x}}\widetilde{g}]) \in \mathscr{E}$.
3. For any $\boldsymbol{\alpha} \in \mathbb{N}_0^N$, $\partial^{\boldsymbol{\alpha}} u * g = u * (\partial^{\boldsymbol{\alpha}} g)$ holds true.
4. $\operatorname{sp}(u * g) \subset \operatorname{sp}(u) + \operatorname{sp}(g)$.

Proof

Item 1 holds true, since $\mathscr{E}'$ is a subset of $\mathscr{D}'$ and by Proposition 4.1.23 in combination with Lemma 5.1.2.

Item 2 and **Item 3** are proven by a slight alternation of the proof in [HI83, Theorem 4.1.1]. We regard the function $(\boldsymbol{x}, \boldsymbol{y}) \mapsto g(\boldsymbol{x} - \boldsymbol{y})$ and, given $|h| \leq 1$ and $i \in 1, \ldots, N$, the vector $h\boldsymbol{e_i}$. Due to Taylor's theorem, we find

$$g(\boldsymbol{x} + h\boldsymbol{e_i} - \boldsymbol{y}) = g(\boldsymbol{x} - \boldsymbol{y}) + h\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} g(\boldsymbol{x} - \boldsymbol{y}) + h^2 \int_0^1 dt\,(1-t)\partial_{\boldsymbol{x}}^{2\boldsymbol{e}_i} g(\boldsymbol{x} + t\boldsymbol{e_i} - \boldsymbol{y})\,.$$

For fixed $\boldsymbol{x} \in \mathbb{R}^N$, this leads to

$$\begin{aligned}
\partial^{\boldsymbol{e}_i} u * g(\boldsymbol{x}) &= \partial^{\boldsymbol{e}_i} u[\boldsymbol{y} \mapsto g(\boldsymbol{x} - \boldsymbol{y})] \\
&= \lim_{h \to 0} \frac{u[\boldsymbol{y} \mapsto g(\boldsymbol{x} + h\boldsymbol{e_i} - \boldsymbol{y})] - u[\boldsymbol{y} \mapsto g(\boldsymbol{x} - \boldsymbol{y})]}{h} \\
&= \lim_{h \to 0} u\left[\boldsymbol{y} \mapsto \frac{g(\boldsymbol{x} + h\boldsymbol{e_i} - \boldsymbol{y}) - g(\boldsymbol{x} - \boldsymbol{y})}{h}\right] \\
&= \lim_{h \to 0} u\left[\boldsymbol{y} \mapsto \partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} g(\boldsymbol{x} - \boldsymbol{y}) + h \int_0^1 dt\,(1-t)\partial_{\boldsymbol{x}}^{2\boldsymbol{e}_i} g(\boldsymbol{x} + t\,h\boldsymbol{e_i} - \boldsymbol{y})\right] \\
&= u[\boldsymbol{y} \mapsto \partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} g(\boldsymbol{x} - \boldsymbol{y})] + \lim_{h \to 0} h\,u\left[\boldsymbol{y} \mapsto \int_0^1 dt\,(1-t)\partial_{\boldsymbol{x}}^{2\boldsymbol{e}_i} g(\boldsymbol{x} + t\,h\boldsymbol{e_i} - \boldsymbol{y})\right]\,.
\end{aligned}$$

The first term can either be identified with $u * (\partial^{\boldsymbol{e}_i} g)(\boldsymbol{x})$ or $(\partial^{\boldsymbol{e}_i} u) * g(\boldsymbol{x})$. Due to the continuity of u and its characterization in Definition/Proposition 5.1.4, there exists $C > 0$, $a \in \mathbb{N}_0$ and $n \in \mathbb{N}$. This allows for the following estimate of the second term

$$\begin{aligned}
&\left| u\left[\boldsymbol{y} \mapsto \int_0^1 dt\,(1-t)\partial_{\boldsymbol{x}}^{2\boldsymbol{e}_i} g(\boldsymbol{x} + t\boldsymbol{e_i} - \boldsymbol{y})\right]\right| \\
&\qquad \leq C \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq a} \left\| \boldsymbol{y} \mapsto \int_0^1 dt\,(1-t)\partial_{\boldsymbol{x}}^{2\boldsymbol{e}_i} g(\boldsymbol{x} + t\,h\boldsymbol{e_i} - \boldsymbol{y})\right\|_{\mathsf{K}_n, \infty, \boldsymbol{\alpha}} \\
&\qquad \leq C \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq a} \|g\|_{\{x\} + [0,1] \cdot \{\boldsymbol{e}_i\} - \mathsf{K}_n, \infty, \boldsymbol{\alpha} + 2\boldsymbol{e}_i} \\
&\qquad < \infty
\end{aligned}$$

since $\{x\} + [0,1] \cdot \{\boldsymbol{e}_i\} - \mathsf{K}_n$ is compact. Thus, the second term vanishes which results in the statement.

Item 4 is a straight forward adaption of the proof of [HI83, Theorem 4.1.2].

The convolution of certain kinds of distributions with certain kinds of test functions fulfills certain types of sequential continuity:

Proposition 5.1.6 Sequential Continuity of Convolution I

For $i \to \infty$, the following limits hold true:

1. $u, (u_i)_{n\in\mathbb{N}}$ in $\mathscr{D}'$, $f, (f_i)_{n\in\mathbb{N}}$ in $\mathscr{D}$ with $u_i \xrightarrow{\mathscr{D}'} u$, $f_i \xrightarrow{\mathscr{D}} f$ implies $u_i * f_i \xrightarrow{\mathscr{E}} u * f$.
2. $u, (u_i)_{n\in\mathbb{N}}$ in $\mathscr{E}'$, $f, (f_i)_{n\in\mathbb{N}}$ in $\mathscr{D}$ with $u_i \xrightarrow{\mathscr{E}'} u$, $f_i \xrightarrow{\mathscr{D}} f$ implies $u_i * f_i \xrightarrow{\mathscr{D}} u * f$.
3. $u, (u_i)_{n\in\mathbb{N}}$ in $\mathscr{E}'$, $f, (f_i)_{n\in\mathbb{N}}$ in $\mathscr{E}$ with $u_i \xrightarrow{\mathscr{E}'} u$, $f_i \xrightarrow{\mathscr{E}} f$ implies $u_i * f_i \xrightarrow{\mathscr{E}} u * f$.

In addition to the characterization of the continuity of distributions in Theorem 4.1.20, the proof of these statements is based on the following property of convergent series on $\mathscr{D}$:

Lemma 5.1.7 Convergent Series in $\mathscr{D}$, $\mathscr{D}'$ and $\mathscr{E}'$

Let $(f_i)_{i\in\mathbb{N}}$ in $\mathscr{D}$, $(u_i)_{i\in\mathbb{N}}$ in $\mathscr{D}'$ and $(v_i)_{i\in\mathbb{N}}$ in $\mathscr{E}'$ be convergent series. The following statements are true:

1. There exists compact $\mathsf{K} \subset \mathbb{R}^N$, such that, for all $i \in \mathbb{N}$, the set $\mathrm{sp}(f_i)$ lies in K.
2. For all $n \in \mathbb{N}$, there exists $C_n \geq 0$ and $a_n \in \mathbb{N}$ such that, for all $f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$ and $i \in \mathbb{N}$, the following bound holds true:
$$|u_i[f]| \leq C_n \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N, |\boldsymbol{\alpha}|\leq a_n} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} .$$
3. There exists compact $\mathsf{K} \subset \mathbb{R}^N$, $C \geq 0$ and $a \in \mathbb{N}$ such that, for all $f \in C^\infty(\mathbb{R}^N, \mathbb{C})$ and $i \in \mathbb{N}$, the following bound holds true:
$$|u_i[f]| \leq C \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N, |\boldsymbol{\alpha}|\leq a} \|f\|_{\mathsf{K},\infty,\boldsymbol{\alpha}} .$$

Proof

Item 1 is a consequence of [RSI80, Theorem V.17].

Item 2 follows from a version the uniform boundedness principle, i.e. [RSI80, Theorem V.7]. We follow the ideas used in the proof of [HI83, Theorem 2.1.8] by noticing that, for given $n \in \mathbb{N}$, the restriction $u_i|_{C_c^\infty(\tilde{\mathsf{K}},\mathbb{C})}$ maps the Fréchet space $\big(C_c^\infty(\mathsf{K}_n, \mathbb{C}), (\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}\big)$ continuously to the Banach space $(\mathbb{C}, |\cdot|)$. Due to the weak convergence of u_i to u, for all $f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$, the set $\big\{|u_i[f]| \,\big|\, i \in \mathbb{N}\big\}$ is bounded. Then, [RSI80, Theorem V.7] implies that there exist $C_n \geq 0$ and $a_n \in \mathbb{N}$ such that, for all $f \in C_c^\infty(\mathsf{K}_n, \mathbb{C})$, we find
$$\Big|u_i|_{C_c^\infty(\tilde{\mathsf{K}},\mathbb{C})}[f]\Big| \leq C_n \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N, |\boldsymbol{\alpha}|\leq a_n} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}}$$
and thus the statement.

Item 3 holds due to the fact, that u_i maps the Féchet space $\big(C^\infty(\mathbb{R}^N, \mathbb{C}), (\|\cdot\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}})_{n\in\mathbb{N},\boldsymbol{\alpha}\in\mathbb{N}_0^N}\big)$ continuously to the Banach space $(\mathbb{C}, |\cdot|)$. The weak-convergence of u_i to u implies, for given $f \in C^\infty(\mathbb{R}^N, \mathbb{C})$, the boundedness of the set $\big\{|u_i[f]| \,\big|\, i \in \mathbb{N}\big\}$. Then, [RSI80, Theorem V.7] implies, that there exist $n \in \mathbb{N}$, $C \geq 0$ and $a \in \mathbb{N}$ such that, for all $f \in C^\infty(\mathbb{R}^N, \mathbb{C})$, we find
$$|u_i[f]| \leq C \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^N, |\boldsymbol{\alpha}|\leq a} \|f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}}$$

and thus the statement.

Proof Proposition 5.1.6

The proof of all these statements follows the same strategy: First, we show the pointwise convergence of the convolutions according to the pattern of [HI83, Theorem 2.1.8]. This is lifted by the Arzelà-Ascoli theorem to the type of convergence indicated in each case.

Item 1: Let us fix $i \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ and regard $\boldsymbol{x} \in \mathbb{R}^N$. First, we notice that there exists $n \in \mathbb{N}$ such that $\{x\} - \mathrm{sp}(f_i) \subset \mathsf{K}_n$ leading to

$$\begin{aligned}
&\lim_{i\to\infty} \left|\partial^{\boldsymbol{\alpha}}\big(u_i * f_i(\boldsymbol{x}) - u * f(\boldsymbol{x})\big)\right| \\
&\overset{(i)}{=} \lim_{i\to\infty} \left|\big(u_i * (\partial^{\boldsymbol{\alpha}} f_i)(\boldsymbol{x}) - u * (\partial^{\boldsymbol{\alpha}} f) = (\boldsymbol{x})\big)\right| && (i) \text{ by Proposition 4.1.23} \\
&\leq \lim_{i\to\infty} \Big(|u_i * \partial^{\boldsymbol{\alpha}}(f_i - f)(\boldsymbol{x})| + |(u_i - u) * (\partial^{\boldsymbol{\alpha}} f)(\boldsymbol{x})|\Big) \\
&\overset{(ii)}{=} \lim_{i\to\infty} \Big(\big|u_i\big[\partial^{\boldsymbol{\alpha}}(f_i - f)(\boldsymbol{x} - \boldsymbol{\cdot})\big]\big| + \big|(u_i - u)\big[\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x} - \boldsymbol{\cdot})\big]\big|\Big) && (ii) \text{ by Def./Lemma 4.1.22} \\
&\overset{(iii)}{\leq} \lim_{i\to\infty} C_n \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq b_n} \|\partial^{\boldsymbol{\alpha}}(f_i - f)\|_{\infty,\boldsymbol{\beta}} \\
&\overset{(iv)}{=} 0 && (iv) \text{ since } f_i \xrightarrow{\mathscr{D}} f
\end{aligned}$$

with (iii) due to $u_i \xrightarrow{\mathscr{D}'} u$ and Item 2 of Lemma 5.1.7.

To show the convergence in $\mathscr{E}$, we regard $n \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$. Finding, for all $j = 1, \ldots, N$, a uniform bound of the partial derivative $\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} u_i * f_i$ of $\partial^{\boldsymbol{\alpha}} u_i * f_i$ on $\mathsf{K}_n = [-n, n]^N$ implies the equicontinuity of $\partial^{\boldsymbol{\alpha}} u_i * f_i$. This, by Arzelà–Ascoli's theorem, results in their uniform convergence on K_n.
Due to $f_i \xrightarrow{\mathscr{D}} f$, there exists, by Item 1 of Lemma 5.1.7, a compact $\mathsf{K} \subset \mathbb{R}^N$ such that $\mathrm{sp}(f_i)$ lies in K. Further, due to the compactness of both K_n and K, one can find $m \in \mathbb{N}$, such that $\mathsf{K} - \mathsf{K}_n$ lies in K_m. We will apply this in step (i) of the following calculation:

$$\begin{aligned}
&\left\|\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} u_i * f_i\right\|_{\mathsf{K}_n,\infty} \\
&= \sup_{\boldsymbol{x}\in\mathsf{K}_n} \partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} u_i * f_i(\boldsymbol{x}) \\
&= \sup_{\boldsymbol{x}\in\mathsf{K}_n} \left|u_i[\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} f_i(\boldsymbol{x} - \boldsymbol{\cdot})]\right| \\
&\overset{(i)}{\leq} C_m \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq b_m} \left\|\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} f_i(\boldsymbol{x} - \boldsymbol{\cdot})\right\|_{\mathsf{K}_m,\infty,\boldsymbol{\beta}} && (i) \text{ by Item 2 of Lemma 5.1.7} \\
&= C_m \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq b_m} \|f_i\|_{\mathsf{K}_m,\infty,\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{e}_j} \\
&\overset{(ii)}{<} \infty && (ii) \text{ by } f_i \xrightarrow{\mathscr{D}} f\,.
\end{aligned}$$

Thus, by the pointwise convergence and the equicontinuity, we find

$$\lim_{i\to\infty} \|u_i * f_i - u * f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} = \lim_{i\to\infty} \|\partial^{\boldsymbol{\alpha}} u_i * f_i - \partial^{\boldsymbol{\alpha}} u * f\|_{\mathsf{K}_n,\infty} = 0$$

which implies, due to the arbitrariness of $n \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$, the statement $u_i * f_i \xrightarrow{\mathscr{E}} u * f$.

Item 2 is a corollary of Item 1 and Lemma 5.1.2.

Item 3: We argue in a similar way as in Item 1. Let us fix $i \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ and regard $\boldsymbol{x} \in \mathbb{R}^N$. By Item 3 of Lemma 5.1.7, there exists compact $\mathsf{K} \subset \mathbb{R}^N$, such that, for all $i \in\in \mathbb{N}$, the support of u_i is a subset of K . We pick $n \in \mathbb{N}$, such that $\{\boldsymbol{x}\} - \mathsf{K} \subset \mathsf{K}_n$ and argue:

$$\lim_{i\to\infty} \left|\partial^{\boldsymbol{\alpha}}\big(u_i * f_i(\boldsymbol{x}) - u * f(\boldsymbol{x})\big)\right|$$

$$\overset{(i)}{=} \lim_{i\to\infty} \left|\left(u_i * (\partial^{\boldsymbol{\alpha}} f_i)(\boldsymbol{x}) - u * (\partial^{\boldsymbol{\alpha}} f) = (\boldsymbol{x})\right)\right| \qquad (i) \text{ by Definition/Lemma 5.1.5}$$

$$\leq \lim_{i\to\infty} \Big(|u_i * \partial^{\boldsymbol{\alpha}}(f_i - f)(\boldsymbol{x})| + |(u_i - u) * (\partial^{\boldsymbol{\alpha}} f)(\boldsymbol{x})|\Big)$$

$$\overset{(i)}{=} \lim_{i\to\infty} \Big(\big|u_i\big[\partial^{\boldsymbol{\alpha}}(f_i - f)(\boldsymbol{x} - \cdot)\big]\big| + \big|(u_i - u)\big[\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x} - \cdot)\big]\big|\Big)$$

$$\overset{(ii)}{\leq} \lim_{i\to\infty} C_n \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq b_n} \|\partial^{\boldsymbol{\alpha}}(f_i - f)\|_{\mathsf{K}_n,\infty,\boldsymbol{\beta}}$$

$$\overset{(iv)}{=} 0 \qquad (iv) \text{ since } f_i \xrightarrow{\mathscr{E}} f\,.$$

Step (ii) stems from $u_i \xrightarrow{\mathscr{E}'} u$ which implies $u_i \xrightarrow{\mathscr{D}'} u$ and, thus, by the application of Item 2 of Lemma 5.1.7.

Similar to the case in Item 1 we pick for fixed $n \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ a natural number m, such that $\mathsf{K} - \mathsf{K}_n \subset \mathsf{K}_m$. We use this fact in the following step (i) by referring once more to Item 2 of Lemma 5.1.7. We find

$$\left\|\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} u_i * f_i\right\|_{\mathsf{K}_n,\infty} = \sup_{\boldsymbol{x}\in\mathsf{K}_n} \partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} u_i * f_i(\boldsymbol{x})$$

$$= \sup_{\boldsymbol{x}\in\mathsf{K}_n} u_i[\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} f_i(\boldsymbol{x} - \cdot)]$$

$$\overset{(i)}{\leq} C_m \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq b_m} \left\|\partial^{\boldsymbol{\alpha}+\boldsymbol{e}_j} f_i\right\|_{\mathsf{K}_m,\infty,\boldsymbol{\beta}}$$

$$= C_m \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq b_m} \|\partial^{\boldsymbol{\alpha}} f_i\|_{\mathsf{K}_m,\infty,\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{e}_j}$$

$$\overset{(ii)}{<} \infty\,. \qquad (ii) \text{ by } f_i \xrightarrow{\mathscr{E}} f$$

Thus, by the pointwise convergence and the equicontinuity, we get

$$\lim_{i\to\infty} \|u_i * f_i - u * f\|_{\mathsf{K}_n,\infty,\boldsymbol{\alpha}} = \lim_{i\to\infty} \|\partial^{\boldsymbol{\alpha}} u_i * f_i - \partial^{\boldsymbol{\alpha}} u * f\|_{\mathsf{K}_n,\infty} = 0$$

which implies, since $n \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ were arbitrary, the statement $u_i * f_i \xrightarrow{\mathscr{E}} u * f$.

We are now ready to extend these results to the convolution of $\mathscr{E}'$ with $\mathscr{D}'$:

Definition Theorem 5.1.8 Convolution of $\mathscr{E}'$ and $\mathscr{D}'$

Given $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$, there exists a unique way to define their convolutions $u * v \in \mathscr{D}'$ and $v * u \in \mathscr{D}'$, such that

1. $u * v = v * u \in \mathscr{D}'$,
2. $\operatorname{sp}(u * v) \subset \operatorname{sp}(u) + \operatorname{sp}(v)$,
3. for all $\partial \in \boldsymbol{\partial}$, the equation $\partial(u * v) = (\partial u) * v = u * (\partial v)$ holds true and,
4. given $(u_i)_{i\in\mathbb{N}}$ in $\mathscr{E}'$ and $(v_i)_{i\in\mathbb{N}}$ in $\mathscr{D}'$ with $u_i \xrightarrow{\mathscr{E}'} u$, $v_i \xrightarrow{\mathscr{D}'} v$, for $i \to \infty$, we find $u_i * v_i \xrightarrow{\mathscr{D}'} u * v$.

Proof

The Definition of the convolution, for given $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$, with the notation established in Definition/Lemma 4.1.22, is provided by:

$$u * v : \mathscr{D} \to \mathbb{C}, f \mapsto u * v[f] := (u * v) * \tilde{f}(\mathbf{0}) := u * (v * \tilde{f})(\mathbf{0}) = u[\widetilde{v * \tilde{f}}] \text{ and}$$

$$v * u : \mathcal{D} \to \mathbb{C}, f \mapsto v * u[f] := (v * u) * \widetilde{f}(\mathbf{0}) := v * (u * \widetilde{f})(\mathbf{0}) = v[\widetilde{u * \widetilde{f}}] .$$

These are well defined, since,

- by Proposition 4.1.23, $v * \widetilde{f}$ lies in $\mathcal{E}$, which can according to Definition/Proposition 5.1.4 be paired wit u, and
- by Definition/Lemma 5.1.5, $u * \widetilde{f}$ lies in $\mathcal{D}$, such that it can be paired with u.

Item 1 and **Item 2** are proven in [HI83, Theorem 4.2.4].

Item 3 holds due to the definition of $u * v$ in combination with Proposition 4.1.23 and the definition of $v * u$ with Item 2 of Definition/Lemma 5.1.5 using further $u * v = v * u$.

Section 5.1.1: Given $f \in \mathcal{D}$, we apply the statements of Proposition 5.1.6:

- The convergence $u_i * v_i[f] \to u * v[f]$ holds true, since $u_i * v_i[f] = u_i * (v_i * \widetilde{f})(\mathbf{0})$ and, thus, by Proposition 5.1.6 of Item 1, $v_i * \widetilde{f} \xrightarrow{\mathcal{E}} v * \widetilde{f}$. This leads, by another application of Proposition 5.1.6 of Item 3, to $u_i * (v_i * \widetilde{f}) \xrightarrow{\mathcal{E}} u * (v * \widetilde{f})$ which in particular implies pointwise convergence.
- The convergence $v_i * u_i[f] \to v * u[f]$ holds true, since $v_i * u_i[f] = v_i * (u_i * \widetilde{f})(\mathbf{0})$ and, thus, by Proposition 5.1.6 of Item 2, $u_i * \widetilde{f} \xrightarrow{\mathcal{D}} u * \widetilde{f}$. This leads, by another application of Proposition 5.1.6 of Item 1, to $v_i * (u_i * \widetilde{f}) \xrightarrow{\mathcal{E}} v * (u * \widetilde{f})$ which in particular implies pointwise convergence.

5.1.2 Convolution via Strictly Compatible Support

Building on the ideas of [HI83] at the end of Section 4.2, we can further extend the notion of convolutions provided by Definition/Theorem 5.1.8 by the following definition:

Definition 5.1.9 Compatible Sets

We call $m \in \mathbb{N}$ subsets $A_1, \ldots, A_m$ of $\mathbb{R}^N$ compatible if the map $+|_{A_1 \times \cdots \times A_m} : A_1 \times \cdots \times A_m \to \mathbb{R}^N, (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \mapsto \boldsymbol{x}_1 + \ldots + \boldsymbol{x}_m$ is proper, i.e., the inverse image of $+|_{A_1 \times \cdots \times A_m}$ of compact sets is compact.

Definition | Theorem 5.1.10 Conv. of two Distr. on Compatible Supports

Given closed, compatible sets A_1, A_2 and $u \in \mathcal{D}'(A_1)$, $v \in \mathcal{D}'(A_2)$, there exists a unique way to define their convolution $u * v \in \mathcal{D}'$ and $v * u \in \mathcal{D}'$ such that

1. $u * v = v * u$,
2. for all $\partial \in \boldsymbol{\partial}$, the equation $\partial(u * v) = (\partial u) * v = u * (\partial v)$ holds true and,
3. given $(u_i)_{i \in \mathbb{N}}$ in $\mathcal{D}'(A_1)$ and $(v_i)_{i \in \mathbb{N}}$ in $\mathcal{D}'(A_2)$ with $u_i \xrightarrow{\mathcal{D}'} u$, $v_i \xrightarrow{\mathcal{D}'} v$, for $i \to \infty$, we find $u_i * v_i \xrightarrow{\mathcal{D}'} u * v$.

Proof

Item 1: Given $n \in \mathbb{N}$, we define the convolution of $u \in \mathcal{D}'(A_1)$ and $v \in \mathcal{D}'(A_2)$ restricted to $C_c^\infty(\mathsf{K}_n, \mathbb{R}^N)$,

- by choosing compact $\mathsf{K}_{n,A_1,A_2} \subset \mathbb{R}^N$, such that $(+|_{A_1 \times A_2})^{-1}(\operatorname{sp} f) \subset \mathsf{K}_{n,A_1,A_2} \times \mathsf{K}_{n,A_1,A_2}$ and
- by selecting $\chi_{n,A_1,A_2} \in \mathcal{D}$, such that $\chi_{n,A_1,A_2} = 1$ on some neighborhood of K_{n,A_1,A_2}.

Given $f \in C_c^\infty(\mathsf{K}_n, \mathbb{R}^N)$ we declare

$$u * v[f] := (\chi_{n,A_1,A_2} u) * (\chi_{n,A_1,A_2} v)[f]$$

where $\chi_{n,A_1,A_2} u$ and $\chi_{n,A_1,A_2} v$ are now distributions of compact support whose convolution is, according to Definition/Theorem 5.1.10, defined. Thus, $u * v$ is a continuous linear form on the domain $\big(C_c^\infty(\mathsf{K}_n, \mathbb{C}), \tau_{C_c^\infty(\mathsf{K}_n)}\big)$. If the definition is independent of the choices of K_{n,A_1,A_2} and χ_{n,A_1,A_2} the convolution is a well-defined a distribution.

We pick different K' and χ'[a]. According to the algorithm above, we find

$$\begin{aligned}\chi u * \chi v[f] - \chi' u * \chi' v[f] &= \chi u * \chi v[f] - \chi' u * \chi v[f] + \chi' u * \chi v[f] - \chi' u * \chi' v[f] \\ &= \Delta u * \chi v[f] - \chi' u * \Delta v[f]\,,\end{aligned}$$

with $\Delta = \chi - \chi'$, and, according to Definition/Theorem 5.1.8 of Item 2, this leads to:

$$\operatorname{sp}(\Delta u * \chi v) \subset \operatorname{sp}(\Delta u) + \operatorname{sp}(\chi v) \subset (\mathsf{K} \cap \mathsf{K}')^{\complement} \cap A_1 + A_2 \text{ since } \Delta|_{\mathsf{K}\cap\mathsf{K}'} = \chi|_{\mathsf{K}\cap\mathsf{K}'} - \chi'|_{\mathsf{K}\cap\mathsf{K}'} = 1 - 1\,.$$

Furthermore, $\big(+\,|_{A_1\times A_2}\big)^{-1}(\operatorname{sp} f)$ is a subset of $(\mathsf{K}\times\mathsf{K}) \cap (\mathsf{K}'\times\mathsf{K}') = (\mathsf{K}\cap\mathsf{K}') \times (\mathsf{K}\cap\mathsf{K}')$. This fact implies $\operatorname{sp}(f) \cap \operatorname{sp}(\Delta u * \chi v) = 0$ since any element in $(\mathsf{K}\cap\mathsf{K}')^{\complement} \cap A_1 + A_2$ can be written by the sum of a tuple $(\boldsymbol{x}, \boldsymbol{y}) \in A_1 \cap (\mathsf{K}\cap\mathsf{K}')^{\complement} \times A_2$. This sum cannot ly inside $\operatorname{sp}(f)$ which would imply $(\boldsymbol{x}, \boldsymbol{y}) \in \big(+\,|_{A_1\times A_2}\big)^{-1}(\operatorname{sp} f) \subset A_1 \cap (\mathsf{K}\cap\mathsf{K}') \times A_2$ reaching a contradiction. The same line of argumentation holds true for $\chi' u * \Delta v[f]$. This implies the independence of the definition of the involved choices.

Item 2 follows by the definition of these convolutions and Item 3 of Definition/Theorem 5.1.8.

Item 3 is a corollary of Section 5.1.1 of Definition/Theorem 5.1.8.

[a] We neglect all the subindices for clarity to the proof.

Definition/Theorem 5.1.10 can be successively applied to a finite set of distributions $u, v, w, \ldots$ to define $u * \big(v * (w * \ldots)\big)$ if their supports are compatible. However, showing the associativity and support properties of the product, if possible, seems quite cumbersome. Instead, we will derive these properties via sequential continuity from the properties of the convolution product on $C_c^\infty(\mathbb{R}^N, \mathbb{C})$. The following lemma shows that C_c^∞ is sequentially dense in a way that does not mess the support too much:

Lemma 5.1.11

Given $\varepsilon > 0$ and $u \in \mathscr{D}'$, there exists a series $(u_i^\varepsilon)_{i\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ such that both $u_i^\varepsilon \to u$ in $\mathscr{D}'$ and additionally $\operatorname{sp}(u_i^\varepsilon) \subset \operatorname{sp}(u) + \mathsf{B}_\varepsilon(\mathbf{0})$.

Proof

By [HI83, Theorem 4.1.5] and its proof.

We therefore tighten the condition for the support of the distributions accordingly by defining the term:

Definition 5.1.12 Strictly Compatible Sets

We call $m \in \mathbb{N}$ subsets $A_1, \ldots, A_m$ of $\mathbb{R}^N$ strictly compatible if there exists $\varepsilon > 0$ such that $A_1^{+\varepsilon}, \ldots, A_m^{+\varepsilon}$ are compatible, where $A_j^{+\varepsilon} := A_j + \mathsf{B}_\varepsilon(\mathbf{0})$.

With this notion, Lemma 5.1.11 and Definition/Theorem 5.1.10 we find directly:

Definition **Theorem 5.1.13** **Conv. of Distr. on Strictly Compatible Supp.**

Given $m \in \mathbb{N}$ strictly compatible subsets $A_1, \dots, A_m$ of $\mathbb{R}^N$ and for each $j = 1, \dots, m$ a distributions u_j in $\mathscr{D}'(A_j)$, then there exists a unique way to define their convolution product $u_1 * \dots * u_m \in \mathscr{D}'$ such that

1. its is associative and commutative,
2. $\mathrm{sp}(u_1 * \dots * u_m) \subset \mathrm{sp}(u_1) + \dots + \mathrm{sp}(u_m)$,
3. for all $\partial \in \boldsymbol{\partial}$, the equation $\partial(u_1 * \dots * u_m) = (\partial u_1) * u_2 * \dots * u_m = \dots = u_1 * \dots * u_{m-1} * (\partial u_m)$ holds true and,
4. given for all $j = 1, \dots, m$ a series $(u_{j,i})_{i \in \mathbb{N}}$ in $\mathscr{D}'(A_j)$ with $u_{j,i} \xrightarrow{\mathscr{D}'} u_j$ we find, for $i \to \infty$, that $u_{1,i} * \dots * u_{m,i} \xrightarrow{\mathscr{D}'} u_1 * \dots * u_m$ holds true.

The following observation shows that the case of a convolution of a distribution with a distribution with compact support is included in this framework:

Lemma 5.1.14

Given any subset $A \subset \mathbb{R}^N$ and compact $\mathsf{K} \subset \mathbb{R}^N$, then they are strictly compatible.

Proof

Given $\varepsilon > 0$, K' and $k, k' > 0$ such that, for all $\boldsymbol{y} \in \mathsf{K}$, the inequality $|\boldsymbol{y}| \leq k$ holds true, and, for all $\boldsymbol{z} \in \mathsf{K}'$, also $|\boldsymbol{z}| \leq k'$. If $\boldsymbol{x} \in A$ and $\boldsymbol{y} \in \mathsf{K} + \mathsf{B}_\varepsilon(\mathbf{0})$ such that $\boldsymbol{x} + \boldsymbol{y} \in \mathsf{K}'$, then we find

$$\|\boldsymbol{x}\| \leq \big|\|\boldsymbol{x} + \boldsymbol{y}\| - \|\boldsymbol{y}\|\big| \leq \|\boldsymbol{x} + \boldsymbol{y}\| + \|\boldsymbol{y}\| \leq k' + k + \varepsilon$$

and, thus, are A and K strictly compatible.

Further a strictly compatible finite family of sets can always be accompanied by a compact set while staying strictly compatible:

Corollary 5.1.15

Consider $m \in \mathbb{N}$ subsets $A_1, \dots, A_m$ of $\mathbb{R}^N$ which are strictly compatible and an additional compact set $\mathsf{K} \subset \mathbb{R}^N$. Then, also $A_1, \dots, A_m, \mathsf{K}$ are strictly compatible.

Proof

Let $\varepsilon > 0$ such that $A_j^{+\varepsilon} := A_j + \mathsf{B}_\varepsilon(\mathbf{0})$ are compatible and define $A^{+\varepsilon} := A_1^{+\varepsilon} + \dots + A_m^{+\varepsilon}$. Given compact $\mathsf{K}' \subset \mathbb{R}^N$, we observe

$$\begin{aligned}&\big(+\,|_{A_1^{+\varepsilon} \times \dots \times A_m^{+\varepsilon} \times \mathsf{K}^\varepsilon}\big)^{-1}(\mathsf{K}')\\ &\qquad = \big(+\,|_{A_1^{+\varepsilon} \times \dots \times A_m^{+\varepsilon}}\big)^{-1}\Big(\pi_1\Big(\big(+\,|_{A^{+\varepsilon} \times \mathsf{K}^\varepsilon}\big)^{-1}(\mathsf{K}')\Big)\Big) \times \pi_2\Big(\big(+\,|_{A^{+\varepsilon} \times \mathsf{K}^\varepsilon}\big)^{-1}(\mathsf{K}')\Big)\end{aligned}$$

and, thus, by Lemma 5.1.14, are the two sets

$$\pi_1\Big(\big(+\,|_{A^{+\varepsilon} \times \mathsf{K}^\varepsilon}\big)^{-1}(\mathsf{K}')\Big) \quad \text{and} \quad \pi_2\Big(\big(+\,|_{A^{+\varepsilon} \times \mathsf{K}^\varepsilon}\big)^{-1}(\mathsf{K}')\Big)$$

are compact. This implies, due to the compatibility of $A_1^{+\varepsilon} + \dots + A_m^{+\varepsilon}$, that further the set

$$\big(+\,|_{A_1^{+\varepsilon} \times \dots \times A_m^{+\varepsilon}}\big)^{-1}\Big(\pi_1\Big(\big(+\,|_{A^{+\varepsilon} \times \mathsf{K}^\varepsilon}\big)^{-1}(\mathsf{K}')\Big)\Big)$$

is compact and thus the statement of the lemma.

5.1.3 Conical Support

For our intended application, the supports of the occurring distributions have special types of supports that are conical. We use the following definition:

Definition 5.1.16 Closed Cones

We call a subset $\Gamma \in \mathbb{R}^N$ a closed cone at $\{0\} \in \mathbb{R}^N$ if Γ is closed and if additional, for all $\boldsymbol{x} \in \Gamma$ and $\lambda \in \mathbb{R}_{\geq 0}$, also $\lambda \boldsymbol{x} \in \Gamma$. This allows for the identification of closed cones at $\{0\}$ with the closed subsets of $\mathsf{S}_1(\mathbf{0}) := \{\boldsymbol{x} \in \mathbb{R}^N | |\boldsymbol{x}| = 1\}$ endowed with its subset topology in $\mathbb{R}^N$.

Given compact $\mathsf{K} \subset \mathbb{R}^N$ and a closed cone Γ at $\{0\}$, we call the set $\Gamma_{\mathsf{K}} := \Gamma + \mathsf{K}$ the closed-Γ cone at K. If $\mathsf{K} = \{\boldsymbol{x}\}$ for $\boldsymbol{x} \in \mathbb{R}^N$, we abbreviate $\Gamma_{\boldsymbol{x}} := \Gamma_{\{\boldsymbol{x}\}}$.

Closed cones have some nice properties when it comes to the compatibility of two cones, which we will now analyze. First of all, the compatibility of cones at $\mathbf{0}$ can be characterized in the following way:

Lemma 5.1.17 Compatibility of Two Closed Cones

Given two closed cones $\Gamma, \Gamma' \subset \mathbb{R}^N$ at $\{0\}$, then they are compatible if, and only if, $\Gamma \cap (-\Gamma') = \{\mathbf{0}\}$.

Proof

The statement of this lemma is equivalent to:

$$\Gamma \text{ and } \Gamma' \text{ are not compatible} \quad \Longleftrightarrow \quad \Gamma \cap (-\Gamma') \neq \{\mathbf{0}\}\,.$$

$\Longrightarrow$**:** Let Γ and Γ' be not compatible, then there exists $(\boldsymbol{x}_n, \boldsymbol{y}_n)_{n\in\mathbb{N}} \in (\Gamma \times \Gamma')^{\mathbb{N}}$ and $C > 0$, such that,

- for all $n \in \mathbb{N}$, $\|\boldsymbol{x}_n + \boldsymbol{y}_n\|$ is bounded by C, and
- either $\lim_{n\to\infty} \|\boldsymbol{x}_n\| = \infty$ or otherwise $\lim_{n\to\infty} \|\boldsymbol{y}_n\| = \infty$.

These imply both $\lim_{n\to\infty} \|\boldsymbol{x}_n\| = \infty$ and $\lim_{n\to\infty} \|\boldsymbol{y}_n\| = \infty$. $\big(\boldsymbol{x_n}/\|\boldsymbol{x}_n\|\big)_{n\in\mathbb{N}}$ possesses a convergent subsequence denoted by $\big(\boldsymbol{x_{n_i}}/\|\boldsymbol{x}_{n_i}\|\big)_{i\in\mathbb{N}}$. Its limit point, called $\hat{\boldsymbol{x}}$, lies due to the closedness of Γ itself in Γ. Furthermore, since

$$\left\| \frac{\boldsymbol{x_{n_i}}}{\|\boldsymbol{x}_{n_i}\|} + \frac{\boldsymbol{y_{n_i}}}{\|\boldsymbol{x}_{n_i}\|} \right\| \leq \frac{c}{\|\boldsymbol{x}_{n_i}\|}$$

and $\lim_{i\to\infty} \|\boldsymbol{x}_{n_i}\| = \infty$, the sequence $\big(\boldsymbol{y_{n_i}}/\|\boldsymbol{x}_{n_i}\|\big)_{i\in\mathbb{N}}$ limits $-\hat{\boldsymbol{x}}$ which is, again due to the closeness of Γ', an element of Γ'. Since clearly $\|\hat{\boldsymbol{x}}\| = 1$ holds true, we find $\Gamma \cap (-\Gamma') \neq \{\mathbf{0}\}$.

$\Longleftarrow$**:** Let $\boldsymbol{x} \in \Gamma \cap (-\Gamma') \setminus \{\mathbf{0}\}$, we find $(+|_{\Gamma\times\Gamma'})^{-1}(\{\mathbf{0}\}) \supset \{(\lambda\boldsymbol{x}, -\lambda\boldsymbol{x}) \in \Gamma \times \Gamma' | \lambda \geq 0\}$ which is not compact.

Further, with this characterization, we can see that if two closed cones are compatible at $\{0\}$, they are also strictly compatible:

Proposition 5.1.18 Strict Compatibility of Closed Cones

Given two closed cones $\Gamma, \Gamma' \subset \mathbb{R}^N$ with $\Gamma \cap (-\Gamma') = \{\mathbf{0}\}$ and compact $\mathsf{K}, \mathsf{K}' \subset \mathbb{R}^N$, then Γ_{K} and $\Gamma'_{\mathsf{K}'}$ are strictly compatible.

The following lemma will help us to prove the preceding proposition:

Lemma 5.1.19

Given a closed cone Γ at $\mathbf{0}$ and some compact set K, then there exists, for all $\varepsilon > 0$, another compact set $\tilde{\mathsf{K}}$, such that $\Gamma_{\mathsf{K}} \subset \Gamma^{\cdot\varepsilon} \cup \tilde{\mathsf{K}}$ where $\Gamma^{\cdot\varepsilon} := \mathbb{R}_{\geq 0} \cdot \big\{\boldsymbol{a} \in \mathsf{S}_1(\mathbf{0}) \big| \exists \boldsymbol{b} \in \Gamma \cap \mathsf{S}_1(\mathbf{0}), \text{such that } \|\boldsymbol{a}-\boldsymbol{b}\| \leq \varepsilon\big\}$.

Proof

Given $\varepsilon > 0$ and $k > 0$, such that, for all $\boldsymbol{x} \in \mathsf{K}$ $|\boldsymbol{x}| \leq k$, we find

$$\Gamma + \mathsf{K} = \big(\Gamma \cap \overline{\mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})} \cup \Gamma \cap \mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})^{\complement}\big) + \mathsf{K} = \big(\Gamma \cap \overline{\mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})} + \mathsf{K}\big) + \big(\Gamma \cap \mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})^{\complement} + \mathsf{K}\big)$$

and deduce from the compactness of $\overline{\mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})}$ that the set $\tilde{\mathsf{K}} := \Gamma \cap \overline{\mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})} + \mathsf{K}$ is compact too.

To show that $\Gamma \cap \mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})^{\complement} + \mathsf{K}$ is a subset of $\Gamma^{\cdot\varepsilon}$, we note that, for any given $c > 0$, the following equation holds true:

$$\Gamma^{\cdot\varepsilon} \cap \mathsf{S}_c(\mathbf{0}) = \big\{\boldsymbol{a} \in \mathsf{S}_c(\mathbf{0}) \big| \exists \boldsymbol{b} \in \Gamma \cap \mathsf{S}_c(\mathbf{0}), \text{such that } \|\boldsymbol{a}-\boldsymbol{b}\| \leq c \cdot \varepsilon\big\}\,. \tag{5.1}$$

Thus given $\boldsymbol{x} \in \Gamma \cap \mathsf{B}_{2k/\varepsilon+k}(\mathbf{0})^{\complement}$ and $\boldsymbol{y} \in \mathsf{K}$, we collect that

(i) $\|\boldsymbol{y}\| \leq k$,

(ii) $\|\boldsymbol{x}+\boldsymbol{y}\| \geq \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \geq 2k/\varepsilon + k - |\boldsymbol{y}| \geq 2k/\varepsilon$, and

(iii) $\big|\|\boldsymbol{x}-\boldsymbol{y}\| - \|\boldsymbol{x}\|\big| \leq \|\boldsymbol{y}\|$.

This leads to the following calculation:

$$\begin{aligned}
\left\|\boldsymbol{x}+\boldsymbol{y} - \tfrac{\|\boldsymbol{x}+\boldsymbol{y}\|}{\|\boldsymbol{x}\|}\boldsymbol{x}\right\| &\leq \left|1 - \tfrac{\|\boldsymbol{x}+\boldsymbol{y}\|}{\|\boldsymbol{x}\|}\right| \|\boldsymbol{x}\| + \|\boldsymbol{x}\| \\
&= \|\boldsymbol{x}+\boldsymbol{y}\| \left(\tfrac{1}{\|\boldsymbol{x}+\boldsymbol{y}\|}\big|\|\boldsymbol{x}-\boldsymbol{y}\| - \|\boldsymbol{x}\|\big| + \tfrac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}+\boldsymbol{y}\|}\right) \\
&\overset{(iii)}{\leq} \|\boldsymbol{x}+\boldsymbol{y}\| 2 \tfrac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}+\boldsymbol{y}\|} \\
&\overset{(i)}{\leq} \|\boldsymbol{x}+\boldsymbol{y}\| 2 \tfrac{k}{\|\boldsymbol{x}+\boldsymbol{y}\|} \\
&\overset{(ii)}{\leq} \|\boldsymbol{x}+\boldsymbol{y}\| 2 \tfrac{k}{2k/\varepsilon} \\
&= \|\boldsymbol{x}+\boldsymbol{y}\| \cdot \varepsilon\,.
\end{aligned}$$

Sine Γ is a cone, we find $\frac{\|\boldsymbol{x}+\boldsymbol{y}\|}{\|\boldsymbol{x}\|}\boldsymbol{x} \in \Gamma \cap \mathsf{S}_{\|\boldsymbol{x}+\boldsymbol{y}\|}(\mathbf{0})$. Thus, by Equation (5.1), we find $\boldsymbol{x}+\boldsymbol{y} \in \Gamma^{\cdot\varepsilon}$.

Proof Proposition 5.1.18

Step One: We show, that there exists $\varepsilon > 0$, such that $\Gamma^{\cdot\varepsilon}$ and $\Gamma'^{\cdot\varepsilon}$ are compatible. This is a consequence of the characterization of cones at $\mathbf{0}$ as closed subsets of $\mathsf{S}_1(\mathbf{0})$ and Lemma 5.1.17 about their compatibility. We find

$$\big(\Gamma \cap \mathsf{S}_1(\mathbf{0})\big) \cap \big(-\Gamma' \cap \mathsf{S}_1(\mathbf{0})\big) = \emptyset$$

and since $\mathsf{S}_1(\mathbf{0})$ is a compact, metric space with metric

$$d : \mathsf{S}_1(\mathbf{0}) \times \mathsf{S}_1(\mathbf{0}) \to \mathbb{R}_{\geq 0}, (\boldsymbol{a},\boldsymbol{b}) \mapsto d(\boldsymbol{a},\boldsymbol{b}) = \|\boldsymbol{a}-\boldsymbol{b}\|\,,$$

there exists open neighborhood $U_1, U_2 \subset \mathsf{S}_1(\mathbf{0})$ of $\Gamma \cap \mathsf{S}_1(\mathbf{0}), -\Gamma' \cap \mathsf{S}_1(\mathbf{0})$ and $\varepsilon > 0$ such that $U_1 \cap U_2 = \emptyset$, $d\big(U_1^{\complement}, \Gamma \cap \mathsf{S}_1(\mathbf{0})\big) > \varepsilon$ and $d\big(U_2^{\complement}, -\Gamma' \cap \mathsf{S}_1(\mathbf{0})\big) > \varepsilon$. Thus, we get both

- $\Gamma \cap \mathsf{S}_1(\mathbf{0}) \subset \bigcup_{\boldsymbol{b} \in \Gamma \cap \mathsf{S}_1(\mathbf{0})} \big\{\boldsymbol{a} \in \mathsf{S}_1(\mathbf{0}) \big| d(\boldsymbol{a},\boldsymbol{b}) \leq \varepsilon\big\} = \Gamma^{\cdot\varepsilon} \cap \mathsf{S}_1(\mathbf{0}) \subset U_1$ and

- $-\Gamma' \cap \mathsf{S}_1(\mathbf{0}) \subset \bigcup_{\boldsymbol{b} \in -\Gamma' \cap \mathsf{S}_1(\mathbf{0})} \{\boldsymbol{a} \in \mathsf{S}_1(\mathbf{0}) | d(\boldsymbol{a}, \boldsymbol{b}) \leq \varepsilon\} = -\Gamma'^{\cdot\varepsilon} \cap \mathsf{S}_1(\mathbf{0}) \subset U_2$

which leads to $\Gamma^{\cdot\varepsilon} \cap \mathsf{S}_1(\mathbf{0}) \cap -\Gamma'^{\cdot\varepsilon} \cap \mathsf{S}_1(\mathbf{0}) = \emptyset$ implying $\Gamma^{\cdot\varepsilon} \cap (-\Gamma'^{\cdot\varepsilon}) = \{\mathbf{0}\}$.

Step Two: We use Lemma 5.1.19 which shows, that there exists compact $\mathsf{K} \subset \mathbb{R}^N$, such that

- $(\Gamma_{\mathsf{K}})^{+\varepsilon} = \Gamma_{\mathsf{K}} + \mathsf{B}_\varepsilon(\mathbf{0}) = \Gamma + \mathsf{K} + \mathsf{B}_\varepsilon(\mathbf{0}) \subset \Gamma + \mathsf{K} + \overline{\mathsf{B}_\varepsilon(\mathbf{0})} \subset \Gamma^{\cdot\varepsilon} \cup \mathsf{K}$ and
- $(\Gamma'_{\mathsf{K}'})^{+\varepsilon} = \Gamma'_{\mathsf{K}'} + \mathsf{B}_\varepsilon(\mathbf{0}) = \Gamma' + \mathsf{K}' + \mathsf{B}_\varepsilon(\mathbf{0}) \subset \Gamma' + \mathsf{K}' + \overline{\mathsf{B}_\varepsilon(\mathbf{0})} \subset \Gamma'^{\cdot\varepsilon} \cup \mathsf{K}$.

Furthermore, given compact $\mathsf{K}' \subset \mathbb{R}^N$, we regard

$$\begin{aligned}(+|_{(\Gamma_{\mathsf{K}})^{+\varepsilon} \times (\Gamma'_{\mathsf{K}'})^{+\varepsilon}})^{-1}(\mathsf{K}') \\ &\subset (+|_{(\Gamma^{\cdot\varepsilon} \cup \mathsf{K}) \times (\Gamma'^{\cdot\varepsilon} \cup \mathsf{K})})^{-1}(\mathsf{K}') \\ &\subset (+|_{\Gamma^{\cdot\varepsilon} \times \Gamma'^{\cdot\varepsilon}})^{-1}(\mathsf{K}') \cup (+|_{\Gamma^{\cdot\varepsilon} \times \mathsf{K}})^{-1}(\mathsf{K}') \cup (+|_{\mathsf{K} \times \Gamma'^{\cdot\varepsilon}})^{-1}(\mathsf{K}') \cup (+|_{\mathsf{K} \times \mathsf{K}})^{-1}(\mathsf{K}')\end{aligned}$$

which is, by the first part and Lemma 5.1.14, the finite union of bounded sets in $\mathbb{R}^N$. Thus, its projection on $(\Gamma_{\mathsf{K}})^{+\varepsilon} \times (\Gamma'_{\mathsf{K}'})^{+\varepsilon}$ is compact in the subset topology.

5.2 Particle Source

The inhomogeneities we want to consider are supposed to represent a source given by a particle, either point-like or extended, moving along a fixed smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ whose velocity is bounded by $v_{\max} < 1$ away from the speed of light, i.e. $\|\dot{\mathbf{r}}(t)\| < v_{\max}$ for all $t \in \mathbb{R}$. Since we have developed the notion of distributions in Section 4.1.2 along the idea of formalizing point sources, we are not surprised that these inhomogeneities can be understood as distributions. We first formulate them in this framework and summarize their expected properties in the subsequence theorem:

Definition 5.2.1 Particle Source

Given some interval $I \subset \mathbb{R}$ and smooth $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ such that there exists $v_{\max} < 1$ and, for all $t \in \mathbb{R}$, the bound $\|\dot{\mathbf{r}}(t)\| < v_{\max}$ holds true. We define

$$\mathbb{1}_I \rho_{\mathbf{r},\delta} : S(\mathbb{R}^4, \mathbb{C}) \to \mathbb{C}, f \mapsto \int_{\mathbb{R}} dt\, q f(t, \mathbf{r}(t)) \mathbb{1}_I(t)$$

which we abbreviate by $\mathbb{1}_I(t)\rho_{\mathbf{r},\delta}(t, \boldsymbol{x}) = \mathbb{1}_I(t) q \delta\big(\boldsymbol{x} - \mathbf{r}(t)\big)$. For $I = \mathbb{R}$, we will denote $\rho_{\mathbf{r},\delta} := \mathbb{1}_{\mathbb{R}} \rho_{\mathbf{r},\delta}$.

For a particle shape $\varrho \in \mathbb{C}_c^\infty(\mathbb{R}^3, \mathbb{C})$, we define $\mathbb{1}_I \rho_{\mathbf{r},\varrho} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} : (t, \boldsymbol{x}) \mapsto \mathbb{1}_I(t) \rho_{\mathbf{r},\varrho}(t, \boldsymbol{x}) := \mathbb{1}_I(t) q \varrho\big(\boldsymbol{x} - \mathbf{r}(t)\big)$.

Furthermore, given $v_{\max} < 1$, we define the following closed cone at $\mathbf{0}$:

$$\Gamma^{v_{\max}} := \left\{(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^3 \big| |\boldsymbol{x}| \leq v_{\max} \cdot t\right\}.$$

Theorem 5.2.2 Properties of Particle Sources

Given the setting of Def. 5.2.1, then the following statements are true:

1. $\mathbb{1}_I \rho_{\mathbf{r},\delta}$ and $\mathbb{1}_I \rho_{\mathbf{r},\varrho}$ lie in $\mathcal{S}'$.
2. $\mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta}) = \mathrm{graph}(\mathbf{r}|_I) \subset \Gamma^{v_{\max}}_{(t_0, \mathbf{r}(t_0))}$ for all $t_0 \in \mathbb{R}$.
3. $\mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\varrho}) = \mathrm{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta}) + \{0\} \times \mathrm{sp}(\varrho) \subset \Gamma^{v_{\max}}_{\{(t_0, \mathbf{r}(t_0))\} + \{0\} \times \mathrm{sp}(\varrho)}$ for all $t_0 \in \mathbb{R}$.

Proof

Item 1 follows immediate by regarding for given $f \in S(\mathbb{R}^4, \mathbb{C})$ the estimate

$$
\begin{aligned}
|\mathbb{1}_I \rho_{\mathbf{r},\varrho}[f]| &\leq \int_{\mathbb{R}^4} dt d^3x\, q |f(t,\boldsymbol{x})| \big|\varrho\big(\boldsymbol{x} - \mathbf{r}(t)\big)\big| \mathbb{1}_I(t) \\
&\leq \int_{\mathbb{R}^4} dt d^3x\, q (1+t^2)^{-1}(1+|\boldsymbol{x}|^5)^{-1} \sup_{(\tilde{t},\tilde{\boldsymbol{x}})\in\mathbb{R}\times\mathbb{R}^3} \Big|(1+\tilde{t}^2)(1+|\tilde{\boldsymbol{x}}|^5) f(t,\boldsymbol{x})\Big| \sup_{\tilde{\boldsymbol{x}}\in\mathbb{R}^3} |\varrho(\tilde{\boldsymbol{x}})| \\
&\leq C \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^N, |\boldsymbol{\beta}|\leq 7} \|f\|_{\infty,\mathbf{0},\boldsymbol{\beta}}
\end{aligned}
$$

which implies $\mathbb{1}_I \rho_{\mathbf{r},\varrho} \in \mathcal{S}'$. The case of $\mathbb{1}_I \rho_{\mathbf{r},\delta}$ follows analogously.

Items 2 and 3 can, for $\cdot = \delta, \varrho$, be read directly from the definition of $\mathbb{1}_I \rho_{\mathbf{r},\cdot}$ and the condition, that, for all $t \in \mathbb{R}$, $\|\dot{\mathbf{r}}(t)\|$ is bounded by $v_{\max}$.

To construct solutions for parts of the particle source along the line of Section 5.1, we need to ensure that the convolution of the Green's functions and the inhomogeneities is well-behaved:

Proposition 5.2.3 Strict Compatibility of $G^{\mathrm{adv/ret}}$ and $\rho_{\mathbf{r}}$

Given some interval $I \subset \mathbb{R}$ and smooth $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ such that there exists $v_{\max} < 1$ and, for all $t \in \mathbb{R}$, the bound $\|\dot{\mathbf{r}}(t)\| < v_{\max}$ holds true. Then, the support of $G^{\mathrm{adv/ret}}$ and $\mathbb{1}_I \rho_{\mathbf{r},\delta}$, and respectively, the support of $\mathbb{1}_I \rho_{\mathbf{r},\varrho}$ are strictly compatible.

Proof

This is a corollary of Proposition 5.1.18 by referring to the support properties of $G^{\mathrm{adv/ret}}$ in Item 5 of Theorem 4.4.3 and of $\mathbb{1}_I \rho_{\mathbf{r},\delta}$, and respectively, of $\mathbb{1}_I \rho_{\mathbf{r},\varrho}$ in Item 2, and respectively, Item 3 or Theorem 5.2.2.

5.3 Particle sourced Solutions

Our preparation for the advanced and delayed Green's functions $G^{\mathrm{adv/ret}}$ and the particle sources $\mathbb{1}_I \rho_{\mathbf{r},\cdot}$ given by an interval $I \subset \mathbb{R}$ and for $\cdot = \delta, \varrho$ leads, through the statements developed in this chapter, to generalizations of distributions:

Proposition 5.3.1 Partial Solutions as Distributions

Given some interval $I \subset \mathbb{R}$ and smooth $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ such that there exists $v_{\max} < 1$ and, for all $t \in \mathbb{R}$, the bound $\|\dot{\mathbf{r}}(t)\| < v_{\max}$ holds true. The following statements hold true:

1. $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ is well-defined as a distribution on space-time.
2. $\mathrm{sp}(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}) \subset \Gamma^{\mathrm{light},\mp}_{\mathrm{graph}(\mathbf{r}|_I)}$
3. $\square G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta} = \mathbb{1}_I \rho_{\mathbf{r},\delta}$

Similar, given additionally $\varrho \in \mathbb{C}_c^\infty(\mathbb{R}^3, \mathbb{C})$, then the following statements hold true:

4. $G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho}$ is well-defined as a distribution on space-time.
5. $\mathrm{sp}(G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho}) \subset \Gamma^{\mathrm{light},\mp}_{\mathrm{graph}(\mathbf{r}|_I)+\{0\}\times\mathrm{sp}(\varrho)}$
6. $\square G^{\mathrm{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho} = \mathbb{1}_I \rho_{\mathbf{r},\varrho}$

Proof

Item 1 holds true, since $v_{\max} < 1$ implies $\Gamma^{\text{light},\mp}_{(0,\mathbf{0})} \cap -\Gamma^{v_{\max}}_{(0,\mathbf{0})} = \{0\}$ and, thus, due to Proposition 5.1.18, have $G^{\text{adv/ret}}$ and $\mathbb{1}_I \rho_{\mathbf{r},\delta}$ compatible support. This results, according to Definition/Theorem 5.1.13, in the well-definedness of convolution.

Item 2 is a further consequence of Definition/Theorem 5.1.13. According to Item 5 of Theorem 4.4.3, $\text{sp}(G^{\text{adv/ret}}) = \Gamma^{\text{light},\mp}_{(t,\boldsymbol{x})}$ and, by Item 2 of Theorem 5.2.2, $\text{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta}) = \text{graph}(\mathbf{r}|_I)$ which leads to:

$$\text{sp}(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}) \subset \text{sp}(G^{\text{adv/ret}}) + \text{sp}(\mathbb{1}_I \rho_{\mathbf{r},\delta}) = \Gamma^{\text{light},\mp}_{(t,\boldsymbol{x})} + \text{graph}(\mathbf{r}|_I) = \Gamma^{\text{light},\mp}_{\text{graph}(\mathbf{r}|_I)} .$$

Item 3 follows by considering Item 2 of Definition/Theorem 5.1.13 in step (i) and Item 2 of Theorem 4.4.3 in step (ii) of the following calculation:

$$\square G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta} \overset{(i)}{=} (\square G^{\text{adv/ret}}) * \mathbb{1}_I \rho_{\mathbf{r},\delta} = \delta * \mathbb{1}_I \rho_{\mathbf{r},\delta} = \mathbb{1}_I \rho_{\mathbf{r},\delta}$$

Items 4 to 6 can be proven in an completely analogous way.

Chapter 6

Regularity of Solutions

The, by the preceding chapter provided, solutions of linear partial differential equations are distributions and, as such, possible of both local as well as global regularity within the bounds set by the regularity theorem 4.1.25. We develop tools for the analysis of their local regularity which are especially useful when trying to supplying the differential equation with initial conditions at some time. Additionally, we will apply these methods to the solutions constructed in Section 5.3, which are sourced by partial chare trajectories.

Section Summaries:

6.1 Time Foliation of Distributions on Space-Time: We provide an in dept introduction to the notion of time foilable distributions on space-time which we motivated in Section 2.3.1. It describes a certain kind of local regularity in the time direction. Furthermore, we provide conditions on distributions on space-time which imply their time foilablity.

6.2 Regularity Particle sourced Solutions: The tools of the preceding section are applied to the solutions of the wave equation of Section 5.3. Afterwards, by the use of a certain transformation used in the proof of [Dec10, Theorem 4.18], we find a representation of the solutions by functions. This illustrates both their local and global regularity.

6.1 Time Foliation of Distributions on Space-Time

Given some linear partial differential equation $\partial\phi = \rho$ understood, for example, in the sense of distributions, the set of solutions for a given inhomogeneity ρ is rarely unique. Thus, these equations are usually supplied by extra conditions which constrain the solution space. One class of conditions arises when foliating the domain $\mathbb{R}^N$ of the generalized functions and forcing certain properties of the solution of the PDE on one hypersurface. We will only consider one special kind of foliation, namely

$$\mathbb{R}^N = \{(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^{N-1} | t \in \mathbb{R}\},$$

and call the first component t of a point $(t, \boldsymbol{x}) \in \mathbb{R}^N$ the foliation parameter time while the later components $\boldsymbol{x} \in \mathbb{R}^{N-1}$ spacial points. $(t, \boldsymbol{x})$ is called a space-time point and $\mathbb{R} \times \mathbb{R}^{N-1}$ space-time. This suggestive nomenclature stems from our desired application of PDEs on spacetime. From the theory of PDE of regular functions we know that, given some PDO ∂ involving n time derivatives, one needs to specify the value of the $0, 1, \ldots, (n-1)$-th derivatives of a solution on some hyperspace $\{t_0\} \times \mathbb{R}^{N-1}$ to uniquely identify it. Since general distributions are defined in an implicit way, acting on objects this domains exceeding hypersurfaces, we cannot expect that the specification of its values on such a set is well defined. Still, in a weak sense, a distribution on spacetime can be foliated into distributions on space, which we will discuss in the following.

6.1.1 Smeared Time Foliation of Distributions

Instead of restricting a distribution $u \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^{N-1})$ to strictly one hypersurface, one can always restrict it smoothly around it in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ of the foliation parameter to yield a distribution on $\mathbb{R}^{N-1}$. This is a consequence of the Schwartz kernel theorem as we will see after stating the theorem:

Theorem 6.1.1 Schwartz Kernel Theorem

Any distribution $u \in \mathscr{D}'(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$ can be regarded as a linear map from $C_c^\infty(\mathbb{R}^{N_1})$ to $\mathscr{D}'(\mathbb{R}^{N_2})$ by

$$f_1 \in C_c^\infty(\mathbb{R}^{N_1}) \mapsto u[f_1 \otimes \cdot\,] = \big(f_2 \in C_c^\infty(\mathbb{R}^{N_2}, \mathbb{C}) \mapsto u[f_1 \otimes f_2]\big) \in \mathscr{D}'(\mathbb{R}^{N_2}),$$

which is continuous as a map from $\mathscr{D}(\mathbb{R}^{N_1}) \to \mathscr{D}'(\mathbb{R}^{N_2})$. Furthermore, any linear, continuous map of this kind defines a distribution on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

Proof

[HI83, Theorem 5.2.1]

To follow along the previous idea, we regard an smooth approximation of unity $\delta_\varepsilon \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ supported on $(-\varepsilon, +\varepsilon)$ and as defined in Def. 4.1.5. We translate it to t_0 by considering $t \mapsto \delta_\varepsilon(t - t_0) \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$. This leads us to the following definition:

Definition 6.1.2 Smeared Time Foliation of Distributions

Given a smooth approximation $(\delta_\varepsilon)_{\varepsilon \in \mathbb{R}_{>0}}$, employing Theorem 6.1.1, we define

$$u^{\delta_\varepsilon} : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^{N-1}), t \mapsto u^{\delta_\varepsilon}(t) := u[\delta_\varepsilon(t - \cdot\,) \otimes \cdot\,].$$

Since u, as a distribution, is differentiable in a weak sense in the time direction, we expect some regularity of its smeared time foliation. One suitable notion is given in the following definition:

Definition 6.1.3 $\mathfrak{C}$-Differentiability in Time

We call a function $u(\cdot)[\cdot] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^{N-1})$ $\mathfrak{C}$-continuously differentiable of order $m \in \mathbb{N}_0$ if for all $f \in C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})$ the evaluation $t \mapsto u(t)[f]$ is m times continuously differentiable. Given $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ with $\boldsymbol{\alpha}_1 \leq m$, we define the function

$$\partial^{\boldsymbol{\alpha}} u(\cdot)[\cdot] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^{N-1}), t \mapsto \left(f \mapsto \partial_t^{\boldsymbol{\alpha}_1}\left(\partial^{(\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_N)}\big(u(t)\big)\right)[f]\right)$$

and call its the $\boldsymbol{\alpha}$-partial derivative of $u(\cdot)[\cdot]$. Furthermore, we define for $m \in \mathbb{N}_0$ the space $\mathfrak{C}^m\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{N-1})\big)$ and the space of $\mathfrak{C}$-smooth functions $\mathfrak{C}^\infty\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{N-1})\big) := \bigcap_{m \in \mathbb{N}} \mathfrak{C}^m\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{N-1})\big)$ by the following expression:

$$\mathfrak{C}^m\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{N-1})\big) := \Big\{u(\cdot)[\cdot] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^{N-1}) \Big| u \text{ is continuously differentiable up to order } m \in \mathbb{N}_0\Big\}.$$

We find:

Lemma 6.1.4 Smeared Foliations are $\mathfrak{C}$-Smooth

Given $u \in \mathscr{D}'(\mathbb{R}^N)$ and a smooth approximation $(\delta_\varepsilon)_{\varepsilon \in \mathbb{R}_{>0}}$, then for all $\varepsilon > 0$ is u^{δ_ε} $\mathfrak{C}$-smooth.

Proof

This is a direct consequence of the smoothness of convolutions between $\mathscr{D}'$ and $\mathscr{D}$ in Proposition 4.1.23.

Since we need to restrict the first $\big(\deg(\partial) - 1\big)$-th orders of partial time derivatives, we analyze the smeared time foliations of time derivatives:

Lemma 6.1.5

Given $u \in \mathscr{D}'(\mathbb{R}^N)$ and $m \in \mathbb{N}$, then $(\partial^{me_1}u)^{\delta_\varepsilon} = \partial^{me_1}u^{\delta_\varepsilon}$ holds true.

Proof

Given $m \in \mathbb{N}_0$, $t \in \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R}^{N-1}, C)$, we compute

$$\begin{aligned}(\partial^{me_1}u)^{\delta_\varepsilon}[f] &= (\partial^{me_1}u)[\delta_\varepsilon(t - \cdot\,) \otimes f] \\ &\overset{(i)}{=} (\partial^{me_1}u) * (\delta \otimes \tilde{f})(t, \mathbf{0}) \\ &\overset{(ii)}{=} \partial_t^m u * (\delta \otimes \tilde{f})(t, \mathbf{0}) \\ &\overset{(i)}{=} \partial_t^m u[\delta_\varepsilon(t - \cdot\,) \otimes f] \\ &= \partial^{me_1}u^{\delta_\varepsilon}[f]\end{aligned}$$

with (i) by Definition/Lemma 4.1.22 of convolutions of $\mathscr{D}'$ and $\mathscr{D}$ using the properties of its derivatives shown in Proposition 4.1.23 in step (ii).

6.1.2 Strict Time Foliation of Distributions

If we want to specify $m = 0, \ldots, n-1$ time derivatives at some time, we require at least that, for $u \in \mathscr{D}'$, the smooth foliation of its time derivatives $\big((\partial^{me_1}u)^{\delta_\varepsilon}(t)\big)_{\varepsilon>0}$ converges in $\mathscr{D}'$to a $\mathfrak{C}$-differentiable function. However, later, we want to analyze a more general setting of space-distribution valued functions on time, in which case the following definition is convenient:

Definition 6.1.6 **Time Foliation of a Distribution**

We call a distribution $u[\cdot] \in \mathscr{D}'(\mathbb{R}^N)$ time foilable, if there exists a function $u(\cdot)[\cdot]$ in the sense of

$$u(\cdot)[\cdot] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^{N-1}), t \mapsto u(t)[\cdot] \quad \text{and} \quad u(t)[\cdot] : \mathscr{D}'(\mathbb{R}^{N-1}) \to \mathbb{C}, f \mapsto u(t)[f]\,,$$

such that for all $f \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ the function $t \mapsto u(t)\big[f(t, \cdot\,)\big]$ is Lebesgue integrable and

$$u[f] = \int_{\mathbb{R}} dt\, u(t)\big[f(t, \cdot\,)\big] \tag{6.1}$$

holds true. The space of time foilable distributions is denoted by $\mathscr{D}'(\mathbb{R} \to \mathbb{R}^{N-1})$.

Given a function $u(\cdot)[\cdot] : \mathbb{R} \to \mathscr{D}'(\mathbb{R}^{N-1})$, it is not necessarily true that it gives rise to a distribution on space-time. If u is $\mathfrak{C}$-continuous, however, this is the case:

Proposition 6.1.7 **Embedding of $\mathfrak{C}^0$ in $\mathscr{D}'$**

$\mathfrak{C}^0\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{N-1})\big) \subset \mathscr{D}'(\mathbb{R} \to \mathbb{R}^{N-1})$ via the canonical embedding of Equation (6.1).

Proof

We need to prove that given $u(\cdot)[\cdot] \in \mathfrak{C}^0\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{N-1})\big)$, the linear map

$$u : \mathscr{D} \to (\mathbb{C}, |\cdot|), f \mapsto \int_{\mathbb{R}} dt\, u(t)\big[f(t, \cdot\,)\big]$$

is continuous. Let $n \in \mathbb{N}$, then, for all $g \in C_c^\infty([-n,n]^{N-1}, \mathbb{C})$, the sets $\{|u(t)[g]| \,|\, t \in [-n,n]\}$ are bounded due to the $\mathfrak{C}$-continuity of $u(\cdot)[\cdot]$. Further, $u(t)[\cdot]|_{C_c^\infty([-n,n]^{N-1},\mathbb{C})}$ maps the Fréchet space $\big(C_c^\infty([-n,n]^{N-1}, \mathbb{C}), (\|\cdot\|_{[-n,n]^{N-1},\infty,\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^N}\big)$ linearly and continuously to the Banach space

$(\mathbb{C}, |\cdot|)$. According to the uniform boundedness principle ([RSI80, Theorem V.7]), there exists $C_n > 0$ and $a_n \in \mathbb{N}_0$ such that

$$|u(t)[g]| \leq C_n \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^{N-1}, |\boldsymbol{\alpha}| \leq a_n} \|g\|_{[-n,n]^{N-1}, \infty, \boldsymbol{\alpha}}$$

for all $g \in C_c^\infty([-n,n]^{N-1}, \mathbb{C})$ and $t \in [-n,n]$. Given $f \in C_c^\infty([-n,n]^N, \mathbb{C})$, this results in

$$\begin{aligned} |u[f]| &\leq \int_{\mathbb{R}} dt \, \big| u(t)\big[f(t,\cdot)\big]\big| \\ &\leq \int_{\mathbb{R}} dt \, C_n \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^N, \alpha_1 = 0, |\boldsymbol{\alpha}| \leq a_n} \sup_{\boldsymbol{x} \in [-n,n]^{N-1}} |\partial^{\boldsymbol{\alpha}} f(t, \boldsymbol{x})| \\ &\leq 2n \, C_n \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^N, \alpha_1 = 0, |\boldsymbol{\alpha}| \leq a_n} \sup_{(t,\boldsymbol{x}) \in [-n,n]^N} |\partial^{\boldsymbol{\alpha}} f(t, \boldsymbol{x})| \end{aligned}$$

which implies $u \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^{N-1})$, i.e. $u(\cdot)[\cdot]$ is indeed a time foliation.

Given a time foilable distribution u, Def. 6.1.6 provides an instruction to retrieve u from its foliation $u(\cdot)[\cdot]$. We want to derive sufficient sets of conditions for the foilablity of distributions. These will be regularity conditions on a certain foliation candidate. For the definition of this object and the derivation of these sufficient sets of conditions for foilablity we need the notion of the tensor product of distributions:

Definition **Theorem 6.1.8** **Tensor Product of Distributions**

Given $u_1 \in \mathscr{D}'(\mathbb{R}^{N_1})$ and $u_2 \in \mathscr{D}'(\mathbb{R}^{N_2})$, there exists a unique distribution denoted by $u = u_1 \otimes u_2 \in \mathscr{D}'(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$, such that

1. for all $f_1 \in C_c^\infty(\mathbb{R}^{N_1}, \mathbb{C})$ and $f_2 \in C_c^\infty(\mathbb{R}^{N_2}, \mathbb{C})$, we have $u[f_1 \otimes f_2] = u_1[f_1] u_2[f_2]$,
2. given $f \in C_c^\infty(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$, then the following equation holds true
$$u[f] = u_1\big[\boldsymbol{x}_1 \mapsto u_2[\boldsymbol{x}_2 \mapsto f(\boldsymbol{x}_1, \boldsymbol{x}_2)]\big] = u_2\big[\boldsymbol{x}_2 \mapsto u_1[\boldsymbol{x}_1 \mapsto f(\boldsymbol{x}_1, \boldsymbol{x}_2)]\big]$$
3. $\mathrm{sp}(u_1 \otimes u_2) = \mathrm{sp}(u_1) \times \mathrm{sp}(u_2)$,
4. and for all $n \in \mathbb{N}$, let $C_{1,n}, C_{2,n} \geq 0$ and $a_{1,n}, a_{2,n} \in \mathbb{N}$ such that
   - $|u_1[f_1]| \leq C_{1,n} \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^{N_1}, |\boldsymbol{\alpha}| \leq a_{1,n}} \|f_1\|_{\mathsf{K}_{1,n}, \infty, \boldsymbol{\alpha}}$ for all $f_1 \in C_c^\infty(\mathsf{K}_{1,n} = [\text{-}n, n]^{N_1}, \mathbb{C})$ and
   - $|u_2[f_2]| \leq C_{2,n} \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^{N_2}, |\boldsymbol{\alpha}| \leq a_{2,n}} \|f_2\|_{\mathsf{K}_{2,n}, \infty, \boldsymbol{\alpha}}$ for all $f_2 \in C_c^\infty(\mathsf{K}_{2,n} = [\text{-}n, n]^{N_2}, \mathbb{C})$.

   Then for all $f \in C_c^\infty(\mathsf{K}_n = [\text{-}n, n]^{N_1 + N_2}, \mathbb{C})$, the following bound holds true:
$$|u[f]| \leq C_{1,n} C_{2,n} \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^{N_1 + N_2}, |\boldsymbol{\alpha}| \leq a_{1,n} + a_{2,n}} \|f\|_{\mathsf{K}_n, \infty, \boldsymbol{\alpha}}$$

Proof

[HI83, Theorem 5.1.1]

Further, we need some statements about their interplay with sequential convergence and convolutions. We start with a proposition about the convergence of tensor products of convergent series:

Proposition 6.1.9 Sequential Continuity of Tensor Products

Given u_1, $(u_{1,i})_{i\in\mathbb{N}}$ in $\mathscr{D}'(\mathbb{R}^{N_1})$ with $u_{1,i} \xrightarrow{\mathscr{D}'} u_1$ and u_2, $(u_{2,i})_{i\in\mathbb{N}}$ in $\mathscr{D}'(\mathbb{R}^{N_2})$ with $u_{2,i} \xrightarrow{\mathscr{D}'} u_2$, then also $u_{1,i} \otimes u_{2,i} \xrightarrow{\mathscr{D}'} u_1 \otimes u_2$ for $i \to \infty$.

To prove this proposition, we show the denseness of finite linear combinations of $f_1 \otimes f_2$ first:

Lemma 6.1.10 Dense Subset via Tensor Product

$\operatorname{span}\left(C_c^\infty(\mathbb{R}^{N_1},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N_2},\mathbb{C})\right)$ is sequentially dense in $\mathscr{D}(\mathbb{R}^{N_1+N_2})$.

Proof

Given $f \in C_c^\infty(\mathbb{R}^{N_1+N_2},\mathbb{C})$, we will construct a convergent series $(f_n)_{n\in\mathbb{N}}$ in $\operatorname{span}\left(C_c^\infty(\mathbb{R}^{N_1},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N_1},\mathbb{C})\right)$ by two steps of approximation. First, we pick one smooth approximation of unity $(\delta_{1,\varepsilon})_{\varepsilon\in\mathbb{R}_{>0}}$ in $C_c^\infty(\mathbb{R}^{N_1},\mathbb{C})$ and one $(\delta_{2,\varepsilon})_{\varepsilon\in\mathbb{R}_{>0}}$ in $C_c^\infty(\mathbb{R}^{N_2},\mathbb{C})$. By [HI83, Theorem 1.3.2], one finds the following convergence and, while due to Item 2 of Definition/Theorem 5.1.8, there exists $\mathsf{K} \subset \mathbb{R}^{N_1+N_2}$ such that:

$$f * (\delta_{1,\varepsilon} \otimes \delta_{2,\varepsilon}) \xrightarrow{\mathscr{D}} f \text{ for } \varepsilon \to 0 \quad \wedge \quad \operatorname{sp}\left(f * (\delta_{1,\varepsilon} \otimes \delta_{2,\varepsilon})\right) \subset \mathsf{K} \text{ for all } 0 < \varepsilon < 1\,. \tag{6.2}$$

Furthermore, for fixed $0 < \varepsilon < 1$ and $0 < \sigma < 1$, we define

$$f_{\varepsilon,\sigma} : \mathbb{R}^{N_1+N_2} \to \mathbb{C}, \boldsymbol{x} \mapsto \sum_{\boldsymbol{k}\in\mathbb{Z}^{N_1+N_2}} f(\boldsymbol{x} - \sigma\boldsymbol{k})\sigma^{N_1+N_2}(\delta_{1,\varepsilon} \otimes \delta_{2,\varepsilon})(\sigma\boldsymbol{k})$$

which converges, given any $0\varepsilon < 1$, according to [HI83, Lemma 4.1.3] in the following way

$$f_{\varepsilon,\sigma} \xrightarrow{\mathscr{D}} f * (\delta_{1,\varepsilon} \otimes \delta_{2,\varepsilon}) \text{ for } \sigma \to 0 \quad \wedge \quad \operatorname{sp}(f_{\varepsilon,\sigma}) \subset \mathsf{K} \text{ for all } 0 < \sigma < 1 \tag{6.3}$$

possibly choosing for K some larger but still compact subset. Next, we want to build a sequence $(f_{\varepsilon_n,\sigma_n})_{n\in\mathbb{N}}$ recursively which converges in $\mathscr{D}$ to f. Given $n \in \mathbb{N}$, we regard $\boldsymbol{\alpha} \in \mathbb{N}_0^{N_1+N_2}$ with $|\boldsymbol{\alpha}| \le n$ and estimate:

$$\|f_{\varepsilon,\sigma} - f\|_{\mathsf{K},\infty,\boldsymbol{\alpha}} \le \|f_{\varepsilon,\sigma} - f * (\delta_{1,\varepsilon} \otimes \delta_{2,\varepsilon})\|_{\mathsf{K},\infty,\boldsymbol{\alpha}} + \|f * (\delta_{1,\varepsilon} \otimes \delta_{2,\varepsilon}) - f\|_{\mathsf{K},\infty,\boldsymbol{\alpha}}\,.$$

We pick $0 < \varepsilon_n < \varepsilon_{n-1} < 1$ such that, for all $\boldsymbol{\alpha} \in \mathbb{N}_0^{N_1+N_2}$ with $|\boldsymbol{\alpha}| \le n$, we have

$$\|f * (\delta_{1,\varepsilon_n} \otimes \delta_{2,\varepsilon_n}) - f\|_{\mathsf{K},\infty,\boldsymbol{\alpha}} \le \tfrac{1}{2n}\,,$$

which is possible due to Equation (6.2). With ε_n at hand, in a second step, we choose $0 < \sigma_n < \sigma_{n-1} < 1$, such that, for all $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ with $|\boldsymbol{\alpha}| \le n$, we infer

$$\|f_{\varepsilon_n,\sigma_n} - f * (\delta_{1,\varepsilon_n} \otimes \delta_{2,\varepsilon_n})\|_{\mathsf{K},\infty,\boldsymbol{\alpha}} \le \tfrac{1}{2n}\,,$$

which is possible due to Equation (6.3). We summarize that, with this construction, we find $(f_{\varepsilon_n,\sigma_n})_{n\in\mathbb{N}}$ in $\operatorname{span}\left(C_c^\infty(\mathbb{R}^{N_1},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N_1},\mathbb{C})\right)$ and a compact set $\mathsf{K} \subset \mathbb{R}^{N_1+N_2}$ such that $\operatorname{sp}(f), \operatorname{sp}(f_{\varepsilon_n,\sigma_n}) \subset \mathsf{K}$ and, for all $\boldsymbol{\alpha} \in \mathbb{N}_0^{N_1+N_2}$, that $\|f_{\varepsilon_n,\sigma_n} - f\|_{\mathsf{K},\infty,\boldsymbol{\alpha}} \to 0$ for $n \to \infty$, i.e. $f_{\varepsilon_n,\sigma_n} \xrightarrow{\mathscr{D}'} f$ for $n \to \infty$.

Proof Proposition 6.1.9

For $f \in \mathscr{D}(\mathbb{R}^{N_1+N_2})$, we pick $(f_j)_{j\in\mathbb{N}}$ in $\operatorname{span}\left(C_c^\infty(\mathbb{R}^{N_1},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N_2},\mathbb{C})\right)$ according to Lemma 6.1.10. We find

$$|u_{1,i} \otimes u_{2,i}[f] - u_1 \otimes u_2[f]|$$

$$\leq \underbrace{|u_{1,i} \otimes u_{2,i}[f] - u_{1,i} \otimes u_{2,i}[f_j]|}_{=:\mathrm{I}} + \underbrace{|u_{1,i} \otimes u_{2,i}[f_j] - u_1 \otimes u_2[f_j]|}_{=:\mathrm{II}} + \underbrace{|u_1 \otimes u_2[f_j] - u_1 \otimes u_2[f]|}_{=:\mathrm{III}}$$

and we estimate the summands individually. For $\varepsilon > 0$, we find:

I Since $f_j \xrightarrow{\mathscr{D}} f$, by Item 1 of Lemma 5.1.7, there exists $n \in \mathbb{N}$, such that $\mathrm{sp}(f_i), \mathrm{sp}(f) \subset \mathsf{K}_n$. Furthermore, due to $u_{1,i} \xrightarrow{\mathscr{D}'} u_1$ and $u_{2,i} \xrightarrow{\mathscr{D}'} u_2$ Item 2 of Lemma 5.1.7 combined with Item 4 of Definition/Theorem 6.1.8 implies that there exists $C_n > 0$ and $a_n \in \mathbb{N}_0^{N_1+N_2}$, such that the estimate

$$\mathrm{I} = |u_{1,i} \otimes u_{2,i}[f - f_j]| \leq |u_{1,i} \otimes u_{2,i}[f]| \leq C_n \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^{N_1+N_2}, |\boldsymbol{\alpha}| \leq a_n} \|f - f_j\|_{\mathsf{K}_n, \infty, \boldsymbol{\alpha}}$$

holds true. We may thus pick $j_\varepsilon \in \mathbb{N}$ such that $\mathrm{I} \leq \varepsilon/3$ for all $i \in \mathbb{N}$.

III A similar argument holds true here.

II Given j_ε, we regard II. Since $f_{j_\varepsilon} \in \mathrm{span}\big(C_c^\infty(\mathbb{R}^{N_1}, \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N_2}, \mathbb{C})\big)$, there exists $m \in \mathbb{N}$, $(c_l)_{l \in (1,\dots,m)}$ in $\mathbb{C}$, $(f^1_{j_\varepsilon,l})_{l \in (1,\dots,m)}$ in $C_c^\infty(\mathbb{R}^{N_1}, \mathbb{C})$ and $(f^2_{j_\varepsilon,l})_{l \in (1,\dots,m)}$ in $C_c^\infty(\mathbb{R}^{N_2}, \mathbb{C})$ such that f_{j_ε} can be represented it in the following way:

$$f_{j_\varepsilon} = \sum_{l=1}^m c_l \, f^1_{j_\varepsilon,l} \otimes f^2_{j_\varepsilon,l}\,.$$

Now, we are able to estimate

$$\begin{aligned} \mathrm{II} = |(u_{1,i} \otimes u_{2,i} - u_1 \otimes u_2)[f_j]| &\leq \sum_{l=1}^m |c_l| \big|(u_{1,i} \otimes u_{2,i} - u_1 \otimes u_2)[f^1_{j_\varepsilon,l} \otimes f^2_{j_\varepsilon,l}]\big| \\ &= \sum_{l=1}^m |c_l| \big|u_{1,i}[f^1_{j_\varepsilon,l}]\, u_{2,i}[f^2_{j_\varepsilon,l}] - u_1[f^1_{j_\varepsilon,l}]\, u_2[f^2_{j_\varepsilon,l}]\big| \end{aligned}$$

which converges to 0 for $i \to \infty$ due to $u_{1,i} \xrightarrow{\mathscr{D}'} u_1$ and $u_{2,i} \xrightarrow{\mathscr{D}'} u_2$ implying the existence of $i_\varepsilon \in \mathbb{N}$ such that $\forall i \geq i_\varepsilon$ $\mathrm{II} \leq \varepsilon/3$

Combining the statements above, we found $i_\varepsilon \in \mathbb{N}$ such that for all $i \geq i_\varepsilon$ $|u_{1,i} \otimes u_{2,i}[f] - u_1 \otimes u_2[f]| \leq \varepsilon$.

As announced, we consider the interaction of tensor products and convolutions, whose behavior on test functions can be lifted to distributions by the subsequent corollary:

Corollary 6.1.11 Convolutions and Tensor Products

Given $u_1, v_1 \in \mathscr{D}'(\mathbb{R}^{N_1})$ and $u_2, v_2 \in \mathscr{D}'(\mathbb{R}^{N_2})$ such that the supports of u_1, v_1 are strictly compatible the supports of u_2, v_2 in the sense of Def. 5.1.12, then $(u_1 \otimes u_2) * (v_1 \otimes v_2) = (u_1 * v_1) \otimes (u_2 * v_2)$.

Proof

We pick $\varepsilon > 0$ such that the pairs $\big(\mathrm{sp}(u_1) + \mathsf{B}_\varepsilon(\mathbf{0}), \mathrm{sp}(v_1) + \mathsf{B}_\varepsilon(\mathbf{0})\big)$ and $\big(\mathrm{sp}(u_2) + \mathsf{B}_\varepsilon(\mathbf{0}), \mathrm{sp}(v_2) + \mathsf{B}_\varepsilon(\mathbf{0})\big)$ are compatible. This implies that further $\mathrm{sp}(u_1 \otimes u_2) + \mathsf{B}_\varepsilon(\mathbf{0})$ and $\mathrm{sp}(v_1 \otimes v_2) + \mathsf{B}_\varepsilon(\mathbf{0})$ are compatible. We consider the approximating series $(u^\varepsilon_{1,i})_{i \in \mathbb{N}}$, $(u^\varepsilon_{2,i})_{i \in \mathbb{N}}$, $(v^\varepsilon_{1,i})_{i \in \mathbb{N}}$ and $(v^\varepsilon_{2,i})_{i \in \mathbb{N}}$ in C_c^∞ of Lemma 5.1.11 and argue

(i) $(u_1 \otimes u_2) * (v_1 \otimes v_2) = \lim_{i \to \infty} (u^\varepsilon_{1,i} \otimes u^\varepsilon_{2,i}) * (v^\varepsilon_{1,i} \otimes v^\varepsilon_{2,i})$,

(ii) $\lim_{i \to \infty} (u^\varepsilon_{1,i} * v^\varepsilon_{1,i}) \otimes (u^\varepsilon_{2,i} * v^\varepsilon_{2,i}) = (u_1 * v_1) \otimes (u_2 * v_2)$ and

(iii) $\lim_{i \to \infty} (u^\varepsilon_{1,i} \otimes u^\varepsilon_{2,i}) * (v^\varepsilon_{1,i} \otimes v^\varepsilon_{2,i}) = \lim_{i \to \infty} (u^\varepsilon_{1,i} * v^\varepsilon_{1,i}) \otimes (u^\varepsilon_{2,i} * v^\varepsilon_{2,i})$

with (i) by Definition/Theorem 5.1.13 Item 4, (ii) due to Proposition 6.1.9, and (iii) follows by Fubini's theorem.

We can now state the definition of the foliation candidate:

Definition 6.1.12 Foliation Candidate and its Representation by Functions

Given $u[\cdot] \in \mathscr{D}'(\mathbb{R}^N)$, we call the family of distributions $u*(\delta\otimes[\tilde{\cdot}])[\cdot] := \{u*(\delta\otimes\widetilde{f})[\cdot]\}_{f\in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})}$ the **foliation candidate** of $u[\cdot]$.

Further, we call a family of functions $\left(v_{\widetilde{f}}(\cdot,\cdot)\right)_{f\in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})}$ a **representation of the foliation candidate by functions** if, for all $f \in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})$

- the function v_f lies in $L^1_{loc}(\mathbb{R}^N,\mathbb{C})$ and
- the equation $v_f[\cdot] = u*(\delta\otimes f)[\cdot]$ holds true in the sense of the embedding $v_f \in L^1_{loc} \subset \mathscr{D}'$.

If there exists such a family, we call the foliation candidate **representable by functions** and denote the family by $u*(\delta\otimes[\tilde{\cdot}])(\cdot,\cdot)$ and the functions $v_{\widetilde{f}}$ by $u*(\delta\otimes\widetilde{f})(\cdot,\cdot)$ for $f \in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})$, and thus in this notation:

$$u*(\delta\otimes[\tilde{\cdot}])(\cdot,\cdot) = \left(u*(\delta\otimes\widetilde{f})(\cdot,\cdot)\right)_{f\in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})}.$$

Furthermore, the tools established in the preceding statements allow us to prove the following theorem. It justifies the nomenclature of $u*(\delta\otimes[\tilde{\cdot}])[\cdot]$ as the foliation candidate:

Theorem 6.1.13 Representation of the Foliation Candidate

Let $u[\cdot] \in \mathscr{D}'(\mathbb{R}\to\mathbb{R}^{N-1})$ and $u(\cdot)[\cdot]$ be foliation of $u[\cdot]$. If, for all $f \in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})$, the function

$$u(\cdot)[\mathsf{T}_\cdot f] : \mathbb{R}^N \to \mathbb{C}, (t,\boldsymbol{x}) \mapsto u(t)[\mathsf{T}_{\boldsymbol{x}} f]$$

lies in $L^1_{loc}(\mathbb{R}^N,\mathbb{C})$, then $\{u(\cdot)[\mathsf{T}_\cdot f]\}_{f\in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})}$ is a representation of the foliation candidate by functions.

Proof

We will show $u(\cdot)[\mathsf{T}_\cdot f] = u*(\delta\otimes\widetilde{f})$ first as an equality of linear forms on $C_c^\infty(\mathbb{R},\mathbb{C})\otimes C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})$. Let $g_1 \in C_c^\infty(\mathbb{R},\mathbb{C})$ and $g_2 \in C_c^\infty(\mathbb{R}^{N-1},C)$, then we calculate

$$\begin{aligned}
u*(\delta\otimes\widetilde{f})[g_1\otimes g_2] &\overset{(i)}{=} \left(u*(\delta\otimes\widetilde{f})\right)*(\widetilde{g_1}\otimes\widetilde{g_2})(0,\mathbf{0}) \\
&\overset{(ii)}{=} u*\left((\delta\otimes\widetilde{f}*\widetilde{g_1}\otimes\widetilde{g_2})\right)(0,\mathbf{0}) \\
&\overset{(iii)}{=} u*\left(\widetilde{g_1}\otimes(\widetilde{f}*\widetilde{g_2})\right)(0,\mathbf{0}) \\
&\overset{(i)}{=} u\big[g_1\otimes\widetilde{(\widetilde{f}*\widetilde{g_2})}\big] \\
&= \int_{\mathbb{R}} dt\, u(t)\big[\widetilde{\widetilde{f}*\widetilde{g_2}}\big]g_1(t) \\
&\overset{(i)}{=} \int_{\mathbb{R}} dt\,\left(u(t)\right)*(\widetilde{f}*\widetilde{g_2})(\mathbf{0})g_1(t) \\
&\overset{(ii)}{=} \int_{\mathbb{R}} dt\,\left(u(t)*\widetilde{f}\right)*\widetilde{g_2}(\mathbf{0})g_1(t) \\
&\overset{(iv)}{=} \int_{\mathbb{R}} dt\left(\int_{\mathbb{R}^3} dx^{N-1}\,\left(u(t)*\widetilde{f}\right)(\boldsymbol{x})g_2(\boldsymbol{x})\right)g_1(t)
\end{aligned}$$

$$\overset{(i)}{=} \int_{\mathbb{R}} dt \left(\int_{\mathbb{R}^3} dx^{N-1}\, u(t)[\mathsf{T}_{\boldsymbol{x}} f] g_2(\boldsymbol{x}) \right) g_1(t)$$
$$\overset{(v)}{=} \int_{\mathbb{R}^4} dt\, dx^{N-1}\, u(t)[\mathsf{T}_{\boldsymbol{x}} f](g_1 \otimes g_2)(t, \boldsymbol{x})$$

where the steps are justified by:

(i) By Definition/Lemma 4.1.22 of the convolution of $\mathscr{D}'$ and $\mathscr{D}$.

(ii) Definition/Theorem 6.1.8 Item 3 implies a compact support of $\widetilde{g_1} \otimes \widetilde{g_2}$. Additionally, Corollary 5.1.15 shows that any number of distributions of which are all but at most one compactly supported have strictly compatible support. Thus, their convolution is by Item 2 Definition/Theorem 5.1.13 associative and commutative. Since the embedding of C^∞ in $\mathscr{D}'$ is injective the equality hold as functions and, thus, pointwisely.

(iii) By the well-behaved interplay of convolution and tensor product of strictly compatible supported distributions in Corollary 6.1.11. By the same argument of step (ii) the equation holds pointwisely when regarding both distributions as C^∞ functions.

(iv) Proposition 4.1.23 implies the smoothness of $\boldsymbol{x} \mapsto \big(u(t) * f\big)(\boldsymbol{x})$ which justifies the step according to the embedding $C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})$ in $\mathscr{D}'(\mathbb{R}^{N-1})$ of Def. 4.1.8.

(v) Since by assumption $u(\,\cdot\,)[\mathsf{T}_{\boldsymbol{\cdot}} f] \in L^1_{loc}(\mathbb{R}^N, \mathbb{C})$ allowing the application of Fubini's theorem.

Since $u * (\delta * \widetilde{f}) \in \mathscr{D}'(\mathbb{R}^N)$ and $u(\,\cdot\,)[\mathsf{T}_{\boldsymbol{\cdot}} f] \in L^1_{loc}(\mathbb{R}^N) \subset \mathscr{D}'(\mathbb{R}^N)$ are continuous linear functions from $\mathscr{D}(\mathbb{R}^N)$ to $(\mathbb{C}, |\cdot|)$ and span $\big(C_c^\infty(\mathbb{R}, \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})\big)$ is dense in $\mathscr{D}(\mathbb{R}^N)$, we get $u * (\delta * \widetilde{f}) = (t, \boldsymbol{x}) \mapsto u(t)[\mathsf{T}_{\boldsymbol{x}} f]$ in $\mathscr{D}'(\mathbb{R}^N)$. Since further the embedding of $L^1_{loc}(\mathbb{R}^N, \mathbb{C})$ is injective in $\mathscr{D}'(\mathbb{R}^N)$ this equality holds true in $L^1_{loc}(\mathbb{R}^N, \mathbb{C})$.

Our sufficient sets of conditions for foilablity are now formulated as the existence of representations of the foliation candidate with certain regularity properties:

Definition 6.1.14 Notions of Regularity for Foliation Candidates

Let $u[\,\cdot\,] \in \mathscr{D}'(\mathbb{R}^N)$ whose foliation candidate $u * (\delta \otimes [\,\widetilde{\cdot}\,])[\,\cdot\,]$ is representable by functions. Given such a representation $u * (\delta \otimes [\,\widetilde{\cdot}\,])(\,\cdot\,, \boldsymbol{\cdot}\,)$.

We call the representation **form-regular** if, for all $f \in C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})$, the following properties:

1. The function $u * (\delta \otimes \widetilde{f})(\,\cdot\,, \boldsymbol{\cdot}\,)$ lies in $L^\infty_{loc}(\mathbb{R}^N, \mathbb{C})$.
2. For all $t \in \mathbb{R}$, the function $u * (\delta \otimes \widetilde{f})(t, \boldsymbol{\cdot}\,) : \mathbb{R}^{N-1} \to \mathbb{C}, \boldsymbol{x} \mapsto u * (\delta \otimes \widetilde{f})(t, \boldsymbol{x})$ is partial differentiable. We denotes for $i \in \{2, \ldots, N\}$ the function $\mathbb{R}^N \to \mathbb{C}, (t, \boldsymbol{x}) \mapsto \partial^{\boldsymbol{e}_i}\big(u * (\delta \otimes \widetilde{f})\big)(t, \boldsymbol{x})$ by $\mapsto \partial^{\boldsymbol{e}_i}\big(u * (\delta \otimes \widetilde{f})\big)(\,\cdot\,, \boldsymbol{\cdot}\,)$.
3. For all $i \in \{2, \ldots, N\}$, the function $\partial^{\boldsymbol{e}_i}\big(u * (\delta \otimes \widetilde{f})\big)(\,\cdot\,, \boldsymbol{\cdot}\,)$ lies in $L^\infty_{loc}(\mathbb{R}^N, \mathbb{C})$.
4. For all $\boldsymbol{x} \in \mathbb{R}^{N-1}$, the following subset of $\mathbb{R}$ possesses Lebesgue measure 0:
$$N_{\boldsymbol{x}}^{u,f} := \big\{ t \in \mathbb{R} \,\big|\, u * (\delta \otimes \widetilde{f})(\,\cdot\,, \boldsymbol{\cdot}\,) \text{ is not continuous at } (t, \boldsymbol{x}) \big\}\,.$$
5. For all $\boldsymbol{x} \in \mathbb{R}^{N-1}$ and $i \in \{2, \ldots, N\}$, the following subset of $\mathbb{R}$ possesses Lebesgue measure 0:
$$N_{\boldsymbol{x}}^{\boldsymbol{e}_i,u,f} := \big\{ t \in \mathbb{R} \,\big|\, \partial^{\boldsymbol{e}_i}\big(u * (\delta \otimes \widetilde{f})\big)(\,\cdot\,, \boldsymbol{\cdot}\,) \text{ is not continuous at } (t, \boldsymbol{x}) \big\}\,.$$

This allows the definition of $u*(\delta\otimes[\,\widetilde{\cdot}\,])[\,\widetilde{\cdot}\,, \boldsymbol{0})\big|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)} \in \mathsf{LF}\big[\mathrm{span}\,\big(C_c^\infty(\mathbb{R}, \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})\big), \mathbb{C}\big]$ by defining, for $n \in \mathbb{N}$, $g_1^{\text{time}}, \ldots, g_n^{\text{time}} \in C_c^\infty(\mathbb{R}, \mathbb{C})$ and $g_1^{\text{space}}, \ldots, g_n^{\text{space}} \in C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})$, the evaluation of $u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0})\big|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ at $\sum_{i=1}^n g_i^{\mathrm{time}} \otimes g_i^{\mathrm{space}}$ as

$$\sum_{i=1}^n \int_{\mathbb{R}} dt\, \widetilde{g_i^{\mathrm{time}}}(t) \big(u * (\delta \otimes \widetilde{g_i^{\mathrm{space}}})\big)(t, \mathbf{0}) \,.$$

We call the representation **distribution-regular** if it fulfills the following properties:

1. For all $g \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, the functions $\mathbb{R} \to \mathbb{C}, t \mapsto u * \big(\delta \otimes \tilde{g}(t, \cdot)\big)(t, \mathbf{0})$ lies in $L^1_{loc}(\mathbb{R}, \mathbb{C})$ where $\tilde{g}(t, \cdot)$ is, for $t \in \mathbb{R}$, understood as the function $\mathbb{R}^{N-1} \to \mathbb{C}, \boldsymbol{x} \mapsto g(\text{-}t, \text{-}\boldsymbol{x})$.
2. The following linear form on $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ lies in $\mathcal{D}'(\mathbb{R}^N)$:

$$u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0}) : C_c^\infty(\mathbb{R}^N, \mathbb{C}) \to \mathbb{C}, g \mapsto \int_{\mathbb{R}} dt\, u * \big(\delta \otimes \tilde{g}(t, \cdot)\big)(t, \mathbf{0}) \,.$$

The distribution $u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0})$ is clearly foilable and $u * (\delta \otimes [\tilde{\cdot}])(\cdot, \mathbf{0}) \in \mathcal{D}'(\mathbb{R} \to \mathbb{R}^{N-1})$ defined by

- $u * (\delta \otimes [\tilde{\cdot}])(\cdot, \mathbf{0}) : \mathbb{R} \to \mathcal{D}'(\mathbb{R}^{N-1}), t \mapsto u * (\delta \otimes [\tilde{\cdot}])(t, \mathbf{0})$ and
- $u * (\delta \otimes [\tilde{\cdot}])(t, \mathbf{0}) : C_c^\infty(\mathbb{R}^{N-1}) \to \mathbb{C}, f \mapsto u * (\delta \otimes \tilde{f})(t, \mathbf{0})$

is a foliation of $u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0})$.

We call the representation **jointly form- and distribution-regular** if it is both form-regular and continuously-regular. The relation of $u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0})\big|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ and $u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0})$ will become clear in Theorem 6.1.17.

The existence of form-regular representation of the foliation candidate implies that the foliation candidate can be canonically identified with the distribution as certain linear forms:

Theorem 6.1.15 Form-Regularity implies Form Equivalence

Let $u[\cdot] \in \mathcal{D}'(\mathbb{R}^N)$ such that there exists a form-regular representation $u * (\delta \otimes [\tilde{\cdot}])(\cdot, \cdot)$ of the foliation candidate via functions. Then, $u[\cdot] = u * (\delta \otimes [\tilde{\cdot}])[\tilde{\cdot}, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ holds true as an equation in $\mathsf{LF}\big[\mathrm{span}\,\big(C_c^\infty(\mathbb{R}, \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})\big), \mathbb{C}\big]$.

To prove this proposition, the following lemma will be helpful.

Lemma 6.1.16 Convolutions on L^p-Spaces

Given a smooth approximation of unity $(\delta_\varepsilon)_{\varepsilon \in \mathbb{R}_{\geq 0}}$.

1. For $h \in L^1_{loc}(\mathbb{R}^N, \mathbb{C})$, we have $h * \delta_\varepsilon \in C^\infty(\mathbb{R}^N, \mathbb{C})$.
2. For a point $\boldsymbol{x} \in \mathbb{R}^N$, a compact neighborhood K of $\boldsymbol{x}$, and $h \in L^\infty(\mathsf{K}, \mathbb{C})$, we have $\lim_{\varepsilon \to 0} h * \delta_\varepsilon(\boldsymbol{x}) = h(\boldsymbol{x})$.
3. For $h \in L^p(\mathbb{R}^N, \mathbb{C})$, $p \in [1, \infty)$, we have $h * \delta_\varepsilon \xrightarrow{L^p} h$ for $\varepsilon \to 0$.

Proof

Item 1 is a corollary of [HI83, Theorem 1.3.2] by noticing that $\bigcup_{\varepsilon < 1} \mathrm{sp}(\delta_\varepsilon)$ is bounded.

Item 2: Let $\epsilon > 0$, $\delta > 0$ such that, for all $\boldsymbol{y} \in \mathsf{B}_\delta(\mathbf{0})$, the estimate $|h(\boldsymbol{x} - \boldsymbol{y}) - h(\boldsymbol{x})| < \epsilon$ holds true.

Furthermore, let $\overline{\varepsilon} > 0$ such that $\varepsilon < \overline{\varepsilon}$ implies $\mathrm{sp}(\delta_\varepsilon) \subset \mathsf{B}_\delta(\mathbf{0})$, Then

$$\begin{aligned}|h * \delta_\varepsilon(\boldsymbol{x}) - h(\boldsymbol{x})| &= \left|\int_{\mathbb{R}^N} d^N y\, h(\boldsymbol{x}-\boldsymbol{y})\delta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})\right| \\ &\leq \int_{\mathbb{R}^N} d^N y\, |h(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})|\delta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \\ &\leq \int_{\mathbb{R}^N} d^N y\, \epsilon\delta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \\ &= \epsilon\end{aligned}$$

and, thus, $\lim_{\varepsilon\to 0} h * \delta_\varepsilon(\boldsymbol{x}) = h(\boldsymbol{x})$.

Item 3 is proven similarly to Item 1 as a corollary of [HI83, Theorem 1.3.2].

Proof

Strategy: Due to linearity it is sufficient to prove the equality $u[\,\cdot\,] = u * (\delta \otimes [\,\widetilde{\cdot}\,])[\,\widetilde{\cdot}\,, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ on $C_c^\infty(\mathbb{R},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})$. Given $g \in C_c^\infty(\mathbb{R},\mathbb{C})$ and $f \in C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})$, we note that, by Definition/Lemma 4.1.22, the convolution $u * (\widetilde{g} \otimes \widetilde{f})$ can be represented by a smooth function, which we denote by $u * (\widetilde{g} \otimes \widetilde{f})(\,\cdot\,,\,\boldsymbol{\cdot}\,)$. The following equality holds true:

$$u[g \otimes f] = u * (\widetilde{g} \otimes \widetilde{f})(0, \mathbf{0})\,.$$

By the properties of convolution and tensor product given in Corollary 6.1.11 and the associativity and commutativity of the convolution according to Item 1 of Definition/Theorem 5.1.13, we also have

$$u * (\widetilde{g} \otimes \widetilde{f}) = u * (\widetilde{g} \otimes \delta) * (\delta \otimes \widetilde{f}) = \widetilde{g} \otimes \delta * \big(u * (\delta \otimes \widetilde{f})\big)$$

as equalities in the sense of distributions. We want to evaluate $\widetilde{g} \otimes \delta * v$ where $v[\,\cdot\,] := u * (\delta \otimes \widetilde{f})$ by using Item 2 of Definition/Theorem 6.1.8 and, thus, approximate $v[\,\cdot\,]$ by $(v_n(\,\cdot\,))_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ such that $v_n[\,\cdot\,] \xrightarrow{\mathscr{D}'} v[\,\cdot\,]$ (**Part I**). By the sequential continuity of the convolution (Item 4 of Definition/Theorem 5.1.13), we infer

$$w_{\mathscr{D}'}[\,\cdot\,]\widetilde{g} \otimes \delta * v = \mathscr{D}'\text{-}\lim_{n\to\infty} w_n[\,\cdot\,]$$

again as an equation of distributions with $w_n[\,\cdot\,] := \widetilde{g} \otimes \delta * v_n$. Since this implies the equality $w_{\mathscr{D}'}[\,\cdot\,] = u * (\widetilde{g} \otimes \widetilde{f})[\,\cdot\,]$, it is represented by the smooth function $w_{\mathscr{D}'}(\,\cdot\,) := u * (\widetilde{g} \otimes \widetilde{f})(\,\cdot\,,\,\boldsymbol{\cdot}\,)$. This implies that, if one takes first the limit $n \to \infty$ of the object in $\mathscr{D}'$ and afterwards regards the result as the pairing of a test function with a $C^\infty(\mathbb{R}^N,\mathbb{C})$-function in the sense of Def. 4.1.8, one can evaluate it at $(0,\mathbf{0})$ leading to the following equality:

$$u[g \otimes f] = w_{\mathscr{D}'}(0, \mathbf{0}).$$

Furthermore, since now $v_n(\,\cdot\,) \in C_c^\infty(\mathbb{R}^N,\mathbb{C})$, as already announced, we apply Item 2 of Definition/Theorem 6.1.8. This leads to a representation $w_n(\,\cdot\,) : \mathbb{R}^N \to \mathbb{C}$ of $w_n[\,\cdot\,]$ whose value at $(t,\boldsymbol{x}) \in \mathbb{R}^N$ is provided by the following complex number:

$$\begin{aligned}w_n(t,\boldsymbol{x}) &:= (\widetilde{g} \otimes \delta * v_n(t,\boldsymbol{x})) \\ &= \widetilde{g} \otimes \delta\big[(s,\boldsymbol{y}) \mapsto v_n(t-s,\boldsymbol{x}-\boldsymbol{y})\big] \\ &= \widetilde{g}\Big[s \mapsto \delta\big[\boldsymbol{y} \mapsto v_n(t-s,\boldsymbol{x}-\boldsymbol{y})\big]\Big] \\ &= \widetilde{g}\big[s \mapsto v_n(t-s,\boldsymbol{x})\big] \\ &= \int_{\mathbb{R}} ds\, \widetilde{g}(s)\, v_n(t-s,\boldsymbol{x}) \\ &= \int_{\mathbb{R}} ds\, g(s-t)\, v_n(s,\boldsymbol{x})\,.\end{aligned}$$

By Proposition 4.1.23, we know that w_n lies in $C^\infty(\mathbb{R}^N, \mathbb{C})$. We will show that, due to the form-regularity of $u * (\delta \otimes [\widetilde{\cdot}])(\,\cdot\,, \boldsymbol{\cdot})$, one can pick the approximation $v_n(\cdot)$, such that $w_n(\cdot)$ converge pointwisely to continuous function w_{pw}, and for all $(t, \boldsymbol{x}) \in \mathbb{R}^N$ the following equality holds true **(Part II)**:

$$w_{\mathrm{pw}}(0, \mathbf{0}) = u * (\delta \otimes \widetilde{g})[\widetilde{f}, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}\,.$$

We summarize the statements proved up to this point:

- The distribution $w_n[\cdot]$ convergences in the topology of distributions to the distribution $w_{\mathscr{D}'}[\cdot]$.
- The smooth function $w_n(\cdot)$ converges pointwisely to the continuous function $w_{\mathrm{pw}}(\cdot)$
- The function $w_n(\cdot)$ is a representation of $w_n[\cdot]$.
- The function $w_{\mathscr{D}'}(\cdot)$ is a representation of $w_{\mathscr{D}'}[\cdot]$.

Since the author is not aware of an argument why the functions $w_{\mathscr{D}'}(\cdot)$ and $w_{\mathrm{pw}}(\cdot)$ should agree pointwisely at $(0, \mathbf{0}) \in \mathbb{R}^N$, which would then finally imply the desired statement

$$u[g \otimes f] = w_{\mathscr{D}'}(0, \mathbf{0}) = w_{\mathrm{pw}}(0, \mathbf{0}) = u * (\delta \otimes \widetilde{g})[\widetilde{f}, \mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}\,,$$

we will show this equality manually. Hereby, we apply the regularity requirements of the spatial partial derivatives of the representation of the foliation candidate in the definition of distribution-regularity (**Part III**).

Definition of the Approximation: For given $n \in \mathbb{N}$ and $f \in C_c^\infty(\mathbb{R}^{N-1}, \mathbb{C})$, to approximate $v[\cdot] = u * (\delta \otimes \widetilde{f})$, we pick

- a smooth approximation of unity "in time", i.e. $(\delta_n^{\mathrm{time}})_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ such that $\mathrm{sp}(\delta_n^{\mathrm{time}}) \subset \mathsf{B}_1(0)$,
- another smooth approximation of unity "in space", i.e. $(\delta_n^{\mathrm{space}})_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^{N-1}, \mathbb{R}_{\geq 0})$ such that $\mathrm{sp}(\delta_n^{\mathrm{space}}) \subset \mathsf{B}_1(\mathbf{0})$,
- smooth cutoffs "in time", i.e. $(\chi_n^{\mathrm{time}})_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}, [0, 1])$ such that $\chi_n^{\mathrm{time}}|_{\mathsf{B}_n(0)} = 1$,
- smooth cutoffs "in space", i.e. $(\chi_n^{\mathrm{space}})_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^{N-1}, [0, 1])$ such that $\chi_n^{\mathrm{space}}|_{\mathsf{B}_n(0)} = 1$,

and note, that $v[\cdot]$ can, due to the form-regularity, be represented by a function $v(\cdot)$ which lies in L^∞_{loc}. This allows us to define the function $v_n(\cdot)$ by defining its value at $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^{N-1}$ as the following complex number:

$$v_n(t, \boldsymbol{x}) := \big(\chi_n^{\mathrm{time}} \otimes \chi_n^{\mathrm{space}}\big)(t, \boldsymbol{x})\big(v \,*\, \delta_n^{\mathrm{time}} \otimes \delta^{\mathrm{space}}\big)(t, \boldsymbol{x})\,.$$

Item 1 of Lemma 6.1.16 and the compact support of $\chi_n^{\mathrm{time}} \otimes \chi_n^{\mathrm{space}}$ that $v_n(\cdot)$ lies in C_c^∞.

Part I: Now, we prove that the convergence $v_n[\cdot] \xrightarrow{\mathscr{D}'} v[\cdot]$ holds true. Let $m \in \mathbb{N}$ and $h \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ such that $\mathrm{sp}(h) \subset \mathsf{B}_m(\mathbf{0})$, then:

$$\begin{aligned}|v_n[h] - v[h]| &\leq \int_{\mathbb{R}^4} dt d^{N-1}x\, |h(t, \boldsymbol{x})| \big|\big(v * (\delta_n^{\mathrm{time}} \otimes \delta_n^{\mathrm{space}}) - v\big)(t, \boldsymbol{x})\big| \\ &\leq \|h\|_\infty \int_{\mathsf{B}_m(\mathbf{0})} dt d^{N-1}x\, \big|\big(v * (\delta_n^{\mathrm{time}} \otimes \delta_n^{\mathrm{space}}) - v\big)(t, \boldsymbol{x})\big|\end{aligned}$$

We perform a restriction of $v(\cdot)$ to a compact domain, by defining $\overline{v}_m(\cdot) := \mathbb{1}_{\mathsf{B}_{m+1}(0)\times\mathsf{B}_{m+1}(\mathbf{0})} v(\cdot)$, such that it does not affect the integral. This results in $\overline{v}_m(\cdot) \in L^1(\mathbb{R}^N, \mathbb{C})$ and we find the estimate

$$|v_n[h] - v[h]| \leq \|h\|_\infty \big\|\overline{v} * (\delta_n^{\mathrm{time}} \otimes \delta_n^{\mathrm{space}}) - \overline{v}\big\|_{L^1(\mathbb{R}^N)}$$

which converges to zero according to Item 3 of Lemma 6.1.16. Since $m \in \mathbb{N}$ and $h \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ were arbitrary, we have shown $v_n[\cdot] \xrightarrow{\mathscr{D}'} v[\cdot]$.

Part II: We pick $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and ain at an application of the dominated convergence theorem in Equation (6.4) below. First, by form-regularity, we note that, $(s, \boldsymbol{y}) \mapsto v(s, \boldsymbol{y})$ is continuous on $N_{\boldsymbol{x}}^{\complement} \times \{\boldsymbol{x}\}$. Moreover, this map lies also in L^∞ on any compact space-time neighborhood of a point $(s, \boldsymbol{x}) \in N_{\boldsymbol{x}}^{\complement} \times \{\boldsymbol{x}\}$ of continuity. By Item 2 of Lemma 6.1.16, we therefore get $\lim_{n\to\infty} v_n(s, \boldsymbol{x}) = v(s, \boldsymbol{x})$ and, thus,

$$\lim_{n\to\infty} g(s-t)v_n(s, \boldsymbol{x}) = g(s-t)v(s, \boldsymbol{x})$$

for all $(t, \boldsymbol{x}) \in \mathbb{R}$ and $s \in N_{\boldsymbol{x}}^{\complement}$. Further, given $\mathsf{K}^{\text{time}} \subset \mathbb{R}$ and $\mathsf{K}^{\text{space}} \subset \mathbb{R}^{N-1}$ compact, we construct an integrable dominating function $d_{\mathsf{K}^{\text{time}},\mathsf{K}^{\text{space}}} : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $|g(s-t)v_n(s, \boldsymbol{x})| \leq d_{\mathsf{K}^{\text{time}},\mathsf{K}^{\text{space}}}(s)$ for all $s \in \mathbb{R}$ and $(t, \boldsymbol{x}) \in \mathsf{K}^{\text{time}} \times \mathsf{K}^{\text{space}}$ by virtue of

- $$\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |g(s-t)\, v_n(s, \boldsymbol{x})| \leq \mathbb{1}_{\text{sp}(g)+\mathsf{K}^{\text{time}}}(s)\, \|g\|_\infty \sup_{(t,\boldsymbol{x})\in(\text{sp}(g)+\mathsf{K}^{\text{time}})\times\mathsf{K}^{\text{space}}} |v_n(t, \boldsymbol{x})|$$
- $$\begin{aligned} |v_n(t, \boldsymbol{x})| &\leq \int_{\mathbb{R}^4} dpd^{N-1}z\, |v(t-p, \boldsymbol{x}-\boldsymbol{z})|\delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z}) \\ &\leq \left(\sup_{(s,\boldsymbol{y})\in\mathsf{B}_1(t)\times\mathsf{B}_1(\boldsymbol{x})} |v(s, \boldsymbol{y})| \right) \int_{\mathbb{R}^4} dpd^{N-1}z\, \delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z}) \\ &= \sup_{(s,\boldsymbol{y})\in\mathsf{B}_1(t)\times\mathsf{B}_1(\boldsymbol{x})} |v(s, \boldsymbol{y})| \,. \end{aligned}$$

Combining these estimates, we find:

$$\begin{aligned} \sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |g(s-t)\, v_n(s, \boldsymbol{x})| &\leq \mathbb{1}_{\text{sp}(g)+\mathsf{K}^{\text{time}}}(s)\, \|g\|_\infty \|v\|_{\infty,(\text{sp}(g)+\mathsf{K}^{\text{time}}+\mathsf{B}_1(0))\times(\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0}))} \\ &=: d_{\mathsf{K}^{\text{time}},\mathsf{K}^{\text{space}}}(s) \,. \end{aligned}$$

By the dominated convergence theorem we find:

$$\lim_{n\to\infty} w_n(t, \boldsymbol{x}) = \lim_{n\to\infty} \int_{\mathbb{R}} ds\, g(s-t)\, v_n(s, \boldsymbol{x}) = \int_{\mathbb{R}} ds\, g(s-t)\, v(s, \boldsymbol{x}) = w_{\text{pw}}(t, \boldsymbol{x}) \tag{6.4}$$

Additionally, as discussed above, the function $(p, \boldsymbol{y}) \mapsto g(s-p)v(s, \boldsymbol{y})$ is continuous at a given point $(t, \boldsymbol{x}) \in \mathbb{R}^N$ for all $s \in N_{\boldsymbol{x}}^{\complement}$ and is dominated by $(t, \boldsymbol{x}, s) \mapsto d_{\mathsf{B}_1(t),\mathsf{B}(\boldsymbol{x})}$ on $\mathsf{B}_1(t) \times \mathsf{B}(\boldsymbol{x}) \times \mathbb{R}$, which leads to the continuity of $w_{\text{pw}}(\,\cdot\,)$ by another application of dominated convergence. Furthermore, this, by the definition of $u * (\delta \otimes [\,\widetilde{\cdot}\,])(\,\cdot\,, \bullet)$ in the case of form-regularity in Def. 6.1.14 Equation (6.4), implies the following equality:

$$w_{\text{pw}}(0, \mathbf{0}) = u * (\delta \otimes \widetilde{g})[\widetilde{f}, \mathbf{0})|_{\text{span}(C_c^\infty \otimes C_c^\infty)} \,.$$

Part III: We start by proving the uniform equicontinuity of $(w_n(\,\cdot\,))_{n\in\mathbb{N}}$ on compact sets by showing that, for all $n \in \mathbb{N}$,

- the function $w_n(\,\cdot\,)$ lies in $C^1(\mathbb{R}^N, \mathbb{C})$ and
- the uniform boundedness of all partial derivatives of $w_n(\,\cdot\,)$ when restricting to arbitrary compact domains of the form $\mathsf{K}^{\text{time}} \times \mathsf{K}^{\text{space}} \subset \mathbb{R}^N$.

Given $n \in \mathbb{N}$, then, by the construction of $v_n(\,\cdot\,)$, lies the function $(t, \boldsymbol{x}) \mapsto g(s-t)v_n(s, \boldsymbol{x})$, for all $s \in \mathbb{R}$, in $C_c^\infty(\mathbb{R}^N, \mathbb{C})$. Furthermore, given $i = 2, \ldots, N$, regard the following estimates which are achieved by the methods of the preceding discussions:

- $$\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_t g(s-t)\, v_n(s, \boldsymbol{x})| \leq \mathbb{1}_{\text{sp}(g)+\mathsf{K}^{\text{time}}}(s) \|\partial g\|_\infty \sup_{(t,\boldsymbol{x})\in(\text{sp}(g)+\mathsf{K}^{\text{time}})\times\mathsf{K}^{\text{space}}} |v_n(t, \boldsymbol{x})|$$
- $$\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} g(s-t)\, v_n(s, \boldsymbol{x})| \leq \mathbb{1}_{\text{sp}(g)+\mathsf{K}^{\text{time}}}(s) \|g\|_\infty \sup_{(t,\boldsymbol{x})\in(\text{sp}(g)+\mathsf{K}^{\text{time}})\times\mathsf{K}^{\text{space}}} |\partial^{\boldsymbol{e}_i} v_n(t, \boldsymbol{x})|$$
- $$|v_n(t, \boldsymbol{x})| \leq \sup_{(s,\boldsymbol{y})\in(\{t\}-\mathsf{B}_1(0))\times(\{\boldsymbol{x}\}\times\mathsf{B}_1(\mathbf{0}))} |v(s, \boldsymbol{y})|$$
- $$|\partial^{\boldsymbol{e}_i} v_n(t, \boldsymbol{x})| \overset{(i)}{\leq} \int_{\mathbb{R}^4} dpd^{N-1}z\, |\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} v(t-p, \boldsymbol{x}-\boldsymbol{z})|\delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z})$$

$$\begin{aligned}&\leq \left(\sup_{(s,\boldsymbol{y})\in(\{t\}-\mathsf{B}_1(0))\times(\{\boldsymbol{x}\}\times\mathsf{B}_1(\mathbf{0}))} |\partial^{\boldsymbol{e}_i} v(s,\boldsymbol{y})|\right)\int_{\mathbb{R}^4} dp d^{N-1}z\, \delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z})\\ &= \sup_{(s,\boldsymbol{y})\in(\{t\}-\mathsf{B}_1(0))\times(\{\boldsymbol{x}\}\times\mathsf{B}_1(\mathbf{0}))} |\partial^{\boldsymbol{e}_i} v(s,\boldsymbol{y})|\,.\end{aligned}$$

The inequality in (i) needs an additional argument, as it involves the commutation of partial derivatives and integration. This is, however, immediate by dominated convergence, since by form-regularity $\boldsymbol{x}\mapsto v(t-p,\boldsymbol{x}-\boldsymbol{z})$ is partial differentiable for all $t,p\in\mathbb{R}$, and $\boldsymbol{z}\in\mathbb{R}^{N-1}$. Further, for all $(p,\boldsymbol{z})\in\mathbb{R}^N$ we find

$$\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} v(t-p,\boldsymbol{x}-\boldsymbol{z})|\delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z}) \leq \|\partial^{\boldsymbol{e}_i} v\|_{\infty,\mathsf{K}^{\text{time}}+\mathsf{B}_1(0)\times\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0})}\delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z})\,,$$

so that $(p,\boldsymbol{z})\mapsto \|\partial^{\boldsymbol{e}_i} v\|_{\infty,\mathsf{K}^{\text{time}}+\mathsf{B}_1(0)\times\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0})}\delta_n^{\text{time}}(p)\delta_n^{\text{space}}(\boldsymbol{z})$ is an integrable dominating function.

Combining the preceding inequalities lead towards the two estimates

- $\displaystyle\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_t g(s-t)\, v_n(s,\boldsymbol{x})| \leq \mathbb{1}_{\text{sp}(g)+\mathsf{K}^{\text{time}}}(s)\|\partial g\|_\infty\|v\|_{\infty,(\text{sp}(g)+\mathsf{K}^{\text{time}}+\mathsf{B}_1(0))\times(\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0}))} =: d_{\boldsymbol{e}_1,\mathsf{K}^{\text{time}},\mathsf{K}^{\text{space}}}(s)$
- $\displaystyle\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} g(s-t)\, v_n(s,\boldsymbol{x})| \leq \mathbb{1}_{\text{sp}(g)+\mathsf{K}^{\text{time}}}(s)\|g\|_\infty\|\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} v\|_{\infty,(\text{sp}(g)+\mathsf{K}^{\text{time}}+\mathsf{B}_1(0))\times(\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0}))} =: d_{\boldsymbol{e}_i,\mathsf{K}^{\text{time}},\mathsf{K}^{\text{space}}}(s)\,,$

which provide dominating functions for another application of the dominated convergence theorem, leading to the partial differentiability of $w_n(\,\cdot\,)$ and, for $(t,\boldsymbol{x})\in\mathbb{R}^4$, to the following equalities:

- $\displaystyle\partial_t w_n(t,\boldsymbol{x}) = \int_{\mathbb{R}} ds\, \partial_t g(s-t)\, v_n(s,\boldsymbol{x})\,,$
- $\displaystyle\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} w_n(t,\boldsymbol{x}) = \int_{\mathbb{R}} ds\, \partial_t g(s-t)\, \partial^{\boldsymbol{e}_i} v_n(s,\boldsymbol{x})\,.$

Applying once more our inequalities leads to the local boundedness of all partial derivatives:

- $\displaystyle\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_t w_n(t,\boldsymbol{x})| \leq \left|\text{sp}(g)+\mathsf{K}^{\text{time}}\right|\|\partial g\|_\infty\|v\|_{\infty,(\text{sp}(g)+\mathsf{K}^{\text{time}}+\mathsf{B}_1(0))\times(\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0}))}$
- $\displaystyle\sup_{(t,\boldsymbol{x})\in\mathsf{K}^{\text{time}}\times\mathsf{K}^{\text{space}}} |\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} w_n(t,\boldsymbol{x})| \leq \left|\text{sp}(g)+\mathsf{K}^{\text{time}}\right|\|g\|_\infty\|\partial_{\boldsymbol{x}}^{\boldsymbol{e}_i} v\|_{\infty,(\text{sp}(g)+\mathsf{K}^{\text{time}}+\mathsf{B}_1(0))\times(\mathsf{K}^{\text{space}}+\mathsf{B}_1(\mathbf{0}))}$

Thereby, for all $n\in\mathbb{N}$, the functions w_n are uniformly equicontinuous on any compact subset of $\mathbb{R}^N$. We are now prepared to prove the pointwise equality of $w_{\mathscr{D}'}(\,\cdot\,)$ and $w_{\text{pw}}(\,\cdot\,)$ on $\mathbb{R}^N$. Fix some point $(t,\boldsymbol{x})$ and a number $\varepsilon>0$. We will pick a number $n\in\mathbb{N}$ and a function h in $C_c^\infty(\mathbb{R}^4,\mathbb{R}_{\geq 0})$, whose integral equals to 1, according to the criteria below, which lead to the following estimate:

$$\begin{aligned}&|w_{\mathscr{D}'}(t,\boldsymbol{x})-w_{\text{pw}}(t,\boldsymbol{x})|\\ &\quad\leq \underbrace{|w_{\mathscr{D}'}(t,\boldsymbol{x})-w_{\mathscr{D}'}[h]|}_{\text{Term I}} + \underbrace{|w_{\mathscr{D}'}[h]-w_n[h]|}_{\text{Term II}} + \underbrace{|w_n[h]-w_n(t,\boldsymbol{x})|}_{\text{Term III}} + \underbrace{|w_n(t,\boldsymbol{x})-w_{\text{pw}}(t,\boldsymbol{x})|}_{\text{Term IV}}\\ &\quad\leq \varepsilon/4+\varepsilon/4+\varepsilon/4+\varepsilon/4\\ &\quad= \varepsilon\,.\end{aligned}$$

The criteria are as follows:

I The support of h must lie in inside the ball $\mathsf{B}_\sigma\big((t,\boldsymbol{x})\big)$ with σ small enough such that, for all $(s,\boldsymbol{y})\in\text{sp}(h)$, the inequality $|w_{\mathscr{D}'}(s,\boldsymbol{y})-w_{\mathscr{D}'}(t,\boldsymbol{x})|\leq\varepsilon/4$ holds true. This is possible by the smoothness of $w_{\mathscr{D}'}$ which leads to the following inequality:

$$|w_{\mathscr{D}'}(t,\boldsymbol{x})-w_{\mathscr{D}'}[h]| \leq \int_{\mathbb{R}^4} h(t,\boldsymbol{y})|w_{\mathscr{D}'}(t,\boldsymbol{x})-w_{\mathscr{D}'}(s,\boldsymbol{y})| \leq \varepsilon/4\|h\|_{L^1} = \varepsilon/4\,.$$

II The number $n \in \mathbb{N}$ must be large enough, such that the inequality $|w_{\mathcal{D}'}[h] - w_n[h]| \leq \varepsilon/4$ holds true. This is possible due to the convergence of $w_n[\,\cdot\,]$ to $w_{\mathcal{D}'}[\,\cdot\,]$ in distributions.

III The support of h must lie inside the ball $\mathsf{B}_{\sigma'}\big((t,\boldsymbol{x})\big)$ with σ' small enough such that, for all $n \in \mathbb{N}$ and $(s,\boldsymbol{y}) \in \mathrm{sp}(h)$, the inequality $|w_n(s,\boldsymbol{y}) - w_n(t,\boldsymbol{x})| \leq \varepsilon/4$ holds true. This is possible due to the uniform equicontinuity on the compact set $\overline{\mathsf{B}_1(t,\boldsymbol{x})}$. It leads, by the same arguments as before, to the inequality $|w_{\mathcal{D}'}(t,\boldsymbol{x}) - w_{\mathcal{D}'}[h]| \leq \varepsilon/4$.

IV The number $n \in \mathbb{N}$ must be large enough such that the inequality $|w_n(t,\boldsymbol{x}) - w_{\mathrm{pw}}(t,\boldsymbol{x})| \leq \varepsilon/4$ holds true. This is possible due to the pointwise convergence of the functions $w_n(\,\cdot\,)$ to $w_{\mathrm{pw}}(\,\cdot\,)$.

Since $\varepsilon > 0$ was arbitrary, we showed $w_{\mathcal{D}'}(t,\boldsymbol{x}) = w_{\mathrm{pw}}(t,\boldsymbol{x})$ concluding the proof.

The notion of distribution-regularity complements the form-regularity in the following sense:

Theorem 6.1.17 Form- and Distribution-Regularity imply Foilablity

Let $u[\,\cdot\,] \in \mathcal{D}'(\mathbb{R}^N)$ such that there exists a jointly form- and distribution-regular representation $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\,\bullet\,)$ of the foliation candidate via functions. Then, $u[\,\cdot\,]$ is foilable and $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\mathbf{0})$ is a foliation of $u[\,\cdot\,]$.

Proof

Since $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\,\bullet\,)$ is form-regular, Theorem 6.1.15 implies that $u[\,\cdot\,] = u * (\delta \otimes [\,\check{\cdot}\,])[\,\check{\cdot}\,,\mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ holds true as an equation in $\mathsf{LF}\big[\mathrm{span}\,\big(C_c^\infty(\mathbb{R},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})\big),\mathbb{C}\big]$. Furthermore, due to the distribution-regularity of the representation, $u * (\delta \otimes [\,\check{\cdot}\,])[\,\check{\cdot}\,,\mathbf{0})$ is defined in $\mathcal{D}'(\mathbb{R}^N)$ and $u * (\delta \otimes [\,\check{\cdot}\,])[\,\check{\cdot}\,,\mathbf{0})|_{\mathrm{span}(C_c^\infty \otimes C_c^\infty)}$ is indeed its restriction to $\mathsf{LF}\big[\mathrm{span}\,\big(C_c^\infty(\mathbb{R},\mathbb{C}) \otimes C_c^\infty(\mathbb{R}^{N-1},\mathbb{C})\big),\mathbb{C}\big]$.

Since the two distributions u and $u * (\delta \otimes [\,\check{\cdot}\,])[\,\check{\cdot}\,,\mathbf{0})$ are continuous and agree on a dense subset of the domain they are equal. Finally, given $g \in C_c^\infty(\mathbb{R}^N,\mathbb{C})$, the following equality holds true:

$$u[g] = \int_{\mathbb{R}} dt\, u * \big(\delta \otimes \widetilde{g}(t,\,\bullet\,)\big)(\text{-}t,\mathbf{0})\,.$$

An interesting special case of Theorem 6.1.17 is given by:

Proposition 6.1.18 $\mathfrak{C}$-differentiable Foliation Candidates

Let $u \in \mathcal{D}'(\mathbb{R}^N)$ such that there exists a representation $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\,\bullet\,)$ of the foliation candidate via functions with the property that $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\mathbf{0})$, defined by

- $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\mathbf{0}) : \mathbb{R} \to \mathcal{D}'(\mathbb{R}^{N-1}), t \mapsto u * (\delta \otimes [\,\check{\cdot}\,])(t,\mathbf{0})$ and
- $u * (\delta \otimes [\,\check{\cdot}\,])(t,\mathbf{0}) : C_c^\infty(\mathbb{R}^{N-1}) \to \mathbb{C}, f \mapsto u * (\delta \otimes \widetilde{f})(t,\mathbf{0})$,

lies,for some $m \in \mathbb{N}$, in $\mathfrak{C}^m\big(\mathbb{R},\mathcal{D}'(\mathbb{R}^{N-1})\big)$. Then, $u[\,\cdot\,] \in \mathfrak{C}^m\big(\mathbb{R},\mathcal{D}'(\mathbb{R}^{N-1})\big) \subset \mathcal{D}'(\mathbb{R} \to \mathbb{R}^{N-1})$ and $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\mathbf{0})$ is a foliation of $u[\,\cdot\,]$.

Proof

We prove the statement by showing that $u * (\delta \otimes [\,\check{\cdot}\,])(\,\cdot\,,\,\bullet\,)$ is form- and distribution-regular.

- Form-regularity is fulfilled due to Def. 6.1.3 of $\mathfrak{C}^1$-continuous differentiability which implies the properties in Items 1 to 5 in Def. 6.1.14.
- Distribution-regularity is a direct consequence of Proposition 6.1.7.

The statement then follows by an application of Theorem 6.1.17.

6.2 Regularity Particle Sourced Solutions

We will analyze the properties of certain convolutions of $G^{\text{adv/ret}}$ with parts of the source $\mathbb{1}_I \rho_{\mathbf{r},\cdot}$ given by an interval $I \subset \mathbb{R}$ and for $\cdot = \delta, \varrho$.

6.2.1 Relation of Point and Extended Charges

Theorem 6.2.1 Sources and Solutions: Point vs. extended Particle

Given the context of Proposition 5.3.1. Then, the following statements are true:

1. $\mathbb{1}_I \rho_{\mathbf{r},\varrho} = \mathbb{1}_I \rho_{\mathbf{r},\delta} * (\delta \otimes \varrho)$
2. $G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho} = \big(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}\big) * (\delta \otimes \varrho)$

Proof

Item 1: First of all, due to Definition/Theorem 6.1.8, $\delta \otimes \varrho$ has a compact support. This leads to the well-definedness of its convolution product with $\mathbb{1}_I \rho_{\mathbf{r},\delta}$. Let us regard $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$ and $g \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, then we follow along the calculation:

$$\begin{aligned}
\mathbb{1}_I \rho_{\mathbf{r},\delta} * (\delta \otimes \varrho)[f \otimes g] &\overset{(i)}{=} \big(\mathbb{1}_I \rho_{\mathbf{r},\delta} * (\delta \otimes \varrho)\big) * (\widetilde{f} \otimes \widetilde{g})(0, \mathbf{0}) \\
&\overset{(ii)}{=} \mathbb{1}_I \rho_{\mathbf{r},\delta} * \big((\delta \otimes \varrho) * (\widetilde{f} \otimes \widetilde{g})\big)(0, \mathbf{0}) \\
&\overset{(iii)}{=} \mathbb{1}_I \rho_{\mathbf{r},\delta} * (\widetilde{f} \otimes \varrho * \widetilde{g})(0, \mathbf{0}) \\
&\overset{(i)}{=} \mathbb{1}_I \rho_{\mathbf{r},\delta}\big[f \otimes \widetilde{\varrho * \widetilde{g}}\big] \\
&\overset{(iv)}{=} \int_{\mathbb{R}} dt\, q f(t) (\varrho * \widetilde{g})\big(-\mathbf{r}(t)\big) \mathbb{1}_I(t) \\
&= \int_{\mathbb{R}} dt\, q f(t) \int_{\mathbb{R}^3} dx^3\, \widetilde{g}(\boldsymbol{x}) \varrho\big(-\mathbf{r}(t) - \boldsymbol{x}\big) \mathbb{1}_I(t) \\
&= \int_{\mathbb{R}^4} dt d^3x\, q f(t) g(\boldsymbol{x}) \varrho\big(\boldsymbol{x} - \mathbf{r}(t)\big) \mathbb{1}_I(t) \\
&\overset{(iv)}{=} \mathbb{1}_I \rho_{\mathbf{r},\varrho}[f \otimes g]\,.
\end{aligned}$$

Steps (i) to (iv) are justified as follows:

(i) By Definition/Lemma 4.1.22 of the convolution of $\mathcal{D}'$ and $\mathcal{D}$.

(ii) Corollary 5.1.15 shows that any number of distributions of which are all but at most one compactly supported have strictly compatible support. Thus, their convolution is, by Item 2 Definition/Theorem 5.1.13, associative and commutative.

(iii) By the well-behaved interplay of convolution and tensor product of strictly compatible supported distributions in Corollary 6.1.11.

(iv) Due to Def. 5.2.1 of $\mathbb{1}_I \rho_{\mathbf{r},\delta}$ and $\mathbb{1}_I \rho_{\mathbf{r},\varrho}$.

Their linearity implies their equality on span $\big(C_c^\infty(\mathbb{R}, \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^3, \mathbb{C})\big)$. According to Lemma 6.1.10, this set is dense in $C_c^\infty(\mathbb{R} \times \mathbb{R}^3, \mathbb{C})$. Thus their continuity implies their equality on all of $C_c^\infty(\mathbb{R}^4, \mathbb{C})$.

Item 2: This is a corollary of Item 1 and the commutativity of the convolution of distributions of Definition/Theorem 5.1.13.

6.2.2 Regularity in the Case of Point Particles

A particular important role is payed by the advanced and retarded times:

Definition **Lemma 6.2.2** **Advanced and retrated space-time points**

Given a smooth function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$, such that there exists $v_{\max} < 1$ and, for all $t \in \mathbb{R}$, the bound $\|\dot{\mathbf{r}}(t)\| < v_{\max}$ holds true. Then, we note that $\left|\operatorname{graph}(\mathbf{r}) \cap \Gamma^{\pm}_{(t,\boldsymbol{x})}\right|$ contains a single point in both $\pm$ cases. We define a tuple of functions via:

$$(r_0^{\pm}, \mathbf{r}^{\pm}) : \mathbb{R}^4 \to \mathbb{R}^4, (t, \boldsymbol{x}) \mapsto \big(r_0^{\pm}(t, \boldsymbol{x}), \mathbf{r}^{\pm}(t, \boldsymbol{x})\big) \text{ where } \Big\{\big(r_0^{\pm}(t, \boldsymbol{x}), \mathbf{r}^{\pm}(t, \boldsymbol{x})\big)\Big\} := \operatorname{graph}(\mathbf{r}) \cap \Gamma^{\text{light},\pm}_{(t,\boldsymbol{x})} .$$

Clearly, $\mathbf{r}\big(r_0^{\pm}\big) = \mathbf{r}^{\pm}$ and the values of these functions are for $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^3$ uniquely defined as the solution of the implicit equations:

- $r_0^{\pm}(t, \boldsymbol{x}) = t \pm \big\|\mathbf{r}\big(r_0^{\pm}(t, \boldsymbol{x})\big) - \boldsymbol{x}\big\|$
- $\mathbf{r}^{\pm}(t, \boldsymbol{x}) = \mathbf{r}\big(t \pm \big\|\mathbf{r}^{\pm}(t, \boldsymbol{x}) - \boldsymbol{x}\big\|\big)$

Furthermore, the restriction $(r_0^{\pm}, \mathbf{r}^{\pm})|_{\mathbb{R}^4 \setminus \operatorname{graph}(\mathbf{r})}$ lies in $C^\infty(\mathbb{R}^4 \setminus \operatorname{graph}(\mathbf{r}), \mathbb{R}^4)$.

Proof

[Har18, Lemma 5.1.1]

Moreover, by applying our tools developed in Section 6.1, we are able to prove the following statement about the pointwise properties of these solutions:

Theorem 6.2.3 **Partial Solutions as Functions**

Given the context of Proposition 5.3.1. Then, $G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ can be represented by a function in $L^1_{loc}(\mathbb{R}^4, \mathbb{R}) \cap C^\infty\big(\mathbb{R}^4 \setminus (\Gamma^{\text{light},\mp}_{\operatorname{graph}(\mathbf{r}|_{\partial I})} \cup \operatorname{graph}(\mathbf{r}|_I)), \mathbb{R}\big)$ and its value at $(t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \operatorname{graph}(\mathbf{r}|_I)$ is given by

$$\big(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}\big)(t, \boldsymbol{x}) = \mathbb{1}_{\Gamma^{\text{light},\mp}_{\operatorname{graph}(\mathbf{r}|_I)}}(t, \boldsymbol{x}) \, \frac{1}{1 \pm \boldsymbol{n}^{\pm}(t, \boldsymbol{x}) \cdot \dot{\mathbf{r}}^{\pm}(t, \boldsymbol{x})} \, \frac{q}{4\pi \|\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{x})\|}$$

with $\boldsymbol{n}^{\pm}(t, \boldsymbol{x}) := \frac{\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{x})}{\|\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{x})\|}$ and $\dot{\mathbf{r}}^{\pm}(t, \boldsymbol{x}) := \dot{\mathbf{r}}\big(r_0^{\pm}(t, \boldsymbol{x})\big)$. In particular, $G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ is time foilable.

Furthermore, given a particle shape $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, then, for all $(t, \boldsymbol{x}) \in \mathbb{R}^4$, we find:

$$\big(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\varrho}\big)(t, \boldsymbol{x}) = \int_{\mathbb{R}^3} d^3y \, \big(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}\big)(t, \boldsymbol{y}) \varrho(\boldsymbol{x} - \boldsymbol{y}) \, .$$

Proof

Strategy: We will apply Theorem 6.1.17 to the distribution $(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta})$ by finding a jointly form- and distribution-regular representation of the foliation candidate $(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}) * (\delta \otimes [\,\tilde{\cdot}\,])[\cdot]$ via functions. The search is based on the observation that, after Item 2 of Theorem 6.2.1, for $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, the following equation applies:

$$(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}) * (\delta \otimes f)[\cdot] = (G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},f})[\cdot] \, .$$

In order of representing $(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},f})[\cdot]$ by some function $v_f \in L^1_{loc}(\mathbb{R}^4, \mathbb{C})$, we approximate $\mathbb{1}_I$ by a series $(\chi_I^n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}, [0, 1])$ which converges pointwisely to $\mathbb{1}_I$. This leads to a representation $v_{f,n} : \mathbb{R}^4 \to \mathbb{C}$ in $C^\infty(\mathbb{R}^4, \mathbb{C})$ of $(G^{\text{adv/ret}} * \chi_I^n \rho_{\mathbf{r},f})[\cdot]$ (**Part I**). Further, for $n \to \infty$, we show

- the distributional convergence of $(G^{\text{adv/ret}} * \chi_I^n \rho_{\mathbf{r},f})[\cdot]$ to $(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},f})[\cdot]$ (**Part II**) and
- the pointwise convergence of $v_{f,n}$. Its limit gets denoted by v_f. We prove further, that it lies in $L^1_{loc}(\mathbb{R}^4, \mathbb{C})$ (**Part III**) and
- the distributional convergence of the embedding of $v_{f,n}$ in $\mathscr{D}'(\mathbb{R}^4)$ to the embedding of v_f in $\mathscr{D}'(\mathbb{R}^4)$ (**Part IV**).

This implies that v_f is a representation of $(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},f})[\,\cdot\,]$ by a function. It remains to be shown in that $(v_{\tilde{f}})_{f \in C_c^\infty(\mathbb{R}^3,\mathbb{C})}$ is indeed a form- (**Part V**) and distribution-regular (**Part VI**) representation. Finally, we show that $G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta}$ is indeed representable by the in the theorem stated function in $L^1_{loc}(\mathbb{R}^4,\mathbb{R}) \cap C^\infty\big(\mathbb{R}^4 \setminus (\Gamma^{\text{light},\mp}_{\text{graph}(\mathbf{r}|_{\partial I})} \cup \text{graph}\,(\mathbf{r}|_I)), \mathbb{R}\big)$ (**Part VII**).

Part I: Since $\chi_I^n \rho_{\mathbf{r},f}$ lies in $C_c^\infty(\mathbb{R}^4,\mathbb{C})$, then, according to Proposition 4.1.23, is the convolution of $G^{\text{adv/ret}}$ and $\chi_I^n \rho_{\mathbf{r},f}$ representable by a smooth functions $v_{f,n}$. Its pointwise evaluation at $(t,\boldsymbol{x}) \in \mathbb{R}^4$ equates to:

$$
\begin{aligned}
v_{f,n}(t,\boldsymbol{x}) &= G^{\text{adv/ret}} * \chi_I^n \rho_{\mathbf{r},f}(t,\boldsymbol{x}) \\
&= G^{\text{adv/ret}}\big[\chi_I^n \rho_{\mathbf{r},f}(t-\,\cdot\,, \boldsymbol{x}-\,\cdot\,)\big] \\
&= \int_{\mathbb{R}} ds\, s^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{y}}) \Theta(\mp s) \frac{1}{4\pi|s|} \chi_I^n(t-s)\, \rho_{\mathbf{r},f}(t-s, \boldsymbol{x}-|s|\hat{\boldsymbol{y}}) \\
&= \int_{\mathbb{R}} ds\, s^2 \int_{\mathsf{S}_1(\mathbf{0})} d\Omega(\hat{\boldsymbol{y}}) \Theta(\mp s) \frac{1}{4\pi|s|} \chi_I^n(t-s)\, qf\big(\boldsymbol{x}-|s|\hat{\boldsymbol{y}}-\mathbf{r}(t-s)\big) \\
&= \int_{\mathbb{R}^3} d^3y\, \frac{1}{4\pi\|\boldsymbol{y}\|} \chi_I^n(t \pm \|\boldsymbol{y}\|)\, qf\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)\,.
\end{aligned}
$$

Part II: The series $(\chi_I^n \rho_{\mathbf{r},f})_{n\in\mathbb{N}}$ of smooth and compactly supported functions lies in $\mathscr{D}'\big(\Gamma^{v_{\max}}_{\{(0,\mathbf{r}(0))\}+\{0\}\times\text{sp}(f)}\big)$ and converges in $\mathscr{D}'$ to $\mathbb{1}_I \rho_{\mathbf{r},f}$ as we will prove by the dominated convergence theorem. Given $g \in C_c^\infty(\mathbb{R}^4,\mathbb{C})$ and $(t,\boldsymbol{x}) \in \mathbb{R}^4$, we can estimate

$$
\big|qf\big(\boldsymbol{x}-\mathbf{r}(t)\big)\chi_I^n(t) g(t,\boldsymbol{x})\big| \leq q\|f\|_\infty |g(t,\boldsymbol{x})|
$$

which provides an integrable dominating function in step (i) of the following calculation:

$$
\begin{aligned}
\lim_{n\to\infty} \chi_I^n \rho_{\mathbf{r},f}[g] &= \lim_{n\to\infty} \rho_{\mathbf{r},f}[\chi_I^n \cdot g] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^4} dt d^3x\, qf\big(\boldsymbol{x}-\mathbf{r}(t)\big) \chi_I^n(t) g(t,\boldsymbol{x}) \\
&\overset{(i)}{=} \int_{\mathbb{R}^4} dt d^3x\, qf\big(\boldsymbol{x}-\mathbf{r}(t)\big) \lim_{n\to\infty} \chi_I^n(t) g(t,\boldsymbol{x}) \\
&= \mathbb{1}_I \rho_{\mathbf{r},f}[g]\,.
\end{aligned}
$$

The sequential continuity of the convolution (see Item 4 of Definition/Theorem 5.1.13) implies, for $n\to\infty$, the convergence of $G^{\text{adv/ret}} * \chi_I^n \rho_{\mathbf{r},f}$ to $G^{\text{adv/ret}} * \chi_I^n \rho_{\mathbf{r},f}$ in distributions.

Part III: To understand the pointwise convergence of $v_{f,n}$ we pick $(t,\boldsymbol{x}) \in \mathbb{R}^4$ and note that, due to $\|\dot{\mathbf{r}}\| \leq v_{\max}$ and the compactness of the support of f, there exists a compact subset $\mathsf{K}_{\{(t,\boldsymbol{x})\},f}$ of $\mathbb{R}^3$, such that, for all $\boldsymbol{y} \in \mathbb{R}^3 \setminus \mathsf{K}_{\{(t,\boldsymbol{x})\},f}$, the expression $f\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)$ equates to zero. Thus, we find an integrable dominating function by the following estimate

$$
\left|\frac{1}{4\pi\|\boldsymbol{y}\|} \chi_I^n(t\pm\|\boldsymbol{y}\|)\, qf\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)\right| \leq \mathbb{1}_{\mathsf{K}_{t,\boldsymbol{x},f}}(\boldsymbol{y}) \frac{q}{4\pi\|\boldsymbol{y}\|}\|f\|_\infty
$$

and we can pull, by dominated convergence and the pointwise convergence of the integrand, the limit $n\to\infty$ through the integration to find:

$$
\begin{aligned}
v_f(t,\boldsymbol{x}) &:= \lim_{\varepsilon\to 0} v_{f,n}(t,\boldsymbol{x}) \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} d^3y\, \frac{1}{4\pi\|\boldsymbol{y}\|} \chi_I^n(t\pm\|\boldsymbol{y}\|)\, qf\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big) \\
&= \int_{\mathbb{R}^3} d^3y\, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t\pm\|\boldsymbol{y}\|)\, qf\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)\,.
\end{aligned}
$$

Given any compact set $\mathsf{K} \subset \mathbb{R}^4$, we can, by the same argumentation as before, find a compact subset $\mathsf{K}_{\mathsf{K},f}$ of $\mathbb{R}^3$, such that, for all $\boldsymbol{y} \in \mathbb{R}^3 \setminus \mathsf{K}_{\mathsf{K},f}$, the expression $f\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)$ equates to zero.

Similarly to before, we estimate

$$\sup_{(t,\boldsymbol{x})\in\mathsf{K}} \left| \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|)\, q f\big(\boldsymbol{x} - \boldsymbol{y} - \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) \right| \leq \mathbb{1}_{\mathsf{K}_{\mathsf{K},f}}(\boldsymbol{y}) \frac{q}{4\pi\|\boldsymbol{y}\|} \|f\|_\infty \tag{6.5}$$

which implies the equality (i) in the following calculation:

$$\begin{aligned}
&\int_{\mathsf{K}} dt d^3x \left| \int_{\mathbb{R}^3} d^3y \, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|)\, q f\big(\boldsymbol{x} - \boldsymbol{y} - \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) \right| \\
&\qquad \leq \int_{\mathsf{K}} dt d^3x \int_{\mathbb{R}^3} d^3y \left| \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|)\, q f\big(\boldsymbol{x} - \boldsymbol{y} - \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) \right| \\
&\qquad \overset{(i)}{\leq} |\mathsf{K}| \left(\int_{\mathbb{R}^3} d^3y \, \mathbb{1}_{\mathsf{K}_{\mathsf{K},f}}(\boldsymbol{y}) \frac{q}{4\pi\|\boldsymbol{y}\|} \right) \|f\|_\infty \\
&\qquad < \infty \,.
\end{aligned}$$

This results in $v_f \in L^1_{loc}(\mathbb{R}^4, \mathbb{C})$.

Part IV: To show that the distributional convergence of the embedding of $v_{f,n}$ in $\mathscr{D}'(\mathbb{R}^4)$ to the embedding of v_f in $\mathscr{D}'(\mathbb{R}^4)$ we need to prove for all $g \in C_c^\infty(\mathbb{R}^4, \mathbb{C})$ the following limit:

$$\lim_{n\to\infty} \int_{\mathbb{R}^4} dt d^3x \, v_{f,n}(t, \boldsymbol{x}) g(t, \boldsymbol{x}) = \int_{\mathbb{R}^4} dt d^3x \, v_f(t, \boldsymbol{x}) g(t, \boldsymbol{x}) \tag{6.6}$$

To use the dominated convergence theorem we construct another dominating function of this integrand $(t, \boldsymbol{x}) \mapsto v_{f,n}(t, \boldsymbol{x}) g(t, \boldsymbol{x})$ in the same way as before by noting that $\mathrm{sp}(g)$ is compact and thus (again due to $\|\dot{\mathbf{r}}\| \leq v_{\max}$) there exists compact $\mathsf{K}_{\mathrm{sp}(g),f} \subset \mathbb{R}^3$ such that for all $\boldsymbol{y} \in \mathbb{R}^3 \setminus \mathsf{K}_{\mathrm{sp}(g),f}$ and $(t, \boldsymbol{x}) \in \mathrm{sp}(g)$ we find $f\big(\boldsymbol{x} - \boldsymbol{y} - \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) = 0$ leading to the estimate

$$\begin{aligned}
|v_{f,n}(t, \boldsymbol{x}) g(t, \boldsymbol{x})| &= \left| \int_{\mathbb{R}^3} d^3y \, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|)\, q f\big(\boldsymbol{x} - \boldsymbol{y} - \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) g(t, \boldsymbol{x}) \right| \\
&\leq \left(\int_{\mathbb{R}^3} d^3y \, \frac{q}{4\pi\|\boldsymbol{y}\|} \, \mathbb{1}_I(t \pm \|\boldsymbol{y}\|) \right) \|f\|_\infty |g(t, \boldsymbol{x})|
\end{aligned}$$

which is integrable on $\mathbb{R}^4$. Furthermore, we proved the pointwise convergence of the integrand already in part III concluding that Equation (6.6) holds true.

Part V: To prove the form-regularity of $(v_{\widetilde{f}})_{f\in C_c^\infty(\mathbb{R}^3,\mathbb{C})}$ we need to show, according to Def. 6.1.14, for all $f \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, the following properties:

1. The function $v_f(\,\cdot\,,\,\boldsymbol{\cdot}\,)$ lies in $L^\infty_{loc}(\mathbb{R}^4, \mathbb{C})$.
2. For all $t \in \mathbb{R}$ is the function $v_f(t, \,\boldsymbol{\cdot}\,) : \mathbb{R}^3 \to \mathbb{C}, \boldsymbol{x} \mapsto v_f(t, \boldsymbol{x})$ partial differentiable. We denote, for all $i \in \{1, \ldots, 3\}$, the function $\mathbb{R}^4 \to \mathbb{C}, (t, \boldsymbol{x}) \mapsto \partial_{x_i} v_f(t, \boldsymbol{x})$ by $\partial_{x_i} v_f(\,\cdot\,,\,\boldsymbol{\cdot}\,)$.
3. For all $i \in \{1, \ldots, 3\}$, the function $\partial_{x_i} v_f(\,\cdot\,,\,\boldsymbol{\cdot}\,)$ lies in $L^\infty_{loc}(\mathbb{R}^4, \mathbb{C})$.
4. For all $\boldsymbol{x} \in \mathbb{R}^3$ possesses the following subset of $\mathbb{R}$ Lebesgue measure 0:
$$N_{\boldsymbol{x}}^{v_f} := \left\{ t \in \mathbb{R} \,\middle|\, v_f(\,\cdot\,,\,\boldsymbol{\cdot}\,) \text{ is not continuous at } (t, \boldsymbol{x}) \right\}.$$
5. For all $\boldsymbol{x} \in \mathbb{R}^3$ and $i \in \{1, \ldots, 3\}$ possesses the following subset of $\mathbb{R}$ Lebesgue measure 0:
$$N_{\boldsymbol{x}}^{i,v_f} := \left\{ t \in \mathbb{R} \,\middle|\, \partial_{x_i} v_f(\,\cdot\,,\,\boldsymbol{\cdot}\,) \text{ is not continuous at } (t, \boldsymbol{x}) \right\}.$$

We remind ourselves, that by defining the function

$$h_f : \mathbb{R}^4 \times \mathbb{R}^3 \to \mathbb{C}, \big((t, \boldsymbol{x}), \boldsymbol{y}\big) \mapsto \begin{cases} \dfrac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|)\, q f\big(\boldsymbol{x} - \boldsymbol{y} - \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) & , \boldsymbol{y} \neq \boldsymbol{0} \\ 0 & , \boldsymbol{y} = \boldsymbol{0} \end{cases}$$

v_f takes at $(t, \boldsymbol{x}) \in \mathbb{R}^4$ the value $\int_{\mathbb{R}^3} d^3y \, h\big((t, \boldsymbol{x}), \boldsymbol{y}\big)$. We check properties 1 to 5 step by step:

1. That $v_f(\,\cdot\,,\,\cdot\,)$ lies in $L^\infty_{loc}(\mathbb{R}^4,\mathbb{C})$ follows directly by the estimate in Equation (6.5) of part III.
2. Given $t\in\mathbb{R}$, the restriction $h|_{\{t\}\times\mathbb{R}^3\times\mathbb{R}^3}$ is, for fixed $\boldsymbol{x}\in\mathbb{R}^3$, Lebesgue integrable in $\boldsymbol{y}$ and, for all fixed $\boldsymbol{y}\in\mathbb{R}^3$, differentiable in $\boldsymbol{x}$. Furthermore, given $i\in\{1,\dots,3\}$ and any compact $\mathsf{K}\subset\mathbb{R}^4$, we pick $\mathsf{K}_{\mathsf{K},f}$ as for the estimate in Equation (6.5) of part III and find analogously the following estimate:
$$\sup_{(t,\boldsymbol{x})\in\mathsf{K}}\left|\frac{1}{4\pi\|\boldsymbol{y}\|}\mathbb{1}_I(t\pm\|\boldsymbol{y}\|)\,q\,\partial_{x_i}f\big(\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)\right|\le\mathbb{1}_{\mathsf{K}_{\mathsf{K},f}}(\boldsymbol{y})\frac{q}{4\pi\|\boldsymbol{y}\|}\|\partial_{x_i}f\|_\infty\,. \tag{6.7}$$
By the dominated convergence theorem is $v_f(t,\,\cdot\,):\mathbb{R}^3\to\mathbb{C},\boldsymbol{x}\mapsto v_f(t,\boldsymbol{x})$ partial differentiable and, for $(t,\boldsymbol{x})\in\mathbb{R}^4$, the equality $\partial_{x_i}v_f(t,\boldsymbol{x})=\int_{\mathbb{R}^3}d^3y\,\partial_{x_i}h\big((t,\boldsymbol{x}),\boldsymbol{y}\big)$ holds true.
3. That $v_f(\,\cdot\,,\,\cdot\,)$ lies in $L^\infty_{loc}(\mathbb{R}^4,\mathbb{C})$ follows directly by the estimate in Equation (6.7).
4. The function h_f is, for fixed $(t,\boldsymbol{x})\in\mathbb{R}^4$, Lebesgue integrable in $\boldsymbol{y}$ and, moreover, for fixed $\boldsymbol{y}\in\mathbb{R}^3$, continuous on $(t,\boldsymbol{x})\in(\mathbb{R}\setminus\partial I)\times\mathbb{R}^3$. Here, ∂I is set of end points of the interval I. Equation (6.5) in part III provides on any compact $\mathsf{K}\subset(\mathbb{R}\setminus\partial I)\times\mathbb{R}^3$ a dominating function. This leads to the continuity of $v_f(\,\cdot\,,\,\cdot\,)$ on $(\mathbb{R}\setminus\partial I)\times\mathbb{R}^3$ and thus the property.
5. By substitution in the preceding argumentation h_f by $\partial_{x_i}h_f$ and Equation (6.5) to Equation (6.7), we find the continuity of $\partial_{x_i}v_f(\,\cdot\,,\,\cdot\,)$ on $(\mathbb{R}\setminus\partial I)\times\mathbb{R}^3$ and thus the property.

Part VI: To show the distribution-regularity of $(v_{\widetilde{f}})_{f\in C_c^\infty(\mathbb{R}^3,\mathbb{C})}$ we need to prove according to Def. 6.1.14 the following properties:

1. For all $g\in C_c^\infty(\mathbb{R}^4,\mathbb{C})$, the functions $\mathbb{R}\to\mathbb{C},t\mapsto v_{g(t,\,\cdot\,)}(t,\mathbf{0})$ lies in $L^1_{loc}(\mathbb{R},\mathbb{C})$. Given $t\in\mathbb{R}$, then $g(t,\,\cdot\,)$ is understood as the function $\mathbb{R}^3\to\mathbb{C},\boldsymbol{x}\mapsto g(t,\boldsymbol{x})$.
2. The following linear form on $C_c^\infty(\mathbb{R}^4,\mathbb{C})$ lies in $\mathscr{D}'(\mathbb{R}^4)$:
$$v_{[\,\cdot\,]}[\,\cdot\,,\mathbf{0}):C_c^\infty(\mathbb{R}^4,\mathbb{C})\to\mathbb{C},g\mapsto\int_{\mathbb{R}}dt\,v_{g(t,\,\cdot\,)}(t,\mathbf{0})\,.$$

We check their fulfillment.:

1. When regarding $g\in C_c^\infty(\mathbb{R}^4,\mathbb{C})$ we show the integrability of the following function:
$$\mathbb{R}\to\mathbb{C},t\mapsto v_{g(t,\,\cdot\,)}(t,\mathbf{0})=\int_{\mathbb{R}^3}d^3y\,\frac{1}{4\pi\|\boldsymbol{y}\|}\mathbb{1}_I(t\pm\|\boldsymbol{y}\|)\,q\,g\big(t,-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)\,.$$
Due to the inequality $\|\dot{\mathbf{r}}\|\le v_{\max}$ and the compactness of the support of g, there exists a compact set $\mathsf{K}_g\subset\mathbb{R}^4$, such that, for all $(t,\boldsymbol{y})\in\mathbb{R}^4\setminus\mathsf{K}_g$, the expression $g\big(t,\boldsymbol{x}-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)$ equates to zero. This leads to the following bound
$$\left|\frac{1}{4\pi\|\boldsymbol{y}\|}\mathbb{1}_I(t\pm\|\boldsymbol{y}\|)\,q\,g\big(t,-\boldsymbol{y}-\mathbf{r}(t\pm\|\boldsymbol{y}\|)\big)\right|\le\frac{1}{4\pi\|\boldsymbol{y}\|}\,\mathbb{1}_{\mathsf{K}_g}(t,\boldsymbol{y})\|g\|_\infty \tag{6.8}$$
where the right-hand side is integrable in $\mathbb{R}^4$.
2. The equality in Equation (6.8) further implies that the absolute value of $v_{[\,\cdot\,]}[\,\cdot\,,\mathbf{0})$ evaluated at some $g\in C_c^\infty(\mathbb{R}^4,\mathbb{C})$ is dominated by:
$$\left(\int_{\mathbb{R}^4}dtd^3y\,\frac{1}{4\pi\|\boldsymbol{y}\|}\,\mathbb{1}_{\mathsf{K}_g}(t,\boldsymbol{y})\right)\|g\|_\infty\,.$$
This proves, by Theorem 4.1.20, the continuity of the linear form $v_{[\,\cdot\,]}[\,\cdot\,,\mathbf{0})$. Thus, it is a distribution.

Part VII: Given given $g\in C_c^\infty(\mathbb{R}^4,\mathbb{C})$, we managed to show the equation
$$(G^{\mathrm{adv/ret}}*\mathbb{1}_I\rho_{\mathbf{r},\delta})[g]=\int_{\mathbb{R}^4}dtd^3y\,\frac{1}{4\pi\|\boldsymbol{y}\|}\mathbb{1}_I(\text{-}t\pm\|\boldsymbol{y}\|)\,q\,\widetilde{g}\big(t,-\boldsymbol{y}-\mathbf{r}(\text{-}t\pm\|\boldsymbol{y}\|)\big)$$

$$= \int_{\mathbb{R}^4} dtd^3y \, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|) \, q \, g\big(t, \boldsymbol{y} + \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big)$$

as a consequence of the application of Theorem 6.1.17. We use the same technique as in the proof of [Dec10, Theorem 4.18] to reformulate the integral on the right-hand side of the preceding equality by considering for given $t \in \mathbb{R}$ the following functions:

$$T_t^\pm : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{y} \mapsto T_t(\boldsymbol{y}) := \boldsymbol{y} + \mathbf{r}(t \pm \|\boldsymbol{y}\|) \, .$$

The following statements hold true:

- For fixed $t \in \mathbb{R}$, the map $T_t^\pm$ is bijective. Given $\boldsymbol{z} \in \mathbb{R}^3$, the equation $\boldsymbol{z} = \boldsymbol{y} + \mathbf{r}(t \pm \|\boldsymbol{y}\|)$ is equivalent to the equation $(\boldsymbol{z} - \boldsymbol{y}) = \mathbf{r}\big(t \pm \|(\boldsymbol{z} - \boldsymbol{y}) - \boldsymbol{z}\|\big)$ and has according to Definition/Lemma 6.2.2 the unique solution $\boldsymbol{z} - \boldsymbol{y} = \mathbf{r}^\pm(t, \boldsymbol{z})$ which is again equivalent to $\boldsymbol{y} = \boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})$. This implies that $(T_t^\pm)^{-1} : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{z} \mapsto \boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})$ is the inverse of $T_t^\pm$.
- According to the discussion in [Dec10, Theorem 4.18], $(T_t^\pm)^{-1}$ is continuously differentiable on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ and the value of its Jacobian determinant at $\boldsymbol{z} \in \mathbb{R}^3$ is provided by the following equation:

$$\det D(T_t^\pm)^{-1}(\boldsymbol{z}) = \left(1 \pm \frac{\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})}{\|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\|} \cdot \dot{\mathbf{r}}\big(r_0^\pm(t, \boldsymbol{z})\big)\right) = \frac{1}{1 \pm \boldsymbol{n}^\pm(t, \boldsymbol{z}) \cdot \dot{\mathbf{r}}^\pm(t, \boldsymbol{z})} \, .$$

$T_t^\pm$ is thus regular enough to perform a substitution $\boldsymbol{z} := T_t(\boldsymbol{y})$ in equation (ii) of the calculation

$$\begin{aligned}
&(G^{\text{adv/ret}} * \mathbb{1}_I \rho_{\mathbf{r},\delta})[g] \\
&\quad = \int_{\mathbb{R}^4} dtd^3y \, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|) \, q \, g\big(t, \boldsymbol{y} + \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) \\
&\quad \overset{(i)}{=} \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3y \, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_I(t \pm \|\boldsymbol{y}\|) \, q \, g\big(t, \boldsymbol{y} + \mathbf{r}(t \pm \|\boldsymbol{y}\|)\big) \\
&\quad \overset{(ii)}{=} \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3z \, \mathbb{1}_I\big(t \pm \big\|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\big\|\big) \, \frac{1}{1 \pm \boldsymbol{n}^\pm(t, \boldsymbol{z}) \cdot \dot{\mathbf{r}}^\pm(t, \boldsymbol{z})} \, \frac{q}{4\pi\|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\|} \, g(t, \boldsymbol{z}) \\
&\quad \overset{(i)}{=} \int_{\mathbb{R}^3} dtd^3z \, \mathbb{1}_I\big(t \pm \big\|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\big\|\big) \, \frac{1}{1 \pm \boldsymbol{n}^\pm(t, \boldsymbol{z}) \cdot \dot{\mathbf{r}}^\pm(t, \boldsymbol{z})} \, \frac{q}{4\pi\|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\|} \, g(t, \boldsymbol{z}) \\
&\quad \overset{(iii)}{=} \int_{\mathbb{R}^3} dtd^3z \, \mathbb{1}_{\Gamma^{\text{light},\mp}_{\text{graph}(\mathbf{r}|_I)}}(t, \boldsymbol{z}) \, \frac{1}{1 \pm \boldsymbol{n}^\pm(t, \boldsymbol{z}) \cdot \dot{\mathbf{r}}^\pm(t, \boldsymbol{z})} \, \frac{q}{4\pi\|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\|} \, g(t, \boldsymbol{z})
\end{aligned}$$

while providing the following justification for the remaining equalities:

(i) is due to Tonelli's Theorem.

(ii) follows by noting, that for $(t, \boldsymbol{z}) \in \mathbb{R}^4$ both

- the equation $t \pm \|\boldsymbol{z} - \mathbf{r}^\pm(t, \boldsymbol{z})\| = t \pm \big\|\boldsymbol{z} - \mathbf{r}\big(r_0^\pm(t, \boldsymbol{z})\big)\big\| = \mathbf{r}_0^\pm(t, \boldsymbol{z})$ holds true and
- the the statements $\mathbf{r}_0^\pm(t, \boldsymbol{z}) \in I$ is equivalent to $(t, \boldsymbol{z}) \in \Gamma^{\text{light},\mp}_{\text{graph}(\mathbf{r}|_I)}$.

Chapter 7

Selection of the Potential

The chapter provides the proofs backing the statements in Section 2.4.2.

Section Summaries:

7.1 Dressing: For trajectories of constant velocity the expression for the functional representation of the function will be simplified even further.

7.2 Germination of the Dressing: We provide the convergence behavior of the by an compact interval determined piece of particle trajectory sourced solutions constructed in Section 5.3 when sending on of the boundary of this interval to plus or minus infinity.

7.1 Dressing

Definition Lemma 7.1.1

If the trajectory takes the form of $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \boldsymbol{x}_0 + \boldsymbol{v}t$ for $\boldsymbol{x}_0, \boldsymbol{v} \in \mathbb{R}^3$ with $\|\boldsymbol{v}\| < 1$, then, for $(t, \boldsymbol{x}) \in \mathbb{R}^4$, we find that the following statements hold true:

1. $r_0^{\pm}(t,\boldsymbol{x}) = \gamma^2(\boldsymbol{v})\big(t - (\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big) \pm \gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\|$.
2. $\big\|\boldsymbol{x} - \mathbf{r}^{\pm}(t,\boldsymbol{x})\big\| = \pm r_0^{\pm}(t,\boldsymbol{x}) \mp t = \pm\gamma^2(\boldsymbol{v})\big(t - (\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big) + \gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\| \mp t$.
3. $\big(\boldsymbol{x} - \mathbf{r}^{\pm}(t,\boldsymbol{x})\big)\cdot\dot{\mathbf{r}}^{\pm}(t,\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v} - \gamma^2(\boldsymbol{v})\big(t - (\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big) \mp \gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\|$.

Hereby, we define $\gamma(\boldsymbol{v}) := (1 - \|\boldsymbol{v}\|^2)^{-1/2}$ and $\mathsf{L}(\boldsymbol{v}) := \mathbb{1} + (\gamma(\boldsymbol{v}) - 1)/\|\boldsymbol{v}\|^2 \boldsymbol{v}\otimes\boldsymbol{v}$.

Proof

Given $(t,\boldsymbol{x}) \in \mathbb{R}^4$, we define $\boldsymbol{y} := \boldsymbol{x} - \boldsymbol{x}_0$ and choose $\boldsymbol{e}_\parallel, \boldsymbol{e}_\perp \in \mathsf{S}_1(\mathbf{0})$, $r \geq 0$ and $a, b \in \mathbb{R}$ such that the following statements hold true:

- $\boldsymbol{e}_\parallel \cdot \boldsymbol{e}_\perp = 0$.
- $\boldsymbol{y} = r\,\boldsymbol{e}_\parallel$.
- $\boldsymbol{v} = a\,\boldsymbol{e}_\parallel + b\,\boldsymbol{e}_\perp$.

Item 1: Furthermore, by Definition/Lemma 6.2.2, the equation $\big|t - r_0^{\pm}(t,\boldsymbol{x})\big|^2 = \|\boldsymbol{x} - \mathbf{r}(t)\|^2 = \|\boldsymbol{x} - \boldsymbol{x}_0 + \boldsymbol{v}t\|^2$ holds true. By abbreviating $\tau_\pm := r_0^{\pm}(t,\boldsymbol{x})$, this leads to the following calculation:

$$(t - \tau_\pm)^2 = \big\|r\,\boldsymbol{e}_\parallel - a\,\boldsymbol{e}_\parallel - b\,\boldsymbol{e}_\perp\big\|^2 \iff (t - tau_\pm)^2 = (r - a\tau_\pm)^2 + b^2\tau_\pm^2$$

$$\begin{aligned}&\iff \tau_\pm^2(1-a^2-b^2)-2\tau_\pm^2(t-ar)+t^2-r^2=0\\&\iff \tau_\pm^2-2\tau_\pm^2\frac{t-\boldsymbol{y}\cdot\boldsymbol{v}}{1-\|\boldsymbol{v}\|^2}+\frac{t^2-\|\boldsymbol{y}\|^2}{1-\|\boldsymbol{v}\|^2}=0\,. \end{aligned}\tag{7.1}$$

In preparation, we first note that the equation

$$(t-\boldsymbol{y}\cdot\boldsymbol{v})^2-(1-\|\boldsymbol{v}\|^2)\big(t^2-\|\boldsymbol{y}\|^2\big)=\|\boldsymbol{v}\|^2\,t^2-2\boldsymbol{y}\cdot\boldsymbol{v}\,t+\|\boldsymbol{y}\|^2(1-\|\boldsymbol{v}\|^2)+(\boldsymbol{y}\cdot\boldsymbol{v})^2$$

holds true. Using this result, the solution of Equation (7.1) takes the form

$$\gamma^2(\boldsymbol{v})\left(t-(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\pm\sqrt{\|\boldsymbol{v}\|^2\,t^2-2(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\,t+\|\boldsymbol{x}-\boldsymbol{x}_0\|^2(1-\|\boldsymbol{v}\|^2)+\big((\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big)^2}\right),$$

which is value of $r_0^\pm(t,\boldsymbol{x})$. On the other hand, with $\mathsf{L}(\boldsymbol{v})$ defined as $\mathbb{1}+(\gamma(\boldsymbol{v})-1)/\|\boldsymbol{v}\|^2\boldsymbol{v}\otimes\boldsymbol{v}$, we regard

$$\begin{aligned}\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0)-\gamma(\boldsymbol{v})\boldsymbol{v}t\|^2&=\Big\|\boldsymbol{e}_\parallel\big(r+(\gamma-1)/\|\boldsymbol{v}\|^2r\,a^2-\gamma a\,t\big)+\boldsymbol{e}_\perp b\big((\gamma-1)/\|\boldsymbol{v}\|^2r\,a-\gamma t\big)\Big\|^2\\&=\big(r+(\gamma-1)/\|\boldsymbol{v}\|^2r\,a^2-\gamma a\,t\big)^2+b^2\big((\gamma-1)/\|\boldsymbol{v}\|^2r\,a-\gamma t\big)^2\\&=t^2(\text{Term I})+t\,r(\text{Term II})+r^2(\text{Term III})\end{aligned}$$

and by separate calculations

- $$\begin{aligned}(\text{Term I})\quad&=\gamma^2a^2+\gamma^2b^2\\&=\gamma^2\|\boldsymbol{v}\|^2\end{aligned}$$
- $$\begin{aligned}(\text{Term II})\quad&=2(-\gamma a)\big(1+(\gamma-1)/\|\boldsymbol{v}\|^2a^2\big)-2\gamma(\gamma-1)/\|\boldsymbol{v}\|^2a\,b^2\\&=-2\gamma a+2\gamma/\|\boldsymbol{v}\|^2a^3-2\gamma^2/\|\boldsymbol{v}\|^2a^3+2\gamma/\|\boldsymbol{v}\|^2a\,b^2-2\gamma^2/\|v\|^2a\,b^2\\&=-2\gamma a\big(1-(a^2+b^2)/\|\boldsymbol{v}\|^2\big)-2\gamma^2/\|\boldsymbol{v}\|^2a(a^2+b^2)\\&=-2\gamma^2a\end{aligned}$$
- $$\begin{aligned}(\text{Term III})\quad&=1+2(\gamma-1)/\|\boldsymbol{v}\|^2a^2+(\gamma^2-2\gamma+1)/\|\boldsymbol{v}\|^4a^4+(\gamma^2-2\gamma+1)/\|\boldsymbol{v}\|^4a^2\,b^2\\&=1+(2\gamma-2)/\|\boldsymbol{v}\|^2a^2+(\gamma^2-2\gamma+1)/\|\boldsymbol{v}\|^2a^2\\&=1+(\gamma^2-1)/\|\boldsymbol{v}\|^2a^2\\&=\gamma^2(1-\|\boldsymbol{v}\|^2)\big(1+(1-(1-\|\boldsymbol{v}\|^2))/(1-\|\boldsymbol{v}\|^2)/\|\boldsymbol{v}\|^2a^2\big)\\&=\gamma^2(1-\|\boldsymbol{v}\|^2+a^2)\end{aligned}$$

we find:

$$\begin{aligned}\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0)-\gamma(\boldsymbol{v})\boldsymbol{v}t\|^2&=\gamma^2\left(t^2\|\boldsymbol{v}\|^2-2t\,r\,a+r^2(1-\|\boldsymbol{v}\|^2)+(r\,a)^2\right)\\&=\gamma^2\left(\|\boldsymbol{v}\|^2\,t^2-2(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\,t+\|\boldsymbol{x}-\boldsymbol{x}_0\|^2(1-\|\boldsymbol{v}\|^2)+\big((\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big)^2\right)\end{aligned}$$

Putting all together, our calculation results in:

$$r_0^\pm(t,\boldsymbol{x})=\gamma^2(\boldsymbol{v})\big(t-(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big)\pm\gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0)-\gamma(\boldsymbol{v})\boldsymbol{v}t\|\,.$$

Item 2: According to Definition/Lemma 6.2.2 and Item 1 we calculate:

$$\big\|\boldsymbol{x}-\mathbf{r}^\pm(t,\boldsymbol{x})\big\|=\pm r_0^\pm(t,\boldsymbol{x})\mp t=\pm\gamma^2(\boldsymbol{v})\big(t-(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big)+\gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0)-\gamma(\boldsymbol{v})\boldsymbol{v}t\|\mp t\,.$$

Item 3: We find, by replacing $\mathbf{r}^\pm(t,\boldsymbol{x})=\boldsymbol{x}_0+vr_0^\pm(t,\boldsymbol{x})$ and the results of Item 1, the following equation:

$$\big(\boldsymbol{x}-\mathbf{r}^\pm(t,\boldsymbol{x})\big)\cdot\dot{\mathbf{r}}^\pm(t,\boldsymbol{x})=(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}-\gamma^2(\boldsymbol{v})\big(t-(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{v}\big)\mp\gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{x}_0)-\gamma(\boldsymbol{v})\boldsymbol{v}t\|$$

Theorem 7.1.2 Boosted Coulomb Potential

Given a trajectory of the form $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \boldsymbol{x}_0 + \boldsymbol{v}t$ for $\boldsymbol{x}_0, \boldsymbol{v} \in \mathbb{R}^3$ with $\|\boldsymbol{v}\| < 1$ and for all $(t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \operatorname{graph}(\mathbf{r})$, the following equation holds true:

$$\big(G^{\text{adv/ret}} * \rho_{\mathbf{r},\delta}\big)(t, \boldsymbol{x}) = \frac{q\,\gamma(\boldsymbol{v})}{4\pi\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\|} .$$

Proof

Given $(t, \boldsymbol{x}) \in \mathbb{R}^4$, then Theorem 6.2.3 provides the pointwise evaluation of $G^{\text{adv/ret}} * \rho_{\mathbf{r},\delta}$. We just need to apply the results of Definition/Lemma 7.1.1 to get together with $\dot{\mathbf{r}}^{\pm}(t, \boldsymbol{x}) = \boldsymbol{v}$ to get:

$$\begin{aligned}
&\frac{1}{1 \pm \boldsymbol{n}^{\pm}(t, \boldsymbol{x}) \cdot \dot{\mathbf{r}}^{\pm}(t, \boldsymbol{x})} \frac{q}{4\pi\|\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{z})\|} \\
&\quad= \frac{q}{4\pi} \frac{1}{\|\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{z})\| \pm \big(\boldsymbol{x} - \mathbf{r}^{\pm}(t, \boldsymbol{x})\big) \cdot \dot{\mathbf{r}}^{\pm}(t, \boldsymbol{x})} \\
&\quad= \frac{q}{4\pi} \times \begin{pmatrix} \pm\,\gamma^2(\boldsymbol{v})\big(t - (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{v}\big) + \gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\| \mp t \\ \pm\,(\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{v} \mp \|\boldsymbol{v}\|^2\gamma^2(\boldsymbol{v})\big(t - (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{v}\big) - \|\boldsymbol{v}\|^2\gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\| \end{pmatrix}^{-1} \\
&\quad= \frac{q}{4\pi} \times \begin{pmatrix} \pm\,\gamma^2(\boldsymbol{v})(1 - \|\boldsymbol{v}\|^2)\big(t - (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{v}\big) \mp \big(t - (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{v}\big) \\ +\,(1 - \|\boldsymbol{v}\|^2)\gamma(\boldsymbol{v})\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\| \end{pmatrix}^{-1} \\
&\quad= \frac{q\,\gamma(\boldsymbol{v})}{4\pi\|\mathsf{L}(\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_0) - \gamma(\boldsymbol{v})\boldsymbol{v}t\|} .
\end{aligned}$$

7.2 Germination of the Dressing

In preparation, we regard the following convergence of the particle trajectory sources:

Lemma 7.2.1

Given a smooth trajectory $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ ($\|\dot{\mathbf{r}}(t)\| \le v_{\max} < 1\, \forall t \in \mathbb{R}$) and $t_0 \in \mathbb{R} \cup \{\pm\infty\}$, then, for $T \to \infty$, the following statement hold true:

$$\mathbb{1}_{(-T,t_0)}\rho_{\mathbf{r},\delta} \xrightarrow{\mathscr{D}'} \mathbb{1}_{(-\infty,t_0)}\rho_{\mathbf{r},\delta} \quad \wedge \quad \mathbb{1}_{(t_0,+T)}\rho_{\mathbf{r},\delta} \xrightarrow{\mathscr{D}'} \mathbb{1}_{(t_0,+\infty)}\rho_{\mathbf{r},\delta} .$$

Proof

The differences $\mathbb{1}_{(-T,t_0]}\rho_{\mathbf{r},\delta} - \mathbb{1}_{(-\infty,t_0]}\rho_{\mathbf{r},\delta}$ and $\mathbb{1}_{[t_0,+T)}\rho_{\mathbf{r},\delta} - \mathbb{1}_{[t_0,+\infty)}\rho_{\mathbf{r},\delta}$ are supported outside $\mathsf{B}_T(\mathbf{0}) \subset \mathbb{R}^4$ and thus, given $n \in \mathbb{N}$, for large enough T outside of $[-n, +n]^4$. This implies the convergence in $\mathscr{D}'$.

Theorem 7.2.2 Asymptotic Convergence of the Solutions

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ be a smooth trajectory such that there exists $0 \le v_{\max} < 1$ with the property that $\|\dot{\mathbf{r}}(t)\| \le v_{\max}$ for all $t \in \mathbb{R}$. Then, the following convergences hold true:

Convergence as Distributions on Space-Time:

- $G^{\text{adv/ret}} * \mathbb{1}_{(T,t_0)}\rho_{\mathbf{r},\delta} \xrightarrow{\mathscr{D}'(\mathbb{R}^4)} G^{\text{adv/ret}} * \mathbb{1}_{(-\infty,t_0)}\rho_{\mathbf{r},\delta}$ for $T \to -\infty$
- $G^{\text{adv/ret}} * \mathbb{1}_{(t_0,T]}\rho_{\mathbf{r},\delta} \xrightarrow{\mathscr{D}'(\mathbb{R}^4)} G^{\text{adv/ret}} * \mathbb{1}_{(t_0,\infty)}\rho_{\mathbf{r},\delta}$ for $T \to \infty$

Convergence of the Time Foliation in Distributions on Space: For all $t \in \mathbb{R}$

- $\big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(T,t_0)}\rho_{\mathbf{r},\delta}\big)(t)[\,\cdot\,] \xrightarrow{\mathscr{D}'(\mathbb{R}^3)} \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(-\infty,t_0)}\rho_{\mathbf{r},\delta}\big)(t)[\,\cdot\,]$ for $T \to -\infty$
- $\big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(t_0,T]}\rho_{\mathbf{r},\delta}\big)(t)[\,\cdot\,] \xrightarrow{\mathscr{D}'(\mathbb{R}^3)} \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(t_0,\infty)}\rho_{\mathbf{r},\delta}\big)(t)[\,\cdot\,]$ for $T \to \infty$

Pointwise Convergence: For all $(t,\boldsymbol{x}) \in \mathbb{R}^4 \setminus \mathrm{graph}(\mathbf{r}|_I)$

- $\big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(T,t_0)}\rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x}) \to \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(-\infty,t_0)}\rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x})$ for $T \to -\infty$
- $\big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(t_0,T]}\rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x}) \to \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(t_0,\infty)}\rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x})$ for $T \to \infty$

Proof

Convergence as Distributions on Space-Time: According to Proposition 5.2.3 have all the involved distributions strictly compatible supports. By the combination of $\mathscr{D}'$-sequential continuity of the convolution (see Item 3 of Definition/Theorem 5.1.10) and Lemma 7.2.1 the statement follows.

Convergence of the Time Foliation in Distributions on Space: We take advantage of the expression $\big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(T,t_0)}\rho_{\mathbf{r},\delta}\big)(t)[\,\cdot\,]$ in part III of the proof of Theorem 6.2.3. Given $f \in C_c^\infty(\mathbb{R}^3,\mathbb{C})$, the calculation

$$
\begin{aligned}
\lim_{T\to-\infty} \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(T,t_0)}\rho_{\mathbf{r},\delta}\big)(t)[f] &= \lim_{T\to-\infty} \int_{\mathbb{R}^3} d^3y\, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_{(T,t_0)}(t \pm \|\boldsymbol{y}\|)\, qf\big(-\boldsymbol{y} - \mathbf{r}(t\pm\|\boldsymbol{y}\|)\big) \\
&\overset{(i)}{=} \int_{\mathbb{R}^3} d^3y\, \frac{1}{4\pi\|\boldsymbol{y}\|} \lim_{T\to-\infty} \mathbb{1}_{(T,t_0)}(t \pm \|\boldsymbol{y}\|)\, qf\big(-\boldsymbol{y} - \mathbf{r}(t\pm\|\boldsymbol{y}\|)\big) \\
&= \int_{\mathbb{R}^3} d^3y\, \frac{1}{4\pi\|\boldsymbol{y}\|} \mathbb{1}_{(-\infty,t_0)}(t \pm \|\boldsymbol{y}\|)\, qf\big(-\boldsymbol{y} - \mathbf{r}(t\pm\|\boldsymbol{y}\|)\big) \\
&= \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(-\infty,t_0)}\rho_{\mathbf{r},\delta}\big)(t)[f]
\end{aligned}
$$

holds true. Step (i) is a consequence of the dominated convergence theorem and the estimate in Equation (6.5). The other convergence follows by an analog line of argumentation.

Pointwise Convergence: Due to Theorem 6.2.3, we find that the equation

$$
\begin{aligned}
\lim_{T\to-\infty} \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(T,t_0)}\rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x}) &= \lim_{T\to-\infty} \mathbb{1}_{\Gamma^{\mathrm{light},\mp}_{\mathrm{graph}(\mathbf{r}|_{(T,t_0)})}}(t,\boldsymbol{x})\, \frac{1}{1 \pm \boldsymbol{n}^{\pm}\cdot\dot{\mathbf{r}}^{\pm}(t,\boldsymbol{x})}\, \frac{q}{4\pi\|\boldsymbol{x} - \mathbf{r}^{\pm}(t,\boldsymbol{x})\|} \\
&= \mathbb{1}_{\Gamma^{\mathrm{light},\mp}_{\mathrm{graph}(\mathbf{r}|_{(-\infty,t_0)})}}(t,\boldsymbol{x})\, \frac{1}{1 \pm \boldsymbol{n}^{\pm}\cdot\dot{\mathbf{r}}^{\pm}(t,\boldsymbol{x})}\, \frac{q}{4\pi\|\boldsymbol{x} - \mathbf{r}^{\pm}(t,\boldsymbol{x})\|} \\
&= \big(G^{\mathrm{adv/ret}} * \mathbb{1}_{(-\infty,t_0)}\rho_{\mathbf{r},\delta}\big)(t,\boldsymbol{x})
\end{aligned}
$$

holds true. The other convergence follows by an analog line of argumentation.

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