Lieb-Thirring inequalities

László Erdős*

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1 Introduction

Consider the one particle Schrödinger operator $h = -\Delta + V$ acting on $L^2(\mathbf{R}^d)$ (as always, it is not necessary for this discussion to define the operator domain precisely, we will consider only low lying eigenvalues that can be defined via variational principle from the quadratic form $\mathcal{E}(\psi) = \int |\nabla\psi|^2 + V|\psi|^2)$. We will always assume that $V \in L^{d/2} + L^\infty$ (if $d \geq 3$) or $V \in L^{1+\varepsilon} + L^\infty$ (for $d = 2$) or $V \in L^1 + L^\infty$ (for $d = 1$) and we assume that V vanishes at infinity. Let $E_1 \leq E_2 \leq \ldots < 0$ be the negative eigenvalues of h.

Let H_N be the Hamiltonian of N noninteracting particles, each subject to h, i.e.

$$H_N = \sum_{j=1}^N h_j$$

and assume that the particles are fermions. Then we have shown that the ground state energy E_N of H_N (on the fermionic subspace) is given by the sum of the lowest N eigenvalue

$$E_{gs} = \inf\Big\{\langle\psi, H_N\psi\rangle,\ \|\psi\| = 1, \psi \in \bigwedge_1^N L^2(\mathbf{R}^d)\Big\} = \sum_{j=1}^N E_j$$

or, if there are less than N eigenvalues, then E_{gs} is the sum of all eigenvalues. Moreover, if the number of negative eigenvalues is finite, then this is the number of particles in the absolute ground state (if the number of particles is allowed to change).

*Part of these notes were prepared by using the webnotes by Michael Loss: "Stability of Matter" and the draft of a forthcoming book by Elliott Lieb and Robert Seiringer: "The Stability of Matter in Quantum Mechanics". I am grateful for the authors to make a draft version of the book available to me before publication.

All these justify to consider the so-called **Riesz-means** of negative eigenvalues

$$\sum_{j\,:\,E_j<0} |E_j|^\gamma$$

where $\gamma \geq 0$ is a real parameter.

Remark: With the trace notation

$$\mathrm{Tr}[h]_-^\gamma = \sum_{j\,:\,E_j<0} |E_j|^\gamma$$

where $[h]_-$ denotes the negative part of the operator h. If h were a compact operator, then one could define $[h]_-^\gamma$ by spectral theorem:

$$h = \sum_j E_j |\phi_j\rangle\langle\phi_j|, \qquad [h]_-^\gamma = \sum_j (E_j)_-^\gamma |\phi_j\rangle\langle\phi_j| = \sum_{j\,:\,\lambda_j<0} |E_j|^\gamma |\phi_j\rangle\langle\phi_j|$$

so clearly $\mathrm{Tr}[h]_-^\gamma = \sum_{E_j<0} |E_j|^\gamma$. In the general case, $[h]_-^\gamma$ could be defined via the spectral theorem for unbounded operators. For our case however, the definition $\sum_{E_j<0} |E_j|^\gamma$ will be sufficient.

The most important Riesz mean is the $\gamma = 1$ case, this gives the sum of negative eigenvalues. It is now a fundamental question to give a good lower bound on the sum of negative eigenvalues since this will be a lower bound on the fermionic ground state energy. For convenience we consider an upper bound on the sum of absolute values; it is more convenient to deal with positive quantities. The Riesz mean with $\gamma = 0$ gives the number of eigenvalues.

If V is non-negative, then no negative eigenvalues exist. If V has both negative and positive parts, $V = V_+ - V_-$, then of course both parts influence the negative eigenvalues, but by variational principle we know that the eigenvalues of $-\Delta + V$ can be bounded (from below) by those of $-\Delta - V_-$. The Lieb-Thirring bound will be insensitive to the effect of positive part of the potential, it actually estimates the negative eigenvalues of $-\Delta - V_-$ and only V_- plays a role in the estimate.

Theorem 1.1 *[Lieb-Thirring inequality] Fix $\gamma \geq 0$ and assume that $V_- \in L^{\gamma+d/2}(\mathbf{R}^d)$. Then there is a finite constant $L_{\gamma,d}$ such that*

$$\sum_{E_j<0} |E_j|^\gamma \leq L_{\gamma,d} \int V_-^{\gamma+d/2} \tag{1.1}$$

in the following cases:

$$\gamma \geq \frac{1}{2} \quad \textit{if } d = 1$$
$$\gamma > 0 \quad \textit{if } d = 2$$
$$\gamma \geq 0 \quad \textit{if } d = 3$$

Remarks. (i) The constant $L_{\gamma,d}$ is explicitly known (see e.g. Lieb-Loss Theorem 12.4) This constant is known to be optimal in some cases, but not in the most interesting case of $d = 3$, $\gamma = 1$.

(ii) Simple scaling shows that $\gamma + d/2$ is the *only* possible exponent for which such an inequality may hold.

(iii) The case $\gamma = 0$ and $d \geq 3$ is proven quite differently, it is also known as the Cwickel-Lieb-Rozenblum (CLR) bound.

2 Semiclassical heuristics

Recall that the classical phase space is the collection of points $(x, p) \in \mathbf{R}^d \times \mathbf{R}^d$. The uncertainty principle forbids a quantum state to have a definite position and momentum, e.g. we know that

$$\langle \psi, x^2 \psi \rangle \, \langle \psi, p^2 \psi \rangle \geq \frac{1}{4}$$

(recall that $p = -i\nabla$ is the momentum operator). If we try to fit a Gaussian wave function about a point $x = 0$ with a width σ, i.e. we take

$$\psi(z) = e^{-z^2/2\sigma^2}$$

then its Fourier transform

$$\widehat{\psi}(k) = (2\pi)^d e^{-(2\pi\sigma k)^2/2}$$

will be supported about zero with a width $(2\pi\sigma)^{-1}$. In phase space, a quantum state has to occupy a volume of at least order 1. More precise calculation with Gaussian functions show that the occupied volume is at least $(2\pi)^d$.

In the semiclassical picture of quantum mechanics we assume that there is a quantum state $\psi_{x,k}(z)$ sitting in a phase space volume of size $(2\pi)^d$ about each point (x, k). Sometimes the following convention is used: set $p = 2\pi k$ to be the classical momentum and the coherent states are labelled with $p = 2\pi k$ instead of k; then the phase space volume of each state is at least 1. The energy of that state is given by the value of the classical Hamiltonian at point (x, k):

$$\langle \psi_{x,k}, (-\Delta + V)\psi_{x,k} \rangle \approx (2\pi k)^2 + V(x) = p^2 + V(x) \tag{2.2}$$

This statement in general is wrong, but approximately it is correct in the semiclassical limit, i.e. if we put back the semiclassical parameter $\hbar = h/2\pi$ ("Planck constant"). In this case the Hamiltonian becomes $-\hbar^2\Delta + V$ (recall that the momentum is $-i\hbar\nabla$) and the semiclassical picture predicts that there is one state per $(2\pi\hbar)^d = h^d$ volume. If k denotes the Fourier variable, then the momentum $-i\hbar\nabla$ is given by $2\pi\hbar k = hk$.

For a coherent state that lives both in position space and in momentum space on a scale $h^{1/2}$, e.g.

$$\psi_{x,k}(z) = (\pi\hbar)^{-d/4} e^{2\pi i\hbar^{-1}k\cdot z} e^{-(z-x)^2/2\hbar}$$
$$\widehat{\psi}_{x,k}(\xi) = 2^{d/2}(\pi\hbar)^{d/4} e^{2\pi i(k/\hbar-\xi)\cdot x} e^{-[2\pi(k-\hbar\xi)]^2/2\hbar}$$

(i.e. the position is supported in a $\hbar^{1/2}$-neighborhood of x, and the momentum $2\pi\hbar\xi$ is supported in a $\hbar^{1/2}$-neighborhood of $2\pi k$), the approximation (2.2) will be correct as $h \to \infty$.

Exercise 2.1 *Prove the above statement if V i.e. that*

$$\langle\psi_{x,k}, (-\hbar^2\Delta + V)\psi_{x,k}\rangle = (2\pi k)^2 + V(x) + o(1)$$

as $\hbar \to 0$ if V is a continuous potential.

With this intuition we can hope that

$$\sum_{E_j<0} |E_j|^\gamma \sim \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} [(2\pi k)^2 + V(x)]_-^\gamma dkdx = (2\pi)^{-d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} [p^2 + V(x)]_-^\gamma dpdx$$

after changing back the classical momentum $p = 2\pi k$. Performing the dp integral, we get

$$(2\pi)^{-d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} [p^2 + V(x)]_-^\gamma dpdx = L_{\gamma,d}^{cl}\int_{\mathbf{R}^d} V_-(x)^{\gamma+d/2}\mathrm{d}x$$

with an explicit constant

$$L_{\gamma,d}^{cl} = \int_{\mathbf{R}^d\,:\,|p|\leq 1} (1-p^2)^{\gamma+d/2} = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+1+d/2)}$$

The constant obtained in the Lieb-Thirring inequality $L_{\gamma,d}$ is in general bigger than $L_{\gamma,d}^{cl}$ (it cannot be smaller as it can be shown that for certain semiclassical situations the above calculation is essentially exact). In some cases the Lieb-Thirring inequality (1.1) is known with the constant $L_{\gamma,d}^{cl}$ and in this case it is sharp. In some other cases it is known that (1.1) cannot hold with the semiclassical constant. Again, the most interesting $\gamma = 1, d = 3$ case is open:

Lieb-Thirring Conjecture: The inequality (1.1) holds with $L_{\gamma,d}^{cl}$ for $d = 3$, $\gamma = 1$.

You can read the precise status of this problem in the remarks after Theorem 12.4 in Lieb-Loss.

3 Kinetic energy inequality

An important corollary to the Lieb-Thirring inequality is the following

Theorem 3.1 *Let ψ be a normalized fermionic wave function of N particles in $d = 3$ dimensions. Recall the definition of its one-particle density:*

$$\varrho_\psi(x) = \gamma_\psi^{(1)}(x, x)$$

Then for some universal constant $K > 0$ we have

$$\langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi \rangle \geq K \int \varrho_\psi^{5/3}$$

In other dimensions, we have

$$\langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi \rangle \geq K_d \int \varrho_\psi^{1+2/d}$$

Remark. This inequality expresses a fundamental property of fermionic systems. Suppose, we drop the fermionic condition, and we take $\psi = \otimes_1^N f$ a simple product bosonic function. Then $\varrho_\psi = N|f|^2$, so

$$\int \varrho_\psi^{5/3} = N^{5/3} \int |f|^{10/3}$$

while

$$\langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi \rangle = N \int |\nabla f|^2$$

so the inequality would not hold. In other words, the kinetic energy of N bosons can be of order N, but the kinetic energy of N fermions (confined in a finite volume) is at least of order $N^{5/3}$ since

$$\int \varrho_\psi = N$$

i.e. if ϱ_ψ is supported in a finite volume then

$$\int_\Omega \varrho_\psi^{5/3} \geq C_\Omega \Big(\int_\Omega \varrho_\psi \Big)^{5/3}$$

i.e. we have

$$\langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi\rangle \geq C_\Omega N^{5/3}$$

It is amusing to check that this inequality is essentially sharp, this will be done in the Exercise sheet this week.

Proof of Theorem 3.1. From the ground state energy of the non-interacting fermions we know that

$$\langle \psi, H_N\psi\rangle \geq -\sum_{E_j<0} |E_j|$$

and from Lieb-Thirring inequality ($\gamma = 1$, $d = 3$)

$$\sum_{E_j<0} |E_j| \leq C \int V_-^{5/2}$$

with $C = L_{1,3}$. On the other hand

$$\langle \psi, H_N\psi\rangle = \langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi\rangle + \int V \varrho_\psi$$

Thus

$$\langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi\rangle \geq -\int V \varrho_\psi - C \int V_-^{5/2}$$

This inequality holds for any potential V, now we choose

$$V(x) = -c\varrho_\psi^{2/3}$$

with some c. Thus

$$\langle \psi, \sum_{j=1}^{N} (-\Delta_j)\psi\rangle \geq (c - Cc^{5/2}) \int \varrho_\psi^{5/3} = K \int \varrho_\psi^{5/3}$$

with some positive K (depending on C) if we choose c sufficiently small. (e.g. $c = (2C)^{-2/3}$ gives $K = \frac{1}{2}(2C)^{-2/3}$). □

4 Birman-Schwinger principle

The main step towards the proof of the Lieb-Thirring inequality is the Birman-Schwinger principle that allows to rewrite the negative eigenvalues of a Schrödinger operator into an eigenvalue problem for a compact operator, the Birman-Schwinger operator.

As we remarked in the introduction, we can replace $h = -\Delta + V$ with $h = -\Delta - V_-$, i.e. we can assume that the potential is non-positive.

Let $-e$ be a negative eigenvalue of $-\Delta - V_-$, i.e. $e > 0$ and

$$(-\Delta + e)\psi = V_-\psi$$

where ψ is the normalized eigenfunction. Define $\phi(x) = \sqrt{V_-(x)}\psi(x)$, i.e.

$$(-\Delta + e)\psi = \sqrt{V_-}\phi$$

from which $\psi = (-\Delta + e)^{-1}\sqrt{V_-}\phi$, i.e.

$$\phi = K_e\phi \tag{4.3}$$

where

$$K_e := \sqrt{V_-}(-\Delta + e)^{-1}\sqrt{V_-}$$

is the Birman-Schwinger operator with operator kernel

$$K_e(x, y) := \sqrt{V_-(x)}(-\Delta + e)^{-1}(x, y)\sqrt{V_-(y)}$$

Recall that the kernel of $(-\Delta+e)^{-1}$ is translation invariant and it can be computed via Fourier transform

$$(-\Delta + e)^{-1}(x, y) = \int_{\mathbf{R}^d} \frac{e^{2\pi ik\cdot(x-y)}}{|2\pi k|^2 + e}\mathrm{d}k \tag{4.4}$$

The equation (4.3) indicates that whenever $-e$ is a negative eigenvalue of h, then 1 is an eigenvalue of K_e. We will see in a moment that $\phi \in L^2$ and we also claim that there is a one-to-one correspondance between the negative eigenvalues $-e$ of h and the eigenvalue 1 of K_e.

First we show that $\phi \in L^2$. Assume $d \geq 3$ for simplicity. Since ψ is an eigenfunction of h, we have $\psi \in H^1$, thus

$$\int |\sqrt{V_-}\psi|^2 = \int V_-|\psi|^2 \leq \|V_1\|_{d/2}\|\psi\|^2_{2d/(d-2)} + \|V_2\|_\infty\|\psi\|_2^2 \leq C\|V_-\|_{d/2}\|\nabla\psi\|^2 + \|V_2\|_\infty < \infty$$

by Sobolev inequality and the fact that $V_- = V_1 + V_2$ with $V_1 \in L^{d/2}$ and $V_2 \in L^\infty$. Moreover, this inequality can be written as

$$\int |\sqrt{V_-}\psi|^2 \le C_V \|\psi\|_{H^1}^2$$

i.e. the multiplication with $\sqrt{V_-}$ is a bounded operator from H^1 to L^2.

Now consider

$$B = \sqrt{V_-}(-\Delta + e)^{-1/2}$$

Clearly the second factor maps L^2 to H^1 since

$$\|(-\Delta + e)^{-1/2} f\|_{H^1}^2 = \int (1 + (2\pi k)^2) \Big| \frac{1}{\sqrt{(2\pi k)^2 + e}} \widehat{f}(k) \Big|^2 \mathrm{d}k \le \max\{1, e^{-1}\} \|f\|_2^2$$

using

$$\frac{1 + (2\pi k)^2}{e + (2\pi k)^2} \le \max\{1, e^{-1}\}$$

Since the first factor in B maps H^1 back to L^2, we see that B is a bounded operator on L^2, thus $K_e = BB^*$ is also bounded on L^2.

Now we show the converse, i.e. let $\phi \in L^2$ be an eigenfunction of K_e with eigenvalue 1. Then define $\psi := (-\Delta + e)^{-1}\sqrt{V_-}\phi$. Clearly

$$(-\Delta + e)\psi = \sqrt{V_-}\phi = \sqrt{V_-} K_e \phi = V_-(-\Delta + e)^{-1}\sqrt{V_-}\phi = V_-\psi$$

so ψ will be an eigenfunction of $h = -\Delta - V_-$ (with eigenvalue $-e$) if we can show that $\psi \in L^2$. But

$$\|\psi\|^2 = \langle \sqrt{V_-}\phi, (-\Delta + e)^{-2}\sqrt{V_-}\phi\rangle \le e^{-1}\langle \sqrt{V_-}\phi, (-\Delta + e)^{-1}\sqrt{V_-}\phi\rangle = e^{-1}\langle \phi, K_e\phi\rangle = e^{-1}\|\phi\|^2$$

In the middle we used the operator inequality

$$(-\Delta + e)^{-2} \le e^{-1}(-\Delta + e)^{-1}$$

which can be easily seen by Fourier transform.

In the next section we will see that K_e is a (selfadjoint) compact operator, moreover, it is in some Schatten class, i.e. $\mathrm{Tr} K_e^m < \infty$ for some m. Furthermore, $K_e \to 0$ as $e \to \infty$. For example, in $d = 3$ one can easily compute that

$$\mathrm{Tr} K_e^2 = \mathrm{Tr}\sqrt{V_-}(-\Delta + e)^{-1} V_- (-\Delta + e)^{-1}\sqrt{V_-} \tag{4.5}$$

$$= \int\int V_-(x)|(-\Delta+e)^{-1}(x,y)|^2 V_-(y)\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{16\pi^2}\int\int V_-(x)\frac{e^{-2\sqrt{e}|x-y|}}{|x-y|^2}V_-(y)\mathrm{d}x\mathrm{d}y$$

Using a Schwarz inequality

$$\mathrm{Tr}K_e^2 \le \frac{1}{16\pi^2}\int V_-(x)^2\frac{e^{-2\sqrt{e}|x-y|}}{|x-y|^2}\mathrm{d}x\mathrm{d}y = \frac{C}{\sqrt{e}}\int V_-^2 \tag{4.6}$$

with an explicit constant that can be easily computed but we will not need it. This shows that K_e is Hilbert-Schmidt if $V_- \in L^2$ (at the end, we will use this argument for $W = [V + e/2]_-$ instead of V_-, see (5.10), and if $V_- \in L^{5/2}$, which is a necessary condition for the Lieb-Thirring bound to be meaningful, then $W \in L^2$).

There is one more important fact, namely that K_e is a monotone decreasing operator in e, i.e. $K_{e'} \le K_e$ is $e' > e$, in particular the eigenvalues of K_e are monotone decreasing in e (variational principle). To see this, from Fourier transform it is clear that

$$(-\Delta + e')^{-1} \le (-\Delta + e)^{-1}$$

and thus $K_{e'} \le K_e$. [**WARNING**. Unlike for numbers, if $A \le B$ for two operators and C is a third operator, then $AC \le BC$ does **not** hold, even if $C \ge 0$! However, $C^*AC \le C^*BC$ holds.]

With all these preparations, we can state the main result of this section:

Theorem 4.1 (Birman-Schwinger principle) *For $e > 0$ let*

$N_e :=$ number of eigenvalues of h less than or equal to $-e$

and

$B_e :=$ number of eigenvalues of K_e that are bigger or equal than 1

Then

$$N_e = B_e$$

Proof. Since K_e is a positive compact operator, we can label its eigenvalues in decreasing order, $\lambda_1(e) \ge \lambda_2(e) \ge \ldots \ge 0$. Fix e and let a parameter e' increase monotonically from e to infinity. All eigenvalues $\lambda_j(e')$ decrease and when one of them hits the level 1, then $-e'$ is an eigenvalue of h. Since $K_e \to 0$ as $e \to \infty$, for a sufficiently large e' all eigenvalues of K_e will be below the threshold 1. Thus the number of eigenvalues above 1 of K_e is the same as the number of eigenvalue crossings along this process, i.e. the number of eigenvalues of h below $-e$. In this argument we tacitly assumed that the functions $\lambda_j(e)$ are continuous – this follows from a standard analytic perturbation argument which we skip here. □

5 Proof of the Lieb-Thirring inequality

We will give the proof for the $\gamma > 0$ case; the $\gamma = 0$ case (CLR bound) requires a different argument.

Easy calculus exercise with integration by parts shows that

$$\sum_{E_j<0} |E_j|^\gamma = \sum_j e_j^\gamma = \gamma \int_0^\infty e^{\gamma-1} N_e \mathrm{d}e$$

where $e_j = -E_j = |E_j|$. Using the Birman-Schwinger principle, we need an upper bound on B_e, and clearly

$$B_e \leq \sum_{\lambda_j \geq 1} \lambda_j^m \leq \mathrm{Tr}(K_e)^m$$

for any $m > 0$, where λ_j are the eigenvalues of K_e. Thus we have

$$\sum_{E_j<0} |E_j|^\gamma \leq \gamma \int_0^\infty e^{\gamma-1} \mathrm{Tr}(K_e)^m \mathrm{d}e \tag{5.7}$$

for any $m > 0$.

For $d = 3$ we can choose $m = 2$ and as we already computed in (4.6), we have

$$\mathrm{Tr} K_e^2 \leq \frac{C}{\sqrt{e}} \int V_-^2 \tag{5.8}$$

In different dimensions, we have to choose the exponent m differently. For an integer m we have

$$\mathrm{Tr} K_e^m = \int V_-(x_1) G(x_1 - x_2) V_-(x_2) G(x_2 - x_3) \ldots V_-(x_m) G(x_m - x_1) \mathrm{d}x_1 \ldots \mathrm{d}x_m$$

where

$$G(x - y) = (-\Delta + e)^{-1}(x, y)$$

is the free resolvent kernel (4.4). It is to be noted that the kernel is positive, this fact can be seen, for example, from the heat kernel representation

$$(-\Delta + e)^{-1} = \int_0^\infty e^{-t(-\Delta+e)} \mathrm{d}t = \int_0^\infty e^{-te} e^{t\Delta} \mathrm{d}t$$

and from the explicit formula for the heat kernel

$$e^{t\Delta}(x, y) = (4\pi t)^{-d/2} e^{-(x-y)^2/4t}$$

which is positive.

Using a Hölder inequality with exponents $p_1 = \ldots = p_m = 1/m$ and for the measure

$$\mathrm{d}\mu(x_1, \ldots, x_m) = G(x_1 - x_2)G(x_2 - x_3)\ldots G(x_m - x_1)\mathrm{d}x_1 \ldots \mathrm{d}x_m$$

on $\mathbf{R}^{dm}$, we get

$$\mathrm{Tr}\ K_e^m \leq \int V_-(x_1)^m G(x_1 - x_2)G(x_2 - x_3)\ldots G(x_m - x_1)\mathrm{d}x_1 \ldots \mathrm{d}x_m$$

The $x_2, \ldots x_m$ integrals can be done either explicitly, or via Fourier transform

$$\int G(x_1 - x_2)G(x_2 - x_3)\ldots G(x_m - x_1)\mathrm{d}x_2 \ldots \mathrm{d}x_m = (G \star G \star \ldots \star G)(x_1, x_1)$$

$$= \int_{\mathbf{R}^d} \Big[\frac{1}{(2\pi k)^2 + e}\Big]^m \mathrm{d}k = C_{m,d} e^{-m+d/2}$$

if $m > d/2$, with an explicit constant that can be easily computed but we will not need it. Thus

$$\mathrm{Tr}\ K_e^m \leq C_{m,d} e^{-m+d/2} \int V_-^m \tag{5.9}$$

holds for any integer $m > d/2$.

Actually, (5.9) also holds for any $m \geq 1$, $m > d/2$, but the proof is more involved. It relies on the following inequality which we give without proof

Lemma 5.1 *Let A, B be positive operators, then for any real number $m \geq 1$*

$$Tr\Big[B^{1/2}AB^{1/2}\Big]^m \leq Tr\, B^{m/2}A^m B^{m/2} = Tr\, B^m A^m$$

Applying this lemma to the Birman-Schwinger kernel, we have

$$\mathrm{Tr}\ K_e^m = \mathrm{Tr}\Big[V_-^{1/2}(-\Delta + e)^{-1}V_-^{1/2}\Big]^m \leq \mathrm{Tr} V_-^m (-\Delta + e)^{-m}$$

$$= \int V_-^m(x)(-\Delta + e)^{-m}(0)\mathrm{d}x = \int V_-^m(x)\mathrm{d}x \int_{\mathbf{R}^d} \Big[\frac{1}{(2\pi k)^2 + e}\Big]^m \mathrm{d}k$$

as before for the integer m case. The last integral is finite only if $m > d/2$.

Unfortunately, when (5.9) is plugged into (5.7), we get a divergent $\mathrm{d}e$ integral irrespective of the choice of the exponents. But we can consider the following e dependent potential

$$W_e(x) = [V(x) + e/2]_- = \max\{-V(x) - e/2, 0\} \tag{5.10}$$

Let $N_e(V)$ denote the number of negative eigenvalues of $-\Delta + V$ that lie below $-e$, then

$$N_e(-V_-) = N_{e/2}(-V_- + e/2) \leq N_{e/2}(-W_e) \tag{5.11}$$

since $(-\Delta - V_-)\psi = \lambda\psi$ implies that $(-\Delta - V_- + e/2)\psi = (\lambda + e/2)\psi$, so if $\lambda \leq -e$ is an eigenvalue of $-\Delta - V_-$ then $\lambda + e/2 \leq -e/2$ is an eigenvalue of $(-\Delta - V_- + e/2)$. The inequality in (5.11) follows from the fact that $-W_e \leq -V_- + e/2$, i.e. that $V_- - e/2 \leq [V(x) + e/2]_-$. (Check that $V_- \leq a + [V + a]_-$ for any $a > 0$).

Now we can repeat the previous argument if we replace e by $e/2$ and V_- by W_e (for any fixed e), we get

$$\sum_{E_j<0} |E_j|^\gamma \leq C_{d,m,\gamma} \int_{\mathbf{R}^d} \int_0^\infty e^{\gamma-1}(e/2)^{-m+d/2} W_e(x)^m \mathrm{d}e\mathrm{d}x$$

$$= C'_{d,m,\gamma} \int_{\mathbf{R}^d} \int_0^\infty e^{\gamma-1-m+d/2} [V(x) + e/2]_-^m \mathrm{d}e\mathrm{d}x$$

$$= C'_{d,m,\gamma} \int_{\mathbf{R}^d} \int_0^{2V_-(x)} e^{\gamma-1-m+d/2} (V_-(x) - e/2)^m \mathrm{d}e\mathrm{d}x$$

$$= C''_{d,m,\gamma} \int_{\mathbf{R}^d} V_-^{\gamma+d/2} \int_0^1 u^{\gamma-1-m+d/2}(1-u)^m \mathrm{d}u$$

after a change of variables $e = 2V_-(x)u$ in the last line. The last integral is finite if $m < \gamma + d/2$.

Combining this condition with $m > d/2$, we can choose, for example, $m = (\gamma + d)/2$, which is at least 1 if $d \geq 2$ (and in case of $d = 1$ and $1/2 < \gamma < 1$ we choose $m = 1$) and thus we proved that

$$\sum_{E_j<0} |E_j|^\gamma \leq L_{\gamma,d} \int_{\mathbf{R}^d} V_-^{\gamma+d/2}$$

for any $\gamma > 0$ and $d \geq 2$ or $\gamma > 1/2$ for $d = 1$. This completes the proof of Theorem (1.1) apart from two borderline cases, $d = 1$, $\gamma = 1/2$ and $d \geq 3$, $\gamma = 0$ which require a different argument.

In applications for the stability of matter, the most important case is $d = 3$, $\gamma = 1$. In this case we can choose $m = 2$, the calculation (5.8) suffices (of course W_e has to replace V). In other words, the trace inequality (Lemma 5.1) can be circumvented in this case. □