AB Geometrie und Topologie
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Complex geometry: Hodge theory and Kähler manifolds

WiSem 2026

as of August 30, 2026

The first part of the course will be devoted to some geometric analysis. We will work in the general setting of smooth (real) manifolds and prove the Hodge decomposition theorem for elliptic linear differential operators on compact Riemannian manifolds. Applied to the Laplacian this yields harmonic representatives for de Rham cohomology classes and main applications are finiteness (of dimension) results and Poincaré duality for de Rham cohomology on oriented closed manifolds. Further applications to Riemannian geometry are obtained via the Bochner technique which yields vanishing results for cohomology under suitable curvature assumptions.

The remainder of the course will be devoted to complex geometry. Among the most natural and important examples of complex manifolds are the solution sets of polynomial equations and in particular smooth complex projective varieties. These are even Kähler manifolds, a class of complex manifolds to which we will give special attention. We will first discuss general complex manifolds and their Dolbeault cohomology which is an invariant of the complex structure and not a topological one like de Rham cohomology. In the compact case the harmonic theory, applied to natural elliptic operators provided by the complex structure (Dolbeault Laplacians), yields analogous basic applications to finiteness and (Kodaira-Serre) duality. We then focus on studying the Dolbeault cohomology of compact Kähler manifolds which, due to local identities between various natural geometric differential operators (Kähler identities), has strong additional algebraic structure (Lefschetz theorems) and is also closely related to the de Rham cohomology (Hodge decomposition). In the final part of the course we will prove the Kodaira vanishing theorem for holomorphic line bundles and, based on it, the Kodaira embedding theorem which characterizes the Kähler manifolds which embed into a complex projective space and hence, according to a theorem of Chow (which we will not prove), are projective varieties.

For students of mathematics/physics (master, TMP)

Prerequisites: basic complex analysis (in one variable), differentiable manifolds and differential forms, only very basic functional analysis, no PDE theory

References:

Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer, 1983
Wells, *Differential Analysis on Complex Manifolds*, Springer, 1980
Kodaira, *Complex manifolds and deformation of complex structures*, Springer, 1986
Huybrechts, *Complex geometry: An introduction*, Springer, 2005
Ballmann, *Lectures on Kähler Manifolds*, EMS, 2006
Griffiths, Harris, *Principles of Algebraic Geometry*, Wiley, 1978
Lawson, Michelsohn, *Spin geometry*, Princeton, 1989