

Characteristic classes

SEMINAR WiSEM 2026

as of August 8, 2026

Vector bundles are basic objects in differential as well as complex and algebraic geometry. Among the first examples one encounters are probably the tangent bundle of a smooth manifold and the normal bundle of an immersed smooth submanifold. Characteristic classes are global topological, more specifically, cohomological invariants of vector bundles measuring their “global twistedness”. One derives from them differential-topological invariants for manifolds (characteristic numbers) which have been used, for instance, by Thom to distinguish smooth manifolds up to cobordism and by Milnor to construct his famous exotic 7-spheres.

We will take the topological approach to the theory based on singular cohomology theory. For the groundwork we will mostly rely on Hatcher’s text [Ha2] which avoids the use of cohomology operations (Steenrod squares) as in the classic by Milnor and Stasheff. The latter book will be our source for most of the applications.

If time allows and depending on the interest of the participants we may also incorporate the smooth approach based on de Rham cohomology. Since there one works with real coefficients, Stiefel-Whitney classes (having $\mathbb{Z}_2$ -coefficients) cannot be treated and also for the Chern and Pontrjagin classes (having $\mathbb{Z}$ -coefficients) information is lost. On the other hand, one has the smooth structure at one’s disposal and can represent characteristic cohomology classes by differential forms with geometric meaning (curvature) thereby establishing a strong link between topology and differential geometry. The simplest example is the Gauß-Bonnet formula for oriented closed surfaces equipped with a Riemannian metric which expresses the Euler characteristic as an integral of the Gauß curvature.

Among the topics of the seminar are:

- cohomology of fiber bundles (Leray-Hirsch theorem), in particular of sphere bundles (Thom isomorphism, Gysin sequence)
- homotopy classification of vector bundles, classifying spaces, universal bundles
- construction of characteristic classes (Stiefel-Whitney, Chern and Pontrjagin classes, Euler class)
- characteristic numbers and cobordism
- multiplicative sequences and the Hirzebruch signature theorem
- Milnor’s exotic 7-spheres

References:

- [MS] Milnor, Stasheff, *Characteristic classes*, Princeton, 1974
- [Ha1] Hatcher, *Vector bundles and K-theory*, online version, 2017
- [Ha2] Hatcher, *Algebraic topology*, online version, 2015
- [Mi] Milnor, *On manifolds homeomorphic to the 7-sphere*, Annals of Math., 1956
- [MT] Madsen, Tornehave, *From calculus to cohomology*, 1997
- [Tu] Tu, *Differential geometry*, Springer, 2017

Prerequisites: basic algebraic topology

For students of mathematics or physics (Master, TMP)

Space-time: tuesday 16-18 in room A 252

If you are interested in participating, please write me email (the sooner, the better).