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Aufgabe 1. [8 Punkte]

Consider the following commutative diagram of $\mathbb{R}$-vector spaces

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "k"] & A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "(*)"] & B \arrow[r, "p"] \arrow[d] & C \arrow[d, "c"] \\
K' \arrow[r, "i'"] & A' \arrow[r, "f'"] & B' \arrow[r, "p'"] & C',
\end{tikzcd}$$

where (K, i) (resp., (K', i')) is the kernel of f (resp., f'), and (C, p) (resp., (C', p')) is the cokernel of f (resp., f').

(a) Prove that the square $(*)$ is cartesian if and only if k is an isomorphism and c is a monomorphism. [3 P.]

(b) Prove that the square $(*)$ is cocartesian if and only if k is an epimorphism and c is a isomorphism. [1 P.]

(c) Consider the following diagram of smooth real finite rank vector bundles over a smooth finite dimensional manifold X

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Prove that $A \oplus B' \cong A' \oplus B$. [4 P.]

Solution| a) Assume that $(*)$ is cartesian. Consider the diagram:

$$\begin{tikzcd}
K' \arrow[dr, "h"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "i'"'] & & \\
& A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\
& A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

Since $f' \circ i' = 0$, using universal property of $(*)$ we get a map $h: K' \to A$ such that $f \circ h = 0$, ~~and~~ i.e., $\mathrm{Im}(h') \subseteq K$. Then h is the inverse to k. Next, take $x = p(b) \in C$, $b \in B$ and assume that $c(x) = 0$. Since $c \circ p = p' \circ g$, where g denotes the map $g: B \to B'$ from $(*)$, we conclude that $\exists\, a' \in A'$ such that $f'(a') = g(b)$. Since $(*)$ is cartesian, $\exists\, a \in A$ such that $f(a) = b$. But then $x = p(f(a)) = 0$, i.e., c is a monomorphism.

Assume that k is an isomorphism & c a monomorphism.
If $g(b^*) = f'(a')$ for some $b^* \in B^*$, $a' \in A'$, then
$c(p(b^*)) = 0 \underset{c \text{ mono}}{\Rightarrow} p(b^*) = 0 \Rightarrow b^* = f(a)$ for some $a \in A$

Let g' denote the map $g' : A \to A'$ in $(*)$
Then $f'(g'(a)) = g(f(a)) = g(b) = f'(a')$, i.e. $f'(a' - g'(a)) = 0$
$\Rightarrow \exists\, x' \in K'$ such that $i'(x') = a' - g'(a^*)$ and $x \in K$
such that $k(x) = x'$. Then $g'(a+x) = g'(a) + i(x') = a'$,
and $f(a+x) = f(a) + 0 = b$.
I.e. $\exists$ an elemen $a+x \in A$ such that $f(a+x) = b$, $g'(a+x) = a'$.
Uniqueness of such an element is clear.

b) Apply the argument from "a)" to the category
$Vect_{\mathbb{R}}^{op}$ dual to the category of $\mathbb{R}$-vector spaces.

c) Recall that all short exact sequences of real vector spaces split, i.e., $B \simeq A \oplus C$ and $B' \simeq A' \oplus C'$
Then $A \oplus B' \simeq A \oplus A' \oplus C' \simeq A \oplus A' \oplus C \simeq A' \oplus B$ ∎

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Aufgabe 2. [12 Punkte]

For a smooth finite dimensional manifold X let $N^*(X)$ denote the unoriented real cobordism rinf of X, i.e., the ring of cobordism classes of proper maps to X, and let pt denote a point.

(a) For a standard real d-dimensional sphere S^d in $\mathbb{R}^{d+1}$ compute $N^*(S^d)$ as an algebra over $N^*(\mathrm{pt})$. [5 P.]

(b) Prove that the natural inclusion $i\colon S^d \hookrightarrow \mathbb{R}^{d+1}$ does not admit a smooth retraction, i.e., a map $r\colon \mathbb{R}^{d+1} \to S^d$ with the property $r \circ i = \mathrm{id}$ does not exist. [2 P.]
Remark: as a textbook corollary of the above result one can prove Brouwer's fixed point theorem.

(c) Let $T^d = S^1 \times \ldots \times S^1$ denote compact real d-dimensional torus. Compute $N^*(T^d)$ as an algebra over $N^*(\mathrm{pt})$. [5 P.]

In your solutions use the properties of N^* proven in the Lectures, e.g., functoriality of pullbacks and pushforwards, localization sequence, projective bundle formula, etc. However, avoid using computations and properties of cohomology ring, which you might know from the other courses.

Solution a) Consider some inclusion of a point $pt \overset{i}{\hookrightarrow} S^d$ and the complement open inclusion $j: S^d \setminus pt \hookrightarrow S^d$. Clearly, $S^d \setminus pt \simeq \mathbb{R}^d$ via stereographic projection, and $\mathbb{R}^d$ is contractible. Then $N(S^d \setminus pt) \simeq N(pt)$ by htpy invariance of N. Using localization sequence, we get an exact sequence of $N(pt)$-modules

$$\begin{array}{ccccc} N(pt) & \xrightarrow{i_*} & N(S^d) & \xrightarrow{j^*} & N(S^d \setminus pt) \simeq N(pt) \\ & \searrow^{\simeq} & \downarrow{\scriptstyle p_*} & & \\ & & N(pt) & & \end{array}$$

where $p: S^d \to pt$ natural projection $\Rightarrow$ i_* is injective by commutativity of ~~the~~ triangle. Since j^* is a morphism of $N(pt)$-algebras, it sends 1 to 1 $\Rightarrow$ is surjective. However, $N(pt)$ is a free module $\Rightarrow$ short exact sequence split.

$\Rightarrow N(S^d) \simeq N(pt) \oplus N(pt)$ as $N(pt)$-module

In fact, $N(S^d)$ is freely generated by $j^*(1) = 1$ and $i_*(1) = [pt] \in N^d(pt)$ since push-forward shifts codimension. Then $i_*(1)^2 \in N^{2d}(pt) = 0$, therefore

$$N(S^d) \simeq N(pt)[t]/t^2=0 \quad , \quad \deg(t) = d$$

b) If it does, then $i^* \circ r^* = id : N(S^d) \to N(S^d)$ but the identity map of $N(pt) \oplus N(pt)$ cannot factor through $N(pt)$.

c) Since $S^1 \simeq \mathbb{P}^1$, we can apply projective bundle formula (and induction) to get

$$N(T^n) \simeq N(T^{n-1} \times \mathbb{P}^1) \cong N(T^{n-1})[J]/J^2 =$$

$$\simeq \dots \simeq N(pt)[J_1, \dots, J_n]/J_1^2, \dots, J_n^2$$

(as algebra)

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Aufgabe 3. [10 Punkte]

For a smooth real finite rank vector bundle E over a smooth finite dimensional manifold X let $c_i(E) \in N^i(X)$ denote (unoriented cobordism) Chern classes of E. Let $\mathcal{T}_X$ denote the tangent bundle on X, and $\mathcal{O}_X$ the trivial line bundle on X. Let $\mathbb{P}^d$ denote the real d-dimensional projective space, and τ the tautological line bundle on $\mathbb{P}^d$.

(a) Compute the Chern classes $c_i(\mathcal{T}_{S^d}) \in N^i(S^d)$ of the tangent bundle on the d-dimensional sphere S^d. [4 P.]
Remark: recall that by the classical hairy ball theorem $\mathcal{T}_{S^2}$ is non-trivial.

(b) Recall that after identification $(x, v) \sim (-x, -v)$ in the identity $\mathcal{T}_{S^d} \oplus \mathcal{N}_i = S^d \times \mathbb{R}^{d+1}$ (where $\mathcal{N}_i$ is the normal bundle of the embedding i from Aufgabe 2), one obtains the following isomorphism:

$$\mathcal{T}_{\mathbb{P}^d} \oplus \mathcal{O}_{\mathbb{P}^d} \cong \tau^{\oplus(d+1)}. \tag{1}$$

Use (1) to compute (unoriented cobordism) Chern classes $c_i(\mathcal{T}_{\mathbb{P}^d})$ as elements of $N^*(\mathbb{P}^d) \cong N^*(\mathrm{pt})[\zeta]/\zeta^{d+1}$, where $\zeta = c_1(\tau)$. [6 P.]

Solution: a) Using $T_{S^d} \oplus N_i = S^d \times \mathbb{R}^{d+1} = \mathcal{O}_{S^d}^{d+1}$ (cf. "b)")

and the fact $N_i \simeq \mathcal{O}_{S^d}$ (clearly, normal bundle to S^d has nowhere vanishing section)

we get $c_{tot}(T_{S^d}) \cdot c_{tot}(\mathcal{O}_{S^d}) = c_{tot}(\mathcal{O}_{S^d})^{d+1}$

Since $c_{tot}(\mathcal{O}_{S^d}) = 1$, we get $c_{tot}(T_{S^d}) = 1$,

i.e. $c_i(T_{S^d}) = 0$ for $i > 0$.

b) Applying c_{tot} to (1), we get

$$c_{tot}(T_{\mathbb{P}^d}) \cdot c_{tot}(\mathcal{O}_{\mathbb{P}^d}) = c_{tot}(\tau)^{d+1}$$

Since $c_{tot}(\mathcal{O}_{\mathbb{P}^d}) = 1$ and $c_{tot}(\tau) = 1 + t \cdot c_1(\tau) = 1 + t\zeta$

as an element of $N(\mathbb{P}^d)[t]$, we get

$$c_{tot}(T_{\mathbb{P}^d}) = (1 + t\zeta)^{d+1} = \sum_{i=0}^{d+1} \binom{d+1}{i} t^i \zeta^i$$

Since $N(\mathbb{P}^d) = N(pt)[\zeta]/\zeta^{d+1}$, in particular, $\zeta^{d+1}=0$, we get

$c_0(T_1)=1$ & $c_i(T_{\mathbb{P}^d}) = \binom{d+1}{i}\zeta^i$ for $0<i\leq d$.

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Aufgabe 4. [10 Punkte]

Let X be a 4-dimensional smooth manifold, and E a smooth real rank 2 vector bundle over X.

(a) Compute $c_1(E \otimes E) \in N^1(X)$ in terms of Chern classes $c_i(E) \in N^i(X)$. [4 P.]

(b) Let $\widetilde{\Gamma}$ denote the polynomial ring $(\mathbb{Z}/2\mathbb{Z})[a_{ij}]$ in infinitely many independent polynomial variables a_{ij}, $0 < i, j < \infty$, let $\widetilde{F}(x,y) := x + y + \sum_{i,j} a_{ij} x^i y^j \in \widetilde{\Gamma}[[x,y]]$, and let b_{ijk}, $c_i \in \widetilde{\Gamma}$ be defined via identities

$$\widetilde{F}(\widetilde{F}(x,y),z) - \widetilde{F}(x,\widetilde{F}(y,z)) = \sum_{i,j,k} b_{ijk} x^i y^j z^k \in \widetilde{\Gamma}[[x,y,z]]$$

and $\widetilde{F}(x,x) = \sum_i c_i x^i \in \widetilde{\Gamma}[[x]]$. Recall that the cobordism ring of a point $N^*(\mathrm{pt})$ can be identified with the quotient ring

$$\Gamma = \widetilde{\Gamma}/(a_{ij} - a_{ji},\, b_{ijk},\, c_i).$$

The image of $\widetilde{F}$ under this identification coincides with the formal group law of the cobordism theory $F_N(x,y) \in N^*(\mathrm{pt})[[x,y]]$.

Using the identification $N^*(\mathrm{pt}) \cong \Gamma$, and the splitting principle, compute all (unoriented cobordism) Chern classes $c_i(E \otimes E) \in N^i(X)$ in terms of (unoriented cobordism) Chern classes of E. [6 P.]

Solution a) Using splitting principle, we can find $Y \xrightarrow{f} X$ such that $f^*E \simeq L_1 \oplus L_2$ sum of line bdls. and $f^*: N(X) \to N(Y)$ is injective.

Then $c_{tot}((L_1 \oplus L_2) \otimes (L_1 \oplus L_2)) = c_{tot}(L_1 \otimes L_1 \oplus L_1 \otimes L_2 \oplus L_2 \otimes L_1 \oplus L_2 \otimes L_2) = c_{tot}(L_1 \otimes L_1) \cdot c_{tot}(L_1 \otimes L_2) \cdot c_{tot}(L_2 \otimes L_1) \cdot c_{tot}(L_2 \otimes L_2) = c_{tot}(\mathcal{O}) \cdot c_{tot}(L_1 \otimes L_2)^2 \cdot c_{tot}(\mathcal{O}) = 1 \cdot (1 + t \cdot c_1(L_1 \otimes L_2))^2 \cdot 1 =$

$= 1 + t^2 c_1(L_1 \otimes L_2)^2$

In other words, $c_1(f^*(E \otimes E)) = 0$ therefore

$c_1(E \otimes E) = 0$

b) Observe that X is 4-dimensional, therefore $N^i(X)=0$ for $i>4$. In particular, $\prod c_i(E)^{a_i}=0$ for $\sum i \cdot a_i \geq 4$. We know from lectures that $c_i(E\otimes E)$ can be rewriten as $N(pt)$-linear combination of $\{\prod c_i(E)^{a_i}\}$. Arguing as in "a)", take $Y \xrightarrow{f} X$ such that $f^*E \simeq L_1 \oplus L_2$ and f^* is injective. The above argument shows that we can compute $c_i(f^*(E\otimes E))$ only up to degree ≤ 4. Let us denote "$\mathcal{O}(5)$" any element lying in $\bigoplus_{k\geq 5} N^k(Y)$. Then

$$C_{tot}(f^*(E\otimes E)) = 1 + t^2 \cdot c_1(L_1 \otimes L_2)^2$$

(see "a)"), therefore $c_2, c_3, c_4 = 0$ and

$$c_2(f^*(E\otimes E)) = c_1(L_1\otimes L_2)^2 = \cancel{(F_N(L_1 \otimes L_2)}$$

$$= F_N(c_1(L_1), c_1(L_2))^2 = \cancel{\sum a_{ij}}$$

$$= \left(c_1(L_1) + c_1(L_2) + \sum a_{ij}\, c_1(L_1)^i\, c_1(L_2)^j\right)^2 =$$

$$= c_1(L_1)^2 + c_1(L_2)^2 + \sum a_{ij}^2\, c_1(L_1)^{2i}\, c_1(L_2)^{2j} \ominus$$

since $N(pt)$ is a $\mathbb{Z}/2$-algebra

$$\ominus\ c_1(L_1)^2 + c_1(L_2)^2 + a_{11}^2 c_1(L_1)^2 c_1(L_2)^2 + \mathcal{O}(5) =$$

$$= (c_1(L_1) + c_1(L_2))^2 + a_{11}^2 (c_1(L_1) c_1(L_2))^2 + \mathcal{O}(5) =$$

$$= c_1(L_1 \oplus L_2)^2 + a_{11}^2\, c_2(L_1 \oplus L_2)^2 + \mathcal{O}(5)$$

since Chern classes of $L_1 \oplus L_2$ are elementary symmetric polynomials in $c_1(L_1)$, $c_1(L_2)$

In other words,

$$c_2(f^*(E\otimes E)) = c_1(f^*(E))^2 + a_{11}^2 c_2(f^*(E))^2 + \mathcal{O}(5),$$

therefore $c_2(E\otimes E) = c_1(E)^2 + a_{11}^2 c_2(E)^2$.

(And $c_i(E\otimes E)=0$ for $i=1,3,4$)