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Aufgabe 1. [7 Punkte]

Let R be a commutative ring and $I,\, J \trianglelefteq R$ its ideals.

(a) Show that the diagram

$$\begin{tikzcd}
0 \arrow[r] & IJ \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & I \otimes_R R/J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & R \arrow[r] & R \otimes_R R/J \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

is commutative with exact rows. [2 P.]

(b) Prove that

$$\mathrm{Tor}_1^R(R/I,\, R/J) \cong \frac{I \cap J}{IJ}.$$

[5 P.]

Solution| a) After identification $R \otimes_R R/J \simeq R/J$ the bottom row of the diagram $(*)$ is the short exact sequence

$$0 \to J \to R \to R/J \to 0 \qquad (**)$$

Since $I \otimes_R -$ is right exact, applying it to $(**)$ we obtain a right exact sequence

$$I \otimes_R J \to I \otimes_R R \to I \otimes_R R/J \to 0$$

However, $I \otimes_R R \simeq I$ and $\mathrm{Im}(I \otimes_R J \to I \otimes_R R \simeq I)$ coincides with $I \cdot J$. Therefore $IJ \rightarrowtail I \twoheadrightarrow I \otimes_R R/J$ is a short exact sequence. Commutativity of $(*)$ is obvious.

b) Apply $- \otimes_R R/J$ to $0 \to I \to R \to R/I \to 0$ and consider the corresponding long exact sequence of Tor's:

$$\mathrm{Tor}_1^R(R, R/J) \to \mathrm{Tor}_1^R(R/I, R/J) \to I \otimes_R R/J \to R \otimes_R R/J \to \frac{R}{I} \otimes_R \frac{R}{J} \to 0$$

Since $\mathrm{Tor}_1^R(R, -) \equiv 0$, $\mathrm{Tor}_1^R(R/I, R/J) \simeq \mathrm{Ker}(I \otimes_R R/J \to R \otimes_R R/J)$

Applying the Snake Lemma to $(*)$ we conclude that

$$\mathrm{Ker}(I \to R) \to \mathrm{Ker}(I \otimes_R R/J \to R \otimes_R R/J) \to \mathrm{Coker}(IJ \to J) \xrightarrow{\varphi} \mathrm{Coker}(I \to R)$$

is exact. However, $\mathrm{Ker}(I \to R) = 0$, $\mathrm{Coker}(IJ \to J) \simeq J/IJ$, $\mathrm{Coker}(I \to R) \simeq R/I$ and the map φ is induced by $J \to R$, therefore under these identifications coincides with

the natural map $J/IJ \longrightarrow R/I$. The kernel of this map is clearly $\mathrm{Ker}(J \to R \to R/I)/IJ = I \cap J/IJ$.
In other words, $\mathrm{Tor}_1^R(R/I, R/J) \simeq I \cap J / IJ$.

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Aufgabe 2. (4 × 4 Lemma) [7 Punkte]

Let R be a commutative ring and consider the following commutative diagram of R-modules.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & D'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A''' \arrow[r] \arrow[d] & B''' \arrow[r] \arrow[d] & C''' \arrow[r] \arrow[d] & D''' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd} \qquad (*)$$

Assume that the rows of the diagram are exact, that three of the four columns are also exact, and that the remaining column is a complex.
Prove that under these assumptions in fact all four columns are exact.

Solution: Let $C_{\bullet\bullet}$ be the double complex $(*)$.
Consider the associated spectral sequence E,
$E^0 = C_{\bullet\bullet}$, $d^0 = d^{(*)}_{\text{vertical}}$
Then $E^1 = H(E^0, d^0)$, i.e., coincide with the homology of the columns of $(*)$. By our assumption, only one column has non-zero homology, therefore E^1 is concentrated in one column. This implies that all higher differentials of E are trivial, i.e. $E^1 = E^\infty$.
Next, consider the transposed double complex $C^T_{\bullet\bullet}$ and the corresponding spectral sequence F,
$F^0 = C^T_{\bullet\bullet}$, $d^0 = (d^{(*)}_{\text{horizontal}})^T$
Then $F^1 = H(F^0, d^0)$ coincides with the homology of the rows of $(*)$, i.e., $F^1 = 0 \Rightarrow F^\infty = 0$.
Then, since $F \Rightarrow H(\mathrm{Tot}\, C^T_{\bullet\bullet})$, we conclude that $H(\mathrm{Tot}\, C^T_{\bullet\bullet}) = 0$ but $\mathrm{Tot}\, C_{\bullet\bullet} \simeq \mathrm{Tot}\, C^T_{\bullet\bullet}$ and

$E \Rightarrow \mathrm{Tot}\, C_{\bullet\bullet} = 0$, therefore $E^\infty = 0$.
However, as we already observed, E^∞ coincides with E^1, therefore all columns of $(*)$ are exact.

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Aufgabe 3. [11 Punkte]

Let $R = \mathbb{C}[x, y]/(x \cdot y)$. For elements and ideals of $\mathbb{C}[x, y]$ denote by $\bar{\ }$ their classes in R. Recall that the maximal ideals of R have form $\overline{(x-a,\, y-b)}$ where either $a = 0$ or $b = 0$ by Hilbert's Nullstellensatz. Denote $\mathfrak{m}_0 = \overline{(x,\, y)} \trianglelefteq R$.

(a) Show that $R_\mathfrak{m}$ is regular for any $\mathfrak{m} \in \mathrm{Max}(R) \setminus \{\mathfrak{m}_0\}$. [4 P.]

(b) Construct a free resolution for $R/\mathfrak{m}_0$ over R. [3 P.]

(c) Compute $\mathrm{p.d.}_R(R/\mathfrak{m}_0)$ and $\mathrm{gl.dim.}(R_{\mathfrak{m}_0})$. Prove or disprove that R is regular. [4 P.]

Solution | a) Let $\mathfrak{m} = \overline{(x-a, y)}$, $a \neq 0$. Then $\bar{x} \notin \mathfrak{m}$ (otherwise $\bar{a} \in \mathfrak{m}$ and $\mathfrak{m} = R$), therefore the maximal localization $\lambda_\mathfrak{m} : R \to R_\mathfrak{m}$ factors through the principal localization $\lambda_{\bar{x}} : R \to R_{\bar{x}}$, i.e.

$$\begin{tikzcd} R \arrow[r, "\lambda_{\bar{x}}"] \arrow[rr, bend right, "\lambda_\mathfrak{m}"'] & R_{\bar{x}} \arrow[r] & R_\mathfrak{m} \end{tikzcd}$$

However, $R_{\bar{x}} \simeq \mathbb{C}[x^{\pm 1}, y]/xy \simeq \mathbb{C}[x^{\pm 1}, y]/y \simeq \mathbb{C}[x^{\pm 1}]$. Since $\mathbb{C}[x]$ is regular, its localization $\mathbb{C}[x^{\pm 1}] \simeq R_{\bar{x}}$ is also regular and $R_\mathfrak{m}$ as the localization of $R_{\bar{x}}$ is regular as well.

Similar argument shows that for $\mathfrak{m} = \overline{(x, y-b)}$, $b \neq 0$, $R_\mathfrak{m}$ is regular.

b) Let $F_0 = R$, $\varepsilon : F_0 \to R/\mathfrak{m}_0$ be the natural projection. $\mathrm{Ker}(\varepsilon) = (x, y) \trianglelefteq R$. Let $F_1 = R \oplus R$ and $d_1 : F_1 \to F_0 = R$ be the map (x, y), i.e., for $(a, b)^T \in R \oplus R$, $d_1\binom{a}{b} = xa + yb$. Then $\mathrm{Im}(d_1) = \mathrm{Ker}(\varepsilon)$ and $\mathrm{Ker}(d_1) = yR \oplus xR$. Let $F_2 = R \oplus R$ and $d_2 : F_2 \to F_1$ be $y \oplus x$, i.e., for $\binom{a}{b} \in F_2 = R \oplus R$, $d_2\binom{a}{b} = \binom{y \cdot a}{x \cdot b} \in R \oplus R = F_1$. Then $\mathrm{Im}(d_2) = \mathrm{Ker}(d_1)$

and $\mathrm{Ker}(d_2) = xR \oplus yR$. Continuing this procedure we obtain a resolution

$$\cdots \xrightarrow{y\oplus x} R\oplus R \xrightarrow{x\oplus y} R\oplus R \xrightarrow{y\oplus x} R\oplus R \xrightarrow{x\oplus y} R\oplus R \xrightarrow{y\oplus x} R\oplus R \xrightarrow{(x,y)} R \xrightarrow{\varepsilon} R/\mathfrak{m}_0 \qquad (*)$$

c) Localizing $(*)$ at $\mathfrak{m}_0$ and using exactness of localization we obtain a free resolution $\tilde{F}_\bullet$:

$$\xrightarrow{y\oplus x} A\oplus A \xrightarrow{x\oplus y} A\oplus A \xrightarrow{y\oplus x} A\oplus A \xrightarrow{(x,y)} A \qquad (**)$$

of a module A/M, where $A = R_{\mathfrak{m}_0}$, $M = (\mathfrak{m}_0)_{\mathfrak{m}_0}$ its maximal ideal.

Since $(**)$ has the property $\mathrm{Im}(d_i) \subseteq M \cdot \tilde{F}_{i-1}$, i.e., $p.d._A\, A/M$ equals to the length of $\tilde{F}_\bullet$, $p.d._A(A/M) = \infty$. This implies that $gl.\dim A = p.d._A A/M = \infty$. In particular, A is not regular, therefore R is also not regular.

Finally, assume that $R/\mathfrak{m}_0$ has a finite resolution over R. Localizing it at $\mathfrak{m}_0$, we obtain a finite resolution for A/M over A, which is impossible since $p.d._A A/M = \infty$. Therefore $R/\mathfrak{m}_0$ cannot have a finite resolution and $p.d._R R/\mathfrak{m}_0 = \infty$.

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Aufgabe 4. [11 Punkte]

Let R be a noetherian domain, $K = \mathrm{Frac}(R)$ its field of fractions.

(a) For $I \trianglelefteq R$ let $I^{-1} = \{x \in K \mid xI \subseteq R\}$. Assume that R is not a field, then there exists a non-zero proper ideal $I \trianglelefteq R$ such that $I^{-1} \neq R$ (e.g., a principal ideal). Show that the maximal ideal with this property (exists and) is prime. [2 P.]

(b) Recall that an element $x \in K$ is called R-integral, if x is a root of a monic polynomial with coefficients in R or, equivalently, if the ring $R[x]$ is finitely generated as an R-module. The ring R is integrally closed if all R-integral elements of K belong to R.

Assume that R is integrally closed and for $I \trianglelefteq R$ let $R(I) = \{x \in K \mid xI \subseteq I\}$. Prove that $R(I) = R$. [1 P.]

(c) Assume that R is local, integrally closed, and that $\dim(R) = 1$. Prove that the maximal ideal of R is an invertible fractional ideal. [3 P.]

(d) Assume that R is local, integrally closed, and that $\dim(R) = 1$. Prove that R is regular. [3 P.]

(e) Assume that R is integrally closed and $\dim(R) = 1$ (but R is not necessarily local). Prove that R is regular. [2 P.]

Solution: a) The maximal ideal J with this property exists, e.g., by Zorn's Lemma (R is noetherian, therefore all ascending chains of ideals in R stabilize).

Let $z \in J^{-1} \setminus R$ and let $x, y \in R$ such that $x \notin J$, $xy \in J$.

We have to show that $y \in J$.

First, $zy(J + xR) \subseteq R$, i.e. $zy \in (J + xR)^{-1}$.

But $(J + xR) \supsetneq J \rightsquigarrow (J + xR)^{-1} = R$ by maximality of J

Then $zy \in R$ but then $z(J + yR) \subseteq R$. If $y \notin J$, by maximality of J we again obtain $(J + yR)^{-1} = R$, but then $z \in (J + yR)^{-1} = R$: contradiction. this implies that $y \in J$.

b) Clearly $R(I)$ is a ring. Moreover, $R(I)$ is a fractional ideal ($\forall a \in I \setminus 0 \;\; aR(I) \subseteq I \subseteq R$).

In particular, $R(I)$ is fin. generated as an R-module. Then for any $x \in R(I)$ the ring $R[x]$ is an R-submodule of $R(I)$, therefore also finitely generated as an R-module. This implies that x is R-integral and belongs to R since R is integrally closed. I.e., $R(I) \subseteq R$.

c) Under these assumptions, $\operatorname{Spec} R = \{0, \mathfrak{m}\}$.
Consider J from "b)": since J is non-zero prime ideal, $J = \mathfrak{m}$, but $J^{-1} \neq R$, i.e., $\mathfrak{m}^{-1} \neq R$.
However $\mathfrak{m} = \mathfrak{m} \cdot R \subseteq \mathfrak{m} \cdot \mathfrak{m}^{-1} \subseteq R$,
and by maximality of $\mathfrak{m}$, either $\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}^{-1}$
or $\mathfrak{m} \cdot \mathfrak{m}^{-1} = R$. However, $\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$
implies that $\mathfrak{m}^{-1} \subseteq R(\mathfrak{m}) = R$: a contradiction.
Therefore $\mathfrak{m} \cdot \mathfrak{m}^{-1} = R$, i.e., $\mathfrak{m}$ is invertible.

d) Since ~~dim R~~ $\mathfrak{m}$ is invertible, it is a projective R-module of rank 1. But R is local, therefore $\mathfrak{m}$ is in fact free of rank 1. Then $\mathfrak{m}$ is principal and $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$. Since $\dim R = 1$ as well, we conclude that R is regular.

e) Recall that if R is integrally closed, all its localizations are integrally closed as well.
Indeed, if $x \in K$ is integral over R_S,
i.e., $x^n + \sum_{i=0}^{n-1} \frac{a_i}{s_i} x^i = 0$, $a_i \in R$, $s_i \in S$,
then $(x \prod_{i=0}^{n-1} s_i)^n + \sum_{i=0}^{n-1} a_i' (x \prod_{i=0}^{n-1} s_i)^i = 0$ for
$a_i' \in R$, i.e., $(x \prod_{i=0}^{n-1} s_i)$ is integral over R
and therefore $x \cdot \prod_{i=0}^{n-1} s_i \in R$ and $x \in R_S$.

Now, since $\dim R = 1$, $\forall \mathfrak{m} \in \operatorname{Max} R$, $\dim R_{\mathfrak{m}} = 1$ as well, $R_{\mathfrak{m}}$ is integrally closed as well and $R_{\mathfrak{m}}$ is local, therefore $R_{\mathfrak{m}}$ is regular by "d)".
Since $R_{\mathfrak{m}}$ are regular $\forall \mathfrak{m} \in \operatorname{Max} R$, we conclude that R is regular.