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Lineare Algebra II – Tutoriumsblatt 11

Aufgabe 1.
For $u,\, v \in \mathbb{R}^n$, prove that $||u+v|| = ||u|| + ||v||$ if and only if $\exists\, \alpha \in \mathbb{R}_{\geq 0}$ such that $v = \alpha \cdot u$.

Aufgabe 2.
Find an orthonormal basis for the subspace $U \leq \mathbb{R}^4$ generated by $(1,2,1,3)$, $(4,1,1,1)$ and $(3,1,1,0)$.

Aufgabe 3.
Let $V, \langle\ ,\ \rangle$ be a Euclidean space and $f\colon V \to V$ an $\mathbb{R}$-linear automorphism of V. Prove that the following are equivalent:

1. f is an isometry (i.e., $\forall\, u,\, v \in V$ one has $\langle f(u), f(v)\rangle = \langle u, v\rangle$);
2. $\forall\, u \in V$ one has $||f(u)|| = ||u||$;
3. For any orthonormal Basis $v_1, \ldots, v_n$ of V, the set $f(v_1), \ldots, f(v_n)$ is also an orthonormal basis of V;
4. There exists an orthonormal Basis $v_1, \ldots, v_n$ of V, such that $f(v_1), \ldots, f(v_n)$ is also an orthonormal basis of V.

Aufgabe 4.
Consider the Euclidean space $\mathbb{R}^2$ with the standard scalar product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = u_1 v_1 + u_2 v_2.$$

Recall that the orthogonal group $\mathrm{O}_2(\mathbb{R})$ is the group of isometries of $\mathbb{R}^2$:

$$\mathrm{O}_2(\mathbb{R}) = \{ f \in \mathrm{GL}_2(\mathbb{R}) \mid \langle f(u), f(v)\rangle = \langle u, v\rangle \;\; \forall\, u,\, v \in \mathbb{R}^2 \}.$$

1. Prove that
$$\mathrm{O}_2(\mathbb{R}) = \{ A \in \mathrm{GL}_2(\mathbb{R}) \mid A^t A = E \}.$$
2. For $A \in \mathrm{O}_2(\mathbb{R})$, prove that $\det(A) = \pm 1$.
3. Consider the *special orthogonal group*
$$\mathrm{SO}_2(\mathbb{R}) = \{ A \in \mathrm{O}_2(\mathbb{R}) \mid \det(A) = 1 \}.$$
Prove that each element of $\mathrm{SO}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ as a rotation about the origin.
4. Prove that each element of $\mathrm{O}_2(\mathbb{R}) \setminus \mathrm{SO}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ as a composition of a rotation about the origin followed by the reflection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.