LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Lineare Algebra II – Tutoriumsblatt 8

Aufgabe 1.
Let K be a field, V a finite-dimensional K-vector space with a basis $\mathcal{B}$, and $f\colon V \to V$ an endomorphism. Assume that the matrix $A = [f]_{\mathcal{B}}$ is upper triangular with zeros on the main diagonal:

$$A = \begin{pmatrix} 0 & * & \dots & * & * \\ 0 & 0 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & * \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

1. Prove that f is nilpotent.
   Hint: Compute $\chi_f(X)$.
2. Show that A^k has $k-1$ zero diagonals above the main diagonal.

Aufgabe 2.
Find the Jordan normal form of the following matrices.

1. $\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$;
2. $\begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}$.

Aufgabe 3.
Let K be a field, V a finite-dimensional K-vector space, and $f\colon V \to V$ a nilpotent endomorphism. Let $f^n = 0$ and $f^{n-1} \neq 0$ and take $v \in V \setminus \operatorname{Ker}(f^{n-1})$. Prove that $v, f(v), \dots, f^{n-1}(v)$ are linearly independent.

Aufgabe 4.
Let K be a field, V a finite-dimensional K-vector space, and $f\colon V \to V$ an endomorphism. Assume that f commutes with any other endomorphism f' of V. Prove that $f = \alpha \cdot \mathrm{id}_V$ for some $\alpha \in K$.
Hint: Let $g\colon V \to K$ be a non-trivial K-linear form, $x \in V$ such that $g(x) \neq 0$ and $\alpha := g(f(x))/g(x) \in K$. For $v \in V$ consider and endomorphism $f'(u) := g(u) \cdot v$ to prove that $f(v) = \alpha \cdot v$.