LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Lineare Algebra II – Tutoriumsblatt 7

Aufgabe 1.
Let A be a finitely generated abelian group.

1. For $x_1, \ldots x_k \in A$ generating A and $c_1, \ldots, c_k \in \mathbb{N}$ with g.c.d.$(c_1, \ldots, c_k) = 1$ there exist generators $y_1, \ldots, y_k$ of A with $y_1 = c_1 x_1 + \ldots + c_k x_k$.
   Hint: Argue by induction on $s := c_1 + \ldots + c_k$. In the induction step consider the generating set $x_1, x_2 + x_1, x_3, \ldots, x_k$ and the set of natural numbers $c_1 - c_2, c_2, c_3, \ldots, c_k$.
2. Prove that A is a (finite) direct sum of cyclic subgroups.
   Hint: Argue by induction on the minimal number of generators k of A. Among the generating sets $x_1, \ldots, x_k$ choose the set with minimal possible order of x_1 and prove that $A = \langle x_1 \rangle \oplus \langle x_2, \ldots, x_k \rangle$. To get the latter, assume that there is a "relation" $m_1 x_1 + \ldots + m_k x_k = 0$ in A for $m_i \in \mathbb{N}$ and $m_1 < \operatorname{ord}(x_1)$, consider $d :=$ g.c.d.$(m_1, \ldots, m_k)$ and $c_i := m_i / d$, and apply "1." to get a contradiction with the choice of x_i.
3. Prove that A is isomorphic to $\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}/p_i^{m_i}$ for some r, n, $m_i \in \mathbb{N}$ and p_i primes (not necessarily distinct).
4. Prove that A is isomorphic to $\mathbb{Z}^r \oplus \bigoplus_{i=1}^t \mathbb{Z}/d_i$ for some $r, t, d_i \in \mathbb{N}$ such that $d_1 \mid d_2 \mid \ldots \mid d_t$.
   Remark: Observe that for a finite A one has $\exp(A) = d_t$ and $|A| = \prod_{i=1}^t d_i$.
5. Find d_i for $A \cong \mathbb{Z}/48 \oplus \mathbb{Z}/36$.

Aufgabe 2.
Let K be a field, V a finite-dimensional vector space of dimension n, and $f\colon V \to V$ an endomorphism. Assume that f is *unipotent*, i.e., that $f - \mathrm{id}_V$ is nilpotent. Prove that $\chi_f(X) = (X-1)^n$.

Aufgabe 3.
Let K be a field, V a finite-dimensional vector space, and $f\colon V \to V$ an endomorphism.

1. Prove that $\chi_{f^\vee}(X) = \chi_f(X)$.
   Hint: You can use Aufgabe 4 from Tutoriumsblatt 9 of Lineare Algebra I.
2. Assume that $\chi_f(X) = \mu_f(X)$. Prove that (V, f) is a cyclic space, i.e., $V \cong K[X]/P(X)$ in such a way that f corresponds to the endomorphism induced by multiplication by X.
3. Assume that $V = K[X]/P(X)$ is a cyclic space. Prove that $(V^\vee, f^\vee)$ is cyclic and (V, f) is isomorphic to $(V^\vee, f^\vee)$.
   Hint: You can use Aubgabe 1 und 2 from Tutoriumsblatt 3 and Aufgabe 2 from Übungsblatt 6.
4. Using the classification of $K[X]$-modules, prove that (V, f) is isomorphic to $(V^\vee, f^\vee)$ for any finite-dimensional vector space (i.e., without an assumtion that (V, f) is cyclic).

Aufgabe 4.

Let K be a field, V a finite-dimensional vector space and $f\colon V \to V$ an endomorphism. Let λ be an eigenvalue of f, and consider the sequence of subspaces

$$\operatorname{Ker}(f - \lambda \cdot \mathrm{id}) \subseteq \operatorname{Ker}(f - \lambda \cdot \mathrm{id})^2 \subseteq \operatorname{Ker}(f - \lambda \cdot \mathrm{id})^3 \subseteq \dots$$

Let $m \in \mathbb{N}$ be the minimal number such that $\operatorname{Ker}(f - \lambda \cdot \mathrm{id})^m = \operatorname{Ker}(f - \lambda \cdot \mathrm{id})^{m+1}$. Prove that $\mu_f(X) = (X - \lambda)^m \cdot P(X)$ for some $P(X) \in K[X]$ coprime to $(X - \lambda)$.