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Lineare Algebra II – Tutoriumsblatt 5

Aufgabe 1.
Recall that the polynomial ring in n variables $R[X_1, \dots, X_n]$ over the commutative ring R is defined inductively as a polynomial ring in one variable $(R[X_1, \dots, X_{n-1}])[X_n]$ over the ring $R[X_1, \dots, X_{n-1}]$.

Prove that for a commutative ring R, the map

$$\Phi\colon \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[X_1, \dots, X_n],\, R) \to R^n$$

that sends a ring homomorphism f to the n-tuple $(f(X_1), \dots, f(X_n)) \in R^n$ is in fact a bijection.

Aufgabe 2.
Consider a polynomial ring $S = \mathbb{Z}[X_{ij}, Y_{ij}]_{1\le i,j\le n}$ in $2n^2$ variables. Define the "universal" matrices $A := (X_{ij}) \in \mathrm{M}_n(S)$ and $B := (Y_{ij}) \in \mathrm{M}_n(S)$.

1. Prove that $\det(A \cdot B) = \det(A) \cdot \det(B)$.
   Hint: Consider A and B as matrices over some field.
2. Prove that for any commutative ring R and $M,\ N \in \mathrm{M}_n(R)$ one has $\det(M \cdot N) = \det(M) \cdot \det(N)$.
   Hint: Using Aufgabe 1, reduce the claim to the case of universal matrices A and B.

Aufgabe 3.
For a commutative ring R and $M \in \mathrm{M}_n(R)$ one defines the minor $M_{ij} \in R$ of M as the determinants of matrix obtained by deleting the i-th row and the j-th coloumns of M, and the co-factors $C_{ij} := (-1)^{i+j} M_{ij}$ (similarly to the case, where $R = K$ is a field).

Consider the co-factor matrix $\widetilde{M} = (C_{ij})$, and prove that $M \cdot \widetilde{M}^t = \det(M) \cdot E$.
Hint: Use the same strategy as in Aufgabe 2.

Aufgabe 4.
Let K be a field and $P(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$. Recall that one can define a (formal) derivative $P'(X) := \sum_{i=1}^{n} i \cdot a_i X^{i-1} \in K[X]$.

1. Prove that the formal derivative satisfies the Leibniz rule: $(P \cdot Q)' = P' \cdot Q + P \cdot Q'$.
2. Let $P(X) \in K[X]$ a Polynomial, such that $P(X)$ and $P'(X)$ are co-prime. Prove that all roots of $P(X)$ are simple, that is, $(X - \lambda)^k \nmid P(X)$ for $k > 1$, $\lambda \in K$.
3. Let $P(X)$ be an irreducible Polynomial over a field K of characteristic 0, and assume that K is a subfield of a field E. Prove that $P(X)$ as a polynomial over E can only have simple roots.
4. Are $P(X) := X^p - 1 \in (\mathbb{Z}/p)[X]$ and $P'(X)$ co-prime?