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Lineare Algebra II – Tutoriumsblatt 4

Aufgabe 1.
Let $V = \mathbb{R} \oplus \mathbb{R}$ be a real plane and consider a rotation r_φ of this plane about the origin counterclockwise through an angle φ.

1. Show that r_φ is an $\mathbb{R}$-linear endomorphism, and compute its matrix R_φ in the standard basis.
2. Find the minimal polynomials of r_φ.

Aufgabe 2.
Let K be a field, V a finite dimensional K-vector space, and f an endomorphism of V. Let $P(X) = P_1(X) \cdot P_2(X)$ for coprime $P_1(X)$, $P_2(X) \in K[X]$, such that $P(f) = 0$. Consider the decomposition

$$V \cong \operatorname{Ker} P_1(f) \oplus \operatorname{Ker} P_2(f)$$

from Tutoriumsblatt 3, Aufgabe 4. Denote $U_i := \operatorname{Ker} P_i(f)$, for $i = 1, 2$.

1. Prove that $f(U_i) \subseteq U_i$, for $i = 1, 2.$.
2. Let $f_i := f|_{U_i} \colon U_i \to U_i$ and $\mu_i(X) := \mu_{f_i}(X)$ be the corresponding minimal polynomials, for $i = 1, 2$. For $Q(X) \in K[X]$ prove that $Q(f)|_{U_i} = Q(f_i)$.
3. Prove that $\mu_f(X) = \text{l.c.r.}\big(\mu_1(\mathrm{X}),\, \mu_2(\mathrm{X})\big) \in K[X]$.

Aufgabe 3.
Let K be a field, V a finite dimensional K-vector space, and f an endomorphism of V. Assume that $\chi_f(X) = \prod_{i=1}^{r} (X - \lambda_i)^{n_i}$, where $\lambda_i = \lambda_j \;\Rightarrow\; i = j$. Denote $V_i := \operatorname{Ker}(f - \lambda_i \cdot \mathrm{Id}_V)^{n_i}$ and $f_i := f|_{V_i} \colon V_i \to V_i$. Prove that $\chi_{f_i}(X) = (X - \lambda_i)^{n_i}$.

Aufgabe 4.
Let K be a field and $P(X) \in K[X]$ be an irreducible polynomial. Let $E := K[X]/P(X)$. Prove that K is a subfield of E and consider $P(X)$ as a polynomial over E. Prove that $P(X)$ has a root in E.