Prof. Dr. Fabien Morel
Dr. Andrei Lavrenov, Oliver Hendrichs, Katharina Novikov

Sommersemester 2025
9. Mai 2025

Lineare Algebra II – Tutoriumsblatt 3

Aufgabe 1.
Let K be a field, and $(V,\, f)$ a finite-dimensional K-vector space with an endomorphism, and consider the induces $K[X]$-module structure on V. Prove that the following are equivalent:

1. there exist $v \in V$ such that $\{v,\, f(v),\, f^2(v), \ldots\}$ generate V;
2. V as a $K[X]$-module is generated by one element;
3. there exist a monic polynomial $P(X) \in K[X]$, such that $V \cong K[X]/P(X)$, and f corresponds to the endomorphism of $K[X]/P(X)$ induced by the multiplication by X.

Aufgabe 2.
Under equivalent conditions of Exercise 1, prove that $\chi_f(X) = \mu_f(X) = P(X)$.

Aufgabe 3.
Let R be a commutative ring and M an R-module. Define

$$\operatorname{ann}_R(M) := \{a \in R \mid (a\,\cdot)\colon M \to M \ \text{ is a zero map}\} = \{a \in R \mid \forall\, m \in M \quad am = 0\}.$$

1. Prove that $\operatorname{ann}_R(M)$ is an ideal of R.
2. For a field K, $R = K[X]$, and $(V,\, f)$ a K-vector space with an endomorphism, consider the corresponding $K[X]$-module structure on V, and prove that

$$\operatorname{ann}_{K[X]}(V,\, f) = (\mu_f(X))$$

as ideals in $K[X]$.

Aufgabe 4.
Let K be a field, $P(X) = P_1(X) \cdot P_2(X) \in K[X]$, such that $P_1(X)$ and $P_2(X)$ are coprime, and let $(V,\, f)$ be a K-vector space with an endomorphism such that $P(f) = 0$. Prove that

$$V \cong \operatorname{Ker} P_1(f) \oplus \operatorname{Ker} P_2(f) \cong \operatorname{Im} P_2(f) \oplus \operatorname{Im} P_1(f).$$