LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Lineare Algebra I – Tutoriumsblatt 06

Aufgabe 1.

1. Let X be a set, let K be a field and let $V := \mathrm{Abb}(X, K)$ denote the set of all (set-theoretic) maps from X to K. Prove that the field structure on K induces a structure of a K-vector space on V.
2. Assume that $|X| = n$. Construct a bijective linear map from V to K^n.
3. Prove that the functions 1, $\cos x$, $\cos(2x)$, $\ldots$, $\cos(nx)$ are linearly independent over $\mathbb{R}$ as elements of $V = \mathrm{Abb}(\mathbb{R}, \mathbb{R})$.

Aufgabe 2.
Prove that $\mathbb{R}$ has a natural structure of a $\mathbb{Q}$-vector space, and that 1, $\sqrt{2}$, $\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

Aufgabe 3.

1. Let V be a vector space over a field K, and let $u, v \in V \setminus \{0\}$ be linearly dependent. Prove that $\exists\, \lambda \in K$ such that $u = \lambda v$.
2. Prove that for vectors $v_1, v_2 \in V$ and scalars $\alpha_1, \alpha_2 \in K$ such that $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 v_2 + \alpha_2 v_1$, either $v_1 = v_2$, or $\alpha_1 = \alpha_2$.
3. Let V be a vector space over a field K, and let $u_1, \ldots, u_n, u_{n+1} \in V$ be vectors such that u_{n+1} is a linear combination of $u_1, \ldots, u_n$. Prove that a subspace of V generated by $u_1, \ldots, u_n$ coincides with a subspace of V generated by $u_1, \ldots, u_{n+1}$.
4. Assume that $u_1, \ldots, u_n \in V$ are linearly independent, and $u_{n+1} \in V$ such that $u_1, \ldots, u_{n+1}$ are linearly dependent. Prove that u_{n+1} is a linear combination of $u_1, \ldots, u_n$.

Aufgabe 4.

1. Prove that $(\mathbb{Z}, +)$ does not admit a structure of a vector space over any field K.
2. Prove that an abelian group A admits a structure of $\mathbb{Z}/p\mathbb{Z}$-vector space for a prime p if and only if $\forall\, a \in A$ one has $pa = 0$.