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Lineare Algebra I – Tutoriumsblatt 05

Aufgabe 1.

1. Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.
2. Find $\sum_{k=0}^{n}\binom{n}{k}$.
3. Prove that $\sum_{k=0}^{n}(-1)^k\binom{2n}{2k} = 2^n \cos\frac{n\pi}{2}$.
4. Find $\sum_{k=1}^{n}(-1)^{k+1}\binom{2n}{2k-1}$.

Aufgabe 2.

1. Prove that for any $a,\, b \in \mathbb{Z}$ with the greatest common divisor d there exist $x,\, y \in \mathbb{Z}$ such that $d = ax + by$.
   Hint: Consider the minimal element d' of the set $\{ax + by \mid a,\, b \in \mathbb{Z}\} \cap (\mathbb{N} \setminus \{0\})$ and divide a and b by d' with reminder.
2. Prove that $(\mathbb{Z}/n\mathbb{Z})^\times = \{\overline{a} \mid (a,\, n) = 1\}$.
3. Prove that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime number.

Aufgabe 3.

1. Let $\mathbb{Z}[i] = \{a + bi \mid a,\, b \in \mathbb{Z}\} \subset \mathbb{C}$, and prove that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.
2. Find $\mathbb{Z}[i]^\times$.
   Hint: Use the complex norm.

Aufgabe 4.
Let K be a field such that $\mathbb{R}$ is a subfield of K, and there exists an element $r \in K \setminus \mathbb{R}$ such that $K = \{a + br \mid a,\, b \in \mathbb{R}\}$. Prove that K is isomorphic to $\mathbb{C}$.
Hint: You can use Aufgabe 1 from Übungsblatt 4.