TOPOLOGY V

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Abstract. These are lecture notes for my lecture "Topology V" which I taught in the winter term 2025/26 at LMU Munich.

Contents

1. Recollection/Prerequisites

There will be a biweekly exercise session where we discuss further examples and questions. There will be no formal exercise sheets. If you want to get credits for this course, you can do so under WP13 or WP16 for 3 ECTS. The examination will be an oral exam at the end of the term.

This course will build on the lectures Topology I (WS 23/24), Topology II (SS 24), Topology III (WS 24/25), and Topology IV (SS 25) taught at LMU. We briefly recall the main topics that were covered, so a reader has an impression what will be the assumed background knowledge.

(1) Point-set topology
(2) Homotopy theory: homotopy groups, CW complexes, applications of cellular approximation, cofibrations, Seifert-van Kampen's theorem
(3) Covering theory; Fundamental theorem of covering theory
(4) Singular Homology; Definition, Properties, Applications.
(5) Singular Cohomology; Cup product, Universal coefficient theorems, Künneth theorem
(6) Topological Manifolds: Orientability and Poincaré duality, Applications
(7) Homotopy theory: Fibrations, long exact homotopy sequence, Whitehead's theorem, cellular approximation theorem, homotopy excision theorem, Freudenthal
(8) Hurewicz theorems
(9) Eilenberg–Mac Lane spaces and representability of cohomology
(10) Principal G-bundles
(11) Obstruction theory

Date: February 2, 2026.

(12) Steenrod operations
(13) The Leray–Hirsch theorem
(14) Thom isomorphism for spherical fibrations,
(15) Stiefel–Whitney and Wu classes, Chern classes, Pontryagin classes, the cohomology of BO, BU, remarks on BTop and BG,
(16) Poincare duality complexes and Wu's formulas
(17) A survey on manifolds, tangent bundles, Pontryagin–Thom constructions.

Parts (1)–(4) were covered in Topology I [Lan23], parts (5)–(7) were covered in Topology II [Win24], parts (8)–(13) were covered in Topology III [Lan24], and parts (14)–(17) were covered in [Lan25]. The lecture notes for these courses are available on the course webpage.

Topic (16) is not relevant for this course (in particular all the higher categorical things we used to define Poincaré duality complexes) and Topic (17) will only be used in a minimalistic way. The rough plan for this term is to cover the following, (6) below only if time permits (which it almost surely will not);

(1) Spectral sequences and the Serre spectral sequence
(2) Rational homotopy theory
(3) Some stable homotopy groups of spheres, cohomology of EM spaces
(4) Computation of the rational oriented bordism ring, the signature theorem
(5) Construction of exotic spheres.
(6) Further applications to manifolds; geometric interpretation of cup product, existence of manifolds with certain cell structures, spin$^{\mathbb{C}}$-structures + intersection form on 4-manifolds, (obstructions to the) existence of submanifolds representing homology classes, Rokhlin's theorem

2. SPECTRAL SEQUENCES

2.1. **Definition** A strongly convergent spectral sequence consists of the following data satisfying the following axioms:

(1) a complete and separated filtration F on a graded abelian group M called the *abutment* of the spectral sequence. That is, $M = \{M_n\}_{n\in\mathbb{Z}}$ is a graded abelian group and $F_\bullet M_n$ is a exhaustive[1] and separated[2] filtration on M_n for every $n \in \mathbb{Z}$.
(2) for each $r \geq 1$ a bigraded abelian group $E^r_{p,q}$ equipped with a differential $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$, that is $(d^r)^2 = 0$.
(3) An isomorphism between the homology $H_*(E^r, d^r)$ of (E^r, d^r) and E^{r+1}.
(4) For every pair (p, q), there is an $N(p, q)$ such that for all $r \geq N(p, q)$, $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $d^r\colon E^r_{p+r,q+r-1} \to E^r_{p,q}$ vanish. It follows that $E^{N(p,q)}_{p,q} \cong E^{N(p,q)+s}_{p,q}$ for all $s \geq 0$, so we call this common term $E^\infty_{p,q}$.[3]
(5) An isomorphism between the associated graded $\mathrm{gr}(F_\bullet M)$ of the abutment (which is a bigraded abelian group) and E^∞ (which is also a bigraded abelian group). Explicitly, an isomorphism $F_k(M_n)/F_{k-1}(M_n) \cong E^\infty_{\dots}$.

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[1] That is $\mathrm{colim}_n\, M_n = M_n$

[2] That is $\lim_n M_n = 0$

[3] Sometimes it is appropriate to relax these conditions and to assume only that for all $r \geq N(p, q)$, $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ vanishes. Then one obtains surjections $E^r_{p,q} \to E^{r+1}_{p,q}$ for all such r, and the colimit along these maps again deserves the name $E^\infty_{p,q}$.

A spectral sequence as above is called multiplicative if $F_\bullet M$ is a filtered graded commutative ring, all bigraded abelian groups $E^r_{*,*}$ are bigraded commutative rings and the differential satisfies the Leibniz rule

$$d^r(x \cdot y) = d^r(x) \cdot y + (-1)^{|x|} x \cdot d^r(y)$$

and the isomorphism $\mathrm{gr}(F_\bullet M) \cong E^\infty$ is one of bigraded commutative rings.

2.2. **Warning** Just like exact sequences, a spectral sequence is not itself capable of computing the graded abelian group M, only the associated graded with respect to some filtration on it which is part of the spectral sequence. Concretely, this means that in order to compute M itself, possible extension problems have to be solved. This is something one can then try to do by hand, but the spectral sequence is not a priori of any help in this task.

One way to obtain a spectral sequence is through filtered chain complexes. I recommend reading the relevant part of Weibel's book on the topic [Wei94] or Hatcher's account on spectral sequences [Hat04] or McCleary's book [McC01]. In particular, the main example of a spectral sequence we will use in this course, the Serre spectral sequence, can be constructed from a filtered chain complex. An elegant construction using bisimplicial sets was found by Dress [Dre67].

However, not all spectral sequences that arise in practice arise naturally in this fashion, but they do arise naturally as the spectral sequence associated to a filtered *spectrum*. We briefly explain how a filtered spectrum gives rise to a spectral sequence now, see [Lur17] for details, but beware of the different indexing convention: We will have to make a choice whether a $\mathbb{Z}$-indexed filtration lowers or raises degree. To the best of my knowledge, either choice becomes annoying at some point, so we stick to the one that is closer to what we obtain from the examples that we shall consider, but which differs from the one appearing in [Lur17].

2.3. **Definition** A filtered spectrum is an object of $\mathrm{Fun}((\mathbb{Z}, \geq), \mathrm{Sp}) =: \mathrm{Fil}(\mathrm{Sp})$, where we view $(\mathbb{Z}, \geq)$ as a poset. This poset is in fact canonically a symmetric monoidal category under the sum of integers. Hence, $\mathrm{Fil}(\mathrm{Sp})$ is naturally a symmetric monoidal category under Day convolution so we may form $\mathrm{CAlg}(\mathrm{Fil}(\mathrm{Sp}))$. For $F \in \mathrm{Fil}(\mathrm{Sp})$ we write F_n for its evaluation at n. A filtered spectrum is called separated if $\lim_n F = 0$.

A graded spectrum is an object of $\mathrm{Fun}(\mathbb{Z}^\delta, \mathrm{Sp}) =: \mathrm{Gr}(\mathrm{Sp})$, where we view $\mathbb{Z}^\delta$ as a discrete category. This is also symmetric monoidal under the sum of integers, so $\mathrm{Gr}(\mathrm{Sp})$ also carries a Day convolution symmetric monoidal structure.

2.4. **Remark** Let us gather some facts about the above.

(1) There is a functor $\mathrm{gr}\colon \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$ called the associated graded of a filtration, sending F to $n \mapsto \mathrm{gr}^n(F) = \mathrm{cofib}(F_{n+1} \to F_n)$. This functor preserves colimits and limits and is equipped with a canonical symmetric monoidal structure.

(2) The colimit of a filtration gives rise to a functor $\mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$; we often call this the *underlying* spectrum of a filtered spectrum (and hence think of the filtered spectrum as a filtration on its colimit).

(3) Given a spectrum X, its Whitehead tower $\tau_{\geq \bullet} X$

$$\cdots \to \tau_{\geq n+1} X \to \tau_{\geq n} X \to \tau_{\geq n-1} X \to \dots$$

is a separated filtered spectrum with underlying spectrum X. The association $X \mapsto \tau_{\geq \bullet} X$ refines to a lax symmetric monoidal functor $\mathrm{Sp} \to \mathrm{Fil}(\mathrm{Sp})$, see e.g. [Hed, Prop.

II.1.26]. In particular, if X is a (commutative) algebra in Sp, its Whitehead tower $\tau_{\geq \bullet} X$ is a (commutative) algebra in Fil(Sp).

So let us now fix a separated filtered spectrum $X_\bullet$ with colimit $\operatorname{colim} X_\bullet = X$. Note that there is a fibre sequence

$$\mathrm{gr}^n(X) = X_n/X_{n+1} \to X_{n-1}/X_{n+1} \to X_{n-1}/X_n = \mathrm{gr}^{n-1}(X)$$

and hence a connecting morphism $\mathrm{gr}^{n-1}(X) \to \Sigma \mathrm{gr}^n(X)$.

2.5. **Theorem** *Assume that for each $k \in \mathbb{Z}$,*

(a) *the maps $\pi_k(X_n) \to \pi_k(X)$ are isomorphisms for n sufficiently small and*

(b) *the group $\pi_k(X_n)$ vanishes for n sufficiently large.*

Then there is associated a strongly convergent spectral sequence with

(1) *abutment given by $\pi_*(X)$ with (finite, by assumption) filtration given by $F_n(\pi_*(X)) = \mathrm{Im}(\pi_*(X_n) \to \pi_*(X))$.*

(2) *the first page is given as follows. $E^1_{p,q} = \pi_{p+q}\mathrm{gr}^{-p}(X)$ and the differential is the map induced on homotopy groups by the map $\mathrm{gr}^{n-1}(X) \to \Sigma\mathrm{gr}^n(X)$: This is a map*

$$\pi_{p+q}\mathrm{gr}^{-p}(X) \to \pi_{p-1+q}\mathrm{gr}^{-p+1}(X)$$

as needed.

The higher pages of the spectral sequence can also be described via the so called décalage construction, see [Ant24] *for the details.*

If X is a (commutative) algebra in Fil(Sp)*, then the associated spectral sequence is multiplicative.*

Depending on the particular filtered spectrum we use as input for our spectral sequence, it will be convenient to reindex as to get the usual familiar grading conventions. Let us work this out in the main examples of interest:

2.6. **Example** (The homological Serre spectral sequence) Let $F \to E \to B$ be a fibre sequence in An and let A be a coefficient abelian group. Note that the category Fil(Sp) has all limits and colimits. We may therefore form $\operatorname{colim}_B \tau_{\geq \bullet} C_*(F; A)$ which we claim is a separated filtration with colimit $C_*(E; A)$. Indeed, recall that $C_*(X; A)$ is merely a notation for the colimit over X over the constant functor taking values $A \in \mathrm{Sp}$. Then we find

$$\operatorname*{colim}_n \operatorname*{colim}_B \tau_{\geq n} C_*(F; A) = \operatorname*{colim}_B \operatorname*{colim}_n \tau_{\geq n} C_*(F; A) = \operatorname*{colim}_B C_*(F; A) = C_*(E; A).$$

To see that the filtration is separated, note that $\operatorname{colim}_B \tau_{\geq n} C_*(F; A)$ is itself n-connective, so we find that in the system computing $\lim_n \operatorname{colim}_B \tau_{\geq n} C * (F; A)$, the terms become more and more connective and so the limit vanishes; in fact, this argument shows that (a) and (b) of the assumptions in Theorem 2.5 hold true.

Exercise: The filtration we just introduced leads to a finite filtration on $H_k(E; A) = \pi_k C_*(E; A)$ for each k. Moreover, the associated graded has the following property (hint: use that the associated graded functor commutes with colimits):

$$\mathrm{gr}^n(\operatorname*{colim}_B \tau_{\geq \bullet} C_*(F; A)) = \Sigma^n C_*(B; H_n(F)).$$

We therefore obtain a strongly convergent spectral sequence with abutment $C_*(E; A)$ and first page given by

$$E^1_{s,t} = \pi_{s+t}\mathrm{gr}^{-s}(\operatorname*{colim}_B \tau_{\geq \bullet} C_*(F; A)) = \pi_{s+t}\Sigma^{-s} C_*(B; H_{-s}(F)) = H_{2s+t}(B; H_{-s}(F)).$$

Taking the following reindexing: $p := 2s+t$ and $q = -s$, we obtain $E'_{p,q} = H_p(B; H_q(F))$. Now, the differential in the above spectral sequence goes as follows:

$$H_{2s+t}(B; H_{-s}(F)) = E^1_{s,t} \to E^1_{s-1,t} = H_{2s+t-2}(B; H_{-s+1}(F)$$

so in our reindexing, we obtain

$$H_p(B; H_q(F)) \to H_{p-2}(B; H_{q+1}(F))$$

making the reindexing to $E_{p,q}$ look like the second page of a spectral sequence.

Therefore, after the above reindexing, we obtain a spectral sequence with second page and differential

$$E^2_{p,q} = H_p(B; H_q(F)) \xrightarrow{d^2} H_{p-2}(B; H_{q+1}(F)).$$

This is the homological Serre spectral sequence associated to the fibre sequence $F \to E \to B$.

2.7. **Example** (The cohomological Serre spectral sequence) Again, we consider a fibre sequence $F \to E \to B$ in An and a coefficient abelian group (commutative ring) k. We then form $\lim_B \tau_{\geq \bullet} C^*(F; k)$ which is a (commutative algebra) filtered spectrum. It is separated since limits commute, and has colimit $C^*(E; k)$ since the terms over which we take a colimit have growing coconnectivity, since they are obtained by taking a limit over terms which have growing coconnectivity. This implies again that the filtration we obtain on $\pi_k C^*(E; k)$ is finite for each $k \in \mathbb{Z}$. Moreover, the associated graded now satisfies

$$\mathrm{gr}^n(\lim_B \tau_{\geq \bullet} C^*(F; k)) = \Sigma^n C^*(B; H_n(F; k))$$

so that we obtain a strongly convergent spectral sequence with abutment $C^*(E; k)$ and first page

$$E^1_{s,t} = \pi_{s+t}\mathrm{gr}^{-s}(\lim_B \tau_{\geq \bullet} C^*(F; k)) = \pi_{2s+t} C^*(B; H^s(F; k)) = H^{-2s-t}(B; H^s(F; k)).$$

Performing now the reindexing $p := -2s - t$ and $q = s$, we obtain again a spectral sequence (of cohomological indexing convention) with second page (again, this is just convention) and differential

$$E_2^{p,q} = H^p(B; H^q(F; k)) \to H^{p+2}(B; H^{q-1}(F; k)) = E_2^{p+2,q-1}.$$

This is the cohomological Serre spectral sequence associated to the fibre sequence $F \to E \to B$. If k is a commutative ring, then $C^*(F; k)$ is a commutative algebra in spectra, hence $\tau_{\geq \bullet} C^*(F; k)$ is a commutative algebra in Fil(Sp), and since the forgetful functor from commutative algebras always preserves limits, we find that $\lim_B \tau_{\geq \bullet} C^*(F; k)$ is also a naturally a commutative algebra in filtered spectra. Hence, the cohomological Serre spectral sequence is multiplicative.

2.8. **Example** The Atiyah–Hirzebruch spectral sequence for computing $E_*(X) = \pi_*(X \otimes E)$ where $E \in \mathrm{Sp}$ and $X \in \mathrm{An}$. This is given by the spectral sequence associated with the filtered spectrum $\mathrm{colim}_X \tau_{\geq \bullet} E$, where the colimit is over the constant diagram indexed on X with value $\tau_{\geq \bullet} E$, the Whitehead tower of E. Note that when E is not bounded below, we arrive at a situation in which footnote 3 in Definition 2.1 has to be taken seriously; We obtain a spectral sequence with (reindexed) second page and differentials given by

$$H_p(X; \pi_q(E)) = E^2_{p,q} \to E^2_{p-2,q+1} = H_{p-2}(X; \pi_{q+1}(E))$$

converging to $\pi_{p+q}(X \otimes E) = E_{p+q}(X)$. The convergence is strong (in the sense of a finite filtration on the abutment) if X has bounded above homology or E has bounded below homology.

There exists a similar spectral sequence, the cohomological Atiyah–Hirzebruch spectral sequence, converging to $\pi_*(\mathrm{Map}(X, E))$, but here convergence is even more subtle.

2.9. **Example** (The twisted Serre spectral sequence) Here the setup is as follows. Consider a fibration $F \xrightarrow{i} E \to B$ and a functor $\varphi\colon E \to \mathrm{Sp}$ (The Serre spectral sequence as above is the special case where φ is constant with values A or k and the Atiyah–Hirzebruch spectral sequence above is the special case $F = *$ and the functor $X \to \mathrm{Sp}$ is constant – apologies for the double use of E in these two examples). Then there is a spectral sequence abutting to $\pi_*(\mathrm{colim}_E\, \varphi)$ coming from the filtration $\mathrm{colim}_B\, \tau_{\geq \bullet}\, \mathrm{colim}_F\, \varphi i$. It's (once reindexed) second page then looks like

$$E^2_{p,q} = H_p(B; \pi_q(\operatorname*{colim}_F \varphi i))$$

Example: Consider the fibration BSpin → BSO → $K(\mathbb{Z}/2, 2)$ and the functor BSO → Sp induced by the J-homomorphism. Then we obtain a spectral sequence

$$E^2_{p,q} = H_p(K(\mathbb{Z}/2, 2); \mathrm{MSpin}_q(*)) \Rightarrow \mathrm{MSO}_{p+q}(*).$$

I learned the following from Peter Teichner: This spectral sequence can be used to prove that the signature of smooth spin 4-manifolds is divisible by 16. We should do this in an exercise session once we have the necessary ingredients (which are essentially the computation of all the groups that participate in the spectral sequence computation for $\pi_4(\mathrm{MSO})$). Easier exercise: Use a similar spectral sequence to show that the map $\mathrm{MSpin}[\frac{1}{2}] \to \mathrm{MSO}[\frac{1}{2}]$ is an equivalence. Hint: Show $H_*(K(\mathbb{Z}/2, 2); A) = A$ if A is a $\mathbb{Z}[\frac{1}{2}]$-module.

A similar twisted cohomological Serre spectral sequence exists, but again, the convergence is more subtle in general.

2.10. **Example** (The homological Serre spectral sequence for monoids) The homological Serre spectral sequence can also admit a multiplicative structure as we explain in this example. So let $N \to M$ be a map of monoids in anima with M connected and fibre F. We may view this map as an algebra in the slice category $\mathrm{An}_{/M}$, which carries a monoidal structure $\boxtimes$ informally given by $(N \to M) \boxtimes (N' \to M) = N \times N' \to M \times M \to M$ where the map $M \times M \to M$ is the multiplication of the monoid M. The straightening-unstraightening equivalence then refines to a monoidal equivalence $\mathrm{An}_{/M} \simeq \mathrm{Fun}(M, \mathrm{An})$ where $\mathrm{Fun}(M, \mathrm{An})$ is equipped with the Day convolution monoidal structure.[4] In particular, we find an equivalence

$$\mathrm{Mon}(\mathrm{An})_{/M} \simeq \mathrm{Alg}(\mathrm{An}_{/M}) \simeq \mathrm{Alg}(\mathrm{Fun}(M, \mathrm{An})) = \mathrm{Fun}^{\mathrm{lax}}(M, \mathrm{An})$$

where the superscript lax stands for lax monoidal functors. As a consequence, the monoid map $N \to M$ is classified by a lax monoidal functor $M \to \mathrm{An}$. We may then consider the composite

$$M \to \mathrm{An} \to \mathrm{Sp} \to \mathrm{Fil}(\mathrm{Sp})$$

where An → Sp is given by $X \mapsto C_*(X; k)$ for some fixed ring k and Sp → Fil(Sp) is the Whitehead-tower functor. Both of these functors are lax symmetric monoidal, so we deduce that the above composite is canonically lax monoidal, i.e. an object of $\mathrm{Fun}^{\mathrm{lax}}(M, \mathrm{Fil}(\mathrm{Sp})) =$

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[4] See [Ram22] for vast generalisations of this statement and [Ram22, Corollary D] with $\mathcal{O} = \mathbb{E}_1$ for the case at hand.

$\mathrm{Alg}(\mathrm{Fun}(M, \mathrm{Fil}(\mathrm{Sp})))$. Now, in general, left Kan extension along a monoidal map $M \to M'$ refines to a lax monoidal functor $\mathrm{Fun}(M, \mathcal{C}) \to \mathrm{Fun}(M', \mathcal{C})$ under Day convolution (as soon as the Day convolution exists, which is the case for instance when $\mathcal{C}$ is cocomplete), see e.g. [**?**, proof of Cor. 3.8]. In particular, left Kan extension along $M \to *$, i.e. colim_M refines to a lax monoidal functor. In particular, it induces a functor on algebras

$$\operatorname*{colim}_M \colon \mathrm{Fun}^{\mathrm{lax}}(M, \mathrm{Fil}(\mathrm{Sp})) \to \mathrm{Alg}(\mathrm{Fil}(\mathrm{Sp}))$$

whose underlying filtered spectrum is the one giving rise to the homological Serre spectral sequence if evaluated on the image of $N \to M$ in $\mathrm{Mon}(\mathrm{An})_{/M}$ under the previously described functor $\mathrm{Mon}(\mathrm{An})_{/M} \simeq \mathrm{Fun}^{\mathrm{lax}}(M, \mathrm{An}) \to \mathrm{Fun}^{\mathrm{lax}}(M, \mathrm{Fil}(\mathrm{Sp}))$.

In particular, for $F \to E \to B$ a fibre sequence of anima, we find that the homological Serre spectral sequence for the fibre sequence $\Omega F \to \Omega E \to \Omega B$ is multiplicative.

2.11. **Remark** In the lectures, I drew the relevant pictures for the second pages of the above spectral sequences in the plane and mentioned that the cohomological indexing moves the cohomolgical Serre spectral sequence of this last example, which naturally sits in the lower left quadrant, into the upper right quadrant at the cost of reversing the direction of all differentials. The homological Serre spectral sequence naturally lives in the first quadrant.

2.12. **Remark** A map of filtered spectra gives rise to a map of associated spectral sequences. In particular, given a pullback diagram

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & E \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}$$

with vertical fibre F, we obtain a map between the corresponding Serre spectral sequences. Indeed, say for the homological Serre spectral sequence, note that there is a canonical map

$$\operatorname*{colim}_{B'} \tau_{\geq \bullet} C_*(F) \to \operatorname*{colim}_{B} \tau_{\geq \bullet} C_*(F)$$

of filtered spectra. More generally, a map between fibre sequences

$$\begin{tikzcd}
F' \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & B' \arrow[d] \\
F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

gives rise to a map of filtered spectra

$$\operatorname*{colim}_{B'} \tau_{\geq \bullet} C_*(F') \to \operatorname*{colim}_{B} \tau_{\geq \bullet} C_*(F)$$

giving in turn rise to a map of Serre spectral sequences; similarly for cohomology.

2.13. **Remark** Suppose given a pullback square

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & E \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}$$

with vertical fibre F and associated map

$$\operatorname*{colim}_{B'} \tau_{\geq \bullet} C_*(F) \to \operatorname*{colim}_{B} \tau_{\geq \bullet} C_*(F)$$

of filtered spectra. Then we may form the cofibre of this map of filtered spectra (which is computed levelwise), and consider the spectral sequence associated to this cofibre. Since the associated graded functor commutes with colimits, we obtain a spectral sequence with (reindexed) second page given by:

$$E^2_{p,q} = H_p(B, B'; H_q(F)) \Rightarrow H_{p+q}(E, E')$$

converging from relative homology of $B \to B'$ to relative homology of $E \to E'$; again there is a similar such statement for the cohomological Serre spectral sequence.

The naturality of the Serre spectral sequence for instance implies the following:

2.14. **Lemma** *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibre sequence with F connected. Then*

(1) *for the homological Serre spectral sequence, $E^\infty_{p,0}(\pi)$ canonically identifies with the image of the map $\pi_* \colon H_p(E) \to H_p(B)$. Similarly, $E^\infty_{0,q}(\pi)$ identifies with the image of $i_* \colon H_q(F) \to H_q(E)$.*

(2) *for the cohomological Serre spectral sequence, $E^{p,0}_\infty(\pi)$ identifies with the image of $\pi^* \colon H^p(B) \to H^p(E)$. Similarly, $E^{0,q}_\infty$ identifies with the image of $i^* \colon H^q(E) \to H^q(F)$.*

Proof. Consider the map of fibre sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, equal] \\
* \arrow[r] & B \arrow[r, equal] & B
\end{tikzcd}$$

It induces a map on (co)homological Serre spectral sequences, so we obtain commutative diagrams

$$\begin{tikzcd}
H_p(E) \arrow[r, two heads] \arrow[d, "\pi_*"'] & E^\infty_{p,0}(\pi) \arrow[d, hook] & & E^{p,0}_\infty(\mathrm{id}) \arrow[r, "\cong"] \arrow[d, two heads] & H^p(B) \arrow[d, "\pi^*"] \\
H_p(B) \arrow[r, "\cong"] & E^\infty_{p,0}(\mathrm{id}) & & E^{p,0}_\infty(\pi) \arrow[r, hook] & H^p(E)
\end{tikzcd}$$

showing claim (1). Similarly, we may consider the map of fibre sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F \arrow[r, "\mathrm{pr}"] \arrow[d] & * \arrow[d] \\
F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

which again induces a map on (co)homological Serre spectral sequences, to obtain commutative diagrams

$$\begin{tikzcd}
E^\infty_{0,q}(\mathrm{pr}) \arrow[r, "\cong"] \arrow[d, two heads] & H_q(F) \arrow[d, "i_*"] & & H^q(E) \arrow[r, two heads] \arrow[d, "i^*"'] & E^{0,q}_\infty(\pi) \arrow[d, hook] \\
E^\infty_{0,q}(\pi) \arrow[r, hook] & H_q(E) & & H^q(F) \arrow[r, "\cong"] & E^{0,q}_\infty(\mathrm{pr})
\end{tikzcd}$$

showing claim (2). □

Let us now work through some examples of the Serre spectral sequence.

Exercise. Prove the Leray–Hirsch theorem using the Serre spectral sequence.

2.15. **Example** The cohomological Serre spectral sequence for the fibration $S^1 \to * \to \mathbb{CP}^\infty$. We obtain $H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[x]$ with $|x| = 2$ from the multiplicativity of the spectral sequence.

2.16. **Example** Consider the fibration $\Omega S^n \to * \to S^n$ for $n > 1$. We obtain that $H^*(\Omega S^n;\mathbb{Z}) = \Gamma_{\mathbb{Z}}[x]$ with $|x| = n-1$ if n is odd and $\Gamma_{\mathbb{Z}}[x] \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}[e]$ with $|e| = n-1$ and $|x| = 2(n-1)$ if n is even.

Let us give an application of this computation:

2.17. **Lemma** *There is a fibre sequence*

$$S^{2n-1} \to \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$$

where the first map is the unit of the suspension-loop adjunction and the second map is the Hopf map.

Proof. We recall the construction of the Hopf map: To that end, by adjunction, we may equivalently describe a map $\Sigma\Omega S^{2n} \to S^{4n-1}$. For that, we recall the James splitting [Lan25, Exercise 4 Sheet 2]:

$$\Sigma\Omega S^{2n} \simeq \Sigma(\bigvee_{k\geq 1} S^{k(2n-1)})$$

so that there is a canonical projection map to the $k = 2$ wedge-summand. The so constructed map $H\colon \Omega S^{2n} \to \Omega S^{4n-1}$ induces an isomorphism

$$H^{4n-2}(\Omega S^{4n-1};\mathbb{Z}) \to H^{4n-2}(\Omega S^n;\mathbb{Z}),$$

in other words, it sends the divided power generator of the source to the divided power generator of the target from Example 2.16, see Remark 2.18 below. The Serre spectral sequence therefore implies that the fibre F of the map H has the (co)homology of S^{2n-1}. By Hurewicz, there is then a map $S^{2n-1} \to F$ inducing an isomorphism on $H_{2n-1}(-;\mathbb{Z})$ – this map is then an equivalence by Whitehead's theorem. That the resulting map $S^{2n-1} \simeq F \to \Omega S^{2n}$ is (up to a sign) the unit of the adjunction follows simply since $\pi_{2n-1}\Omega S^{2n} \cong \mathbb{Z}$ and the unit of the adjunction is also a generator of this group. □

2.18. **Remark** First, we note that the same arguments provide a map $H\colon \Omega S^n \to \Omega S^{2n-1}$; again it is adjoint to the map

$$\Sigma\Omega S^n \simeq \Sigma(\bigvee_{k\geq 1} S^{k(n-1)}) \to \Sigma(S^{2n-2}) = S^{2n-1}$$

induced by the projection to the $k = 2$ wedge-summand.

We now argue that in this situation, the induced map

$$H_{2n-2}(\Omega S^n;\mathbb{Z}) \to H_{2n-2}(\Omega S^{2n-1};\mathbb{Z})$$

is an isomorphism. Indeed, by the suspension isomorphism, it suffices to show that the map

$$H_{2n-1}(\Sigma\Omega S^n;\mathbb{Z}) \to H_{2n-1}(\Sigma\Omega S^{2n-1};\mathbb{Z})$$

is an isomorphism. Note that the counit map $\epsilon\colon \Sigma\Omega S^{2n-1} \to S^{2n-1}$ is a one-sided inverse of $\Sigma(\eta)$, where $\eta\colon S^{2n-2} \to \Omega S^{2n-1}$ is the unit of the adjunction; this map is $(4n-2)$-connected by Freudenthal, so we deduce that the counit ϵ induces an isomorphism on H_{2n-1}. Hence it suffices to show that the composite

$$H_{2n-1}(\Sigma\Omega S^n;\mathbb{Z}) \to H_{2n-1}(\Sigma\Omega S^{2n-1};\mathbb{Z}) \to H_{2n-1}(S^{2n-1};\mathbb{Z})$$

is an isomorphism. But by construction, it is the projection to the $k = 2$ wedge summand which indeed induces an isomorphism on H_{2n-1} as required.

Let us now discuss what is known as Serre class theory.

2.19. **Definition** A Serre class is a collection of abelian groups which is closed under extensions, subobjects, and quotients. A derived Serre ring is a Serre class $\mathcal{C}$ such that $A, B \in \mathcal{C}$ implies that $A \otimes B$ and $\mathrm{Tor}(A, B)$ are contained in $\mathcal{C}$.

2.20. **Example** The following examples form derived Serre rings.

1. torsion abelian groups, that is $\ker(-\otimes\mathbb{Q})$, or more generally $\mathcal{P}$-primary torsion abelian groups where $\mathcal{P}$ is a set of primes, that is $\ker(-\otimes\mathbb{Z}[\frac{1}{\mathcal{P}}])$.
2. finite abelian groups.
3. finitely generated abelian groups.
4. trivial abelian groups.

It is important to say p-primary torsion abelian groups in example (3): The collection of $\mathbb{F}_p$-vector spaces do *not* form a Serre class, since they are not closed under extensions in abelian groups. Similarly, torsionfree abelian groups do not form a Serre class, since they are not closed under quotients and for instance $\mathbb{Q}$-vector spaces are not a Serre class since they are not closed under subobjects.

2.21. **Lemma** *Let $F \to E \to B$ be a simple fibre sequence with B connected, that is, where the $\pi_1(B)$-action on $H_*(F)$ is trivial, and $\mathcal{C}$ a derived Serre ring. If two out of the three terms $\tilde{H}_*(B;\mathbb{Z})$, $\tilde{H}_*(E;\mathbb{Z})$, and $\tilde{H}_*(F;\mathbb{Z})$ are contained in $\mathcal{C}$[5] then so is the third.*

Proof. By assumption, the a terms appearing in the homological Serre spectral sequence are untwisted homology groups. Assume that $\tilde{H}_*(B;\mathbb{Z})$ and $\tilde{H}_*(F;\mathbb{Z})$ are contained in $\mathcal{C}$. Then, by the universal coefficient theorem, every term on the second page of the Serre spectral sequence is also contained in $\mathcal{C}$. Hence, the same is true for any higher page, since $\mathcal{C}$ is a Serre class. Consequently, $\tilde{H}_*(E;\mathbb{Z})$ is also contained in $\mathcal{C}$ as it is an iterated extension of finitely many terms on some higher page of the spectral sequence.

Now let us assume that $\tilde{H}_*(E;\mathbb{Z})$ and $\tilde{H}_*(B;\mathbb{Z})$ are contained in $\mathcal{C}$. We deduce that the infinite page of the spectral sequence is contained in $\mathcal{C}$ as it contains the associated graded of a filtration of an abelian group in $\mathcal{C}$. Moreover, there is an exact sequence

$$H_2(B) \to H_1(F) \to E^\infty_{0,1} \to 0$$

so the assumption that $H_2(B)$ lies in $\mathcal{C}$ together with the just made observation implies that $H_1(F) \in \mathcal{C}$. Using again the universal coefficient theorem, we deduce that the whole 1-line of the second page of the spectral sequence lies in $\mathcal{C}$, and hence the 1-line of all higher pages as well. Then we can induct up on k in $H_k(F)$ to see that all $H_k(F)$ are contained in $\mathcal{C}$.

The same argument works if the role of F and B are reversed. □

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[5] By which we mean that $\tilde{H}_k(B;\mathbb{Z})$ is contained in $\mathcal{C}$ for all k.

2.22. **Lemma** *Let $\mathcal{C}$ be any of the derived Serre rings of Example 2.20 and $A \in \mathcal{C}$. Then $H_k(K(A,n);\mathbb{Z})$ lies in $\mathcal{C}$ for all $n, k > 0$.*

Proof. Using the (simple) fibre sequence

$$K(A, n-1) \to * \to K(A,n)$$

and induction, together with Lemma 2.21, we see that it suffices to prove the result for $n = 1$. We now do a case by case study and begin with the following observation. Let A be a finite p-local abelian group (that is p-primary torsion and finite). Then $H_k(BA;\mathbb{Z})$ is again finite p-local. Indeed, by the classification of such groups and the Künneth theorem, it suffices to treat the case $A = \mathbb{Z}/p^k$. Exercise: $H_k(B\mathbb{Z}/n;\mathbb{Z})$ is cyclic of order n if $k > 0$ is odd and trivial if $k > 0$ is even; Hint: there is a fibre sequence $S^1 \to B\mathbb{Z}/n \to B^2\mathbb{Z} \xrightarrow{\cdot n} B^2\mathbb{Z}$.

Now we prove the lemma case by case, starting with (2), i.e. where A is finite. By the classification of finite abelian groups, we find that it is a finite product of p-local finite groups, so by Künneth we deduce from the previous argument that $H_k(BA;\mathbb{Z})$ is finite for all $k > 0$ giving (2). To see (1), since homology commutes with filtered colimits and every abelian group A is the filtered colimit of its finitely generated subgroups A_i (which are contained in $\mathcal{C}$ if A is contained in $\mathcal{C}$), we find that it suffices to treat finitely generated $\mathcal{P}$-torsion groups. But these are all products of finite p-primary torsion groups, so again by Künneth we deduce the claim from the above argument. To see (3), again by the classification and Künneth, it suffices to in addition note that $H_k(B\mathbb{Z};\mathbb{Z})$ is indeed finitely generated since $B\mathbb{Z} = S^1$. Finally (4) is obvious. ☐

2.23. **Theorem** *Let $\mathcal{C}$ be any of the derived Serre rings of Example 2.20 and X a connected simple anima.*[6] *Then the following are equivalent:*

(1) *All homotopy groups of X are contained in $\mathcal{C}$.*
(2) *All reduced homology groups of X are contained in $\mathcal{C}$.*

Proof. We first show that (1) implies (2). Note that X being simple implies that $\tau_{\leq n}X$ is also simple for all n. Since $H_k(X) \cong H_k(\tau_{\leq k+1}(X))$ is suffices to assume that X is itself n-truncated, i.e. equivalent to $\tau_{\leq n}X$. Now we induct on n. The base case is $n = 1$ where $X = BA$ for some abelian group A which was treated in Lemma 2.22. Now assume that we have inductively proven that $H_*(\tau_{\leq n-1}X)$ is contained in $\mathcal{C}$. Considering the (simple) fibration

$$K(\pi_n(X), n) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$$

the claim then follows from Lemma 2.22 and Lemma 2.21.

Conversely, let us assume (2) and show (1). From the Hurewicz theorem we find $\pi_1(X) \cong H_1(X)$ is contained in $\mathcal{C}$. Considering the the fibre sequence

$$\tau_{\geq 2}X \to X \to B\pi_1(X)$$

and using Lemma 2.21 and Lemma 2.22 we find that $H_k(\tau_{\geq 2}X;\mathbb{Z})$ is contained in $\mathcal{C}$ for $k > 0$, and hence by the Hurewicz that $\pi_2(X) \cong \pi_2(\tau_{\geq 2}X) \cong H_2(\tau_{\geq 2}X;\mathbb{Z})$ is contained in $\mathcal{C}$. Cinsidering inductively with the (simple) fibration

$$\tau_{\geq n}X \to \tau_{\geq n-1}X \to K(\pi_{n-1}(X), n-1)$$

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[6] One can also generalise this to connected nilpotent anima.

we find from Lemma 2.22 that $\tilde{H}_*(K(\pi_{n-1}(X), n-1)) \in \mathcal{C}$ and $\tilde{H}_*(\tau_{\geq n-1}X) \in \mathcal{C}$, so from Lemma 2.21 that $\tilde{H}_*(\tau_{\geq n}X) \in \mathcal{C}$; and hence by the Hurewicz theorem again that $\pi_n(X) \in \mathcal{C}$. ☐

2.24. **Corollary** *For all $k, n > 0$, the groups $\pi_k(S^n)$ are finitely generated abelian groups.*

Proof. Indeed, S^n is a simple space with finitely generated homology. ☐

Later, we will also compute some of these groups explicitly.

3. p-LOCAL AND RATIONAL HOMOTOPY THEORY

3.1. **Definition** Let R be a localisation of $\mathbb{Z}$, mainly $\mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$. Recall that there is a functor $\mathrm{An} \to \mathrm{Sp}$ given by $X \mapsto X \otimes R = C_*(X; R)$. This functor is a left adjoint and hence defines a Bousfield localisation $\mathrm{An}_R \subseteq \mathrm{An}$ of R-local anima, a full subcategory of anima. An object X is R-local, i.e. lies in this full subcategory, if every map $A \to B$ inducing an isomorphism on R-homology (we shall refer to such maps as R-homology equivalences) induces an equivalence $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$. We denote the localisation functor by $(-)_R \colon \mathrm{An} \to \mathrm{An}_R$, this turns out to be a localisation at all R-homology equivalences. In case $R = \mathbb{Q}$ we use the terminology rational anima for $\mathbb{Q}$-local anima, and rational equivalence for $\mathbb{Q}$-homology equivalence. In case $R = \mathbb{Z}_{(p)}$ we say p-local anima and p-local equivalence.

Exercise. The map localisation map $X \to X_R$ is an R-homology equivalence. Moreover, let $f \colon A \to B$ be an R-homology equivalence. If A and B are R-local, then f is an equivalence.

3.2. **Remark** An anima X is R-local if and only if for all R-homology equivalences $f \colon A \to B$, the induced map $[B, X] \to [A, X]$ on homotopy classes of maps is a bijection. Indeed, we have $\pi_0\mathrm{Map}(T, X) = [T, X]$, so one implication is immediate. To see the converse, we aim to show that $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is an equivalence. By Yoneda in the homotopy category of anima, it suffices to show that for every anima T, the induced map

$$\mathrm{Map}(T, \mathrm{Map}(B, X) \to \mathrm{Map}(T, \mathrm{Map}(A, X)$$

induces a bijection on π_0. But this map is equivalent to

$$\mathrm{Map}(T \times B, X) \to \mathrm{Map}(T \times A, X)$$

so it suffices to note that $T \times A \to T \times B$ is again an R-homology equivalence by Künneth.

3.3. **Example** Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ be as above, V an R-module and $n \geq 1$. Then $K(V, n)$ is an R-local space. Indeed, if $X \to Y$ induces an isomorphism on R-homology, by Remark 3.2 we need to show that $[B, K(V, n)] \to [A, K(V, n)]$ is a bijection. By representability of cohomology, this is equivalent to the statement that the map $H^*(B; V) \to H^*(A; V)$ is an isomorphism, which in turn follows from the universal coeffcient theorem over the base PID R.

3.4. **Corollary** *Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ and let X be a connected simple anima*[7] *whose homotopy groups are R-modules. Then X is R-local.*

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[7] Much of what we say in this section extends to nilpotent spaces, but to keep the arguments shorter, we restrict to simple spaces. Moreover, for the present claim, it in fact suffices to assume that $\pi_1(X)$ is abelian.

Proof. Let $A \to B$ be a map inducing an isomorphism on R-homology. Since $\mathrm{Map}(T, X) \simeq \lim_n \mathrm{Map}(T, \tau_{\leq n} X)$ and the homotopy groups of $\tau_{\leq n} X$ are again R-modules, it suffices to prove the claim for $\tau_{\leq n} X$. This we prove inductively, the induction start is $\tau_{\leq 1} X = K(\pi_1(X), 1)$ which is R-local by Example 3.3. For the inductive step, consider the fibre sequence

$$K(\pi_n(X), n) \to \tau_{\leq n} X \to \tau_{\leq n-1} X$$

whose fibre and base are R-local, again by Example 3.3 and induction. Hence, in the diagram

$$\begin{tikzcd}
\mathrm{Map}(B, K(\pi_n(X), n)) \arrow[r] \arrow[d] & \mathrm{Map}(B, \tau_{\leq n} X) \arrow[r] \arrow[d] & \mathrm{Map}(B, \tau_{\leq n-1} X) \arrow[d] \\
\mathrm{Map}(A, K(\pi_n(X), n)) \arrow[r] & \mathrm{Map}(A, \tau_{\leq n} X) \arrow[r] & \mathrm{Map}(A, \tau_{\leq n-1} X)
\end{tikzcd}$$

the left most and right most vertical maps are equivalences, and the horizontal sequences are fibre sequences. Hence the middle vertical map is an equivalence as well. ☐

3.5. **Lemma** *Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ and let A be an abelian group. Then the map $K(A, n) \to K(A \otimes R, n)$ is an R-localisation.*

Proof. Indeed, the target is a simple space whose homotopy groups are R-modules and is hence R-local by Corollary 3.10. We need to show that the map under investigation is an R-homology isomorphism. First, writing A as the filtered colimit over all its finitely generated subgroups, and using that homology, $K(-, n)$ and $- \otimes R$ all commute with filtered colimits, it suffices to treat the case where A is finitely generated, and by Künneth, it suffices to treat the case where A is cylic of prime power order or $A = \mathbb{Z}$. We now distinguish cases and begin with $n = 1$, and $A = \mathbb{Z}$ we note that $K(R, 1)$ can be modelled by the filtered colimit of degree k (for suitable k depending on R) maps on S^1, so that the commutation of homology with filtered colimits gives the desired result. For $A = \mathbb{Z}/n\mathbb{Z}$ with $n = q^k$ a prime power, we have argued earlier that $H_*(B\mathbb{Z}/n; \mathbb{Z})$ is cyclic of order n in odd degrees and trivial in even positive degrees. Hence, if $q = p$ and R is $\mathbb{Z}[\frac{1}{p}]$ or $\mathbb{Q}$, then $A \otimes R = 0$ and $\tilde{H}_*(BA; R) = 0$ as needed, while if $R = \mathbb{Z}_{(p)}$, the map $A \to A \otimes R$ is an isomorphism itself. Conversely, if A is cyclic and q-local for $q \neq p$, then $A \otimes R = 0$ for $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, as is $\tilde{H}_*(BA; R)$, while if $A \to A \otimes \mathbb{Z}[\frac{1}{p}]$ is an isomorphism.

For $n \geq 2$, consider $F = \mathrm{fib}(K(A, n) \to K(A \otimes R, n)$. Then F is connected and simple with only 2 possibly non-trivial homotopy groups which are abelian groups which vanish after applying $- \otimes R$ (they are given by kernel and cokernel of the map $A \to A \otimes R$ and R is flat, so preserves kernels and cokernels). Recall that the class of abelian groups which vanish upon applying $- \otimes R$ forms a derived Serre ring as in Example 2.20. We deduce from Theorem 2.23 that $\widetilde{H}_*(F; R) = \widetilde{H}_*(F; \mathbb{Z}) \otimes R = 0$. Then we deduce from the Serre spectral sequence that $H_*(K(A, n); R) \to H_*(K(A \otimes R, n); R)$ is an isomorphism. ☐

3.6. **Lemma** *Let $f \colon X \to Y$ be a map between connected simple spaces and $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$. If f induces an isomorphism on $\pi_*(-) \otimes R$, then it also induces an isomorphism on $H_*(-; R)$, that is, it is an R-local equivalence.*

Proof. Consider the diagram

$$\begin{tikzcd}
\tau_{\geq 2}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & K(\pi_1(X),1) \arrow[d] \\
\tau_{\geq 2}Y \arrow[r] & Y \arrow[r] & K(\pi_1(Y),1)
\end{tikzcd}$$

induced by the map $f\colon X \to Y$. Using the Serre spectral sequence, it will suffice to show that both outer maps induce isomorphisms on $H_*(-;R)$ (exercise). For the left hand one, note that $F = \mathrm{fib}(\tau_{\geq 2}X \to \tau_{\geq 2}Y)$ is connected and has trivial $\pi_*(-) \otimes R$; since the class of abelian groups which vanish upon applying $- \otimes R$ is a derived Serre ring, we deduce from Theorem 2.23 that $\tilde{H}_*(F;R)$ vanishes as well. It then follows from the Serre spectral sequence that $H_*(\tau_{\geq 2}X;R) \to H_*(\tau_{\geq 2}Y;R)$ is an isomorphism. For the right hand one, consider the square

$$\begin{tikzcd}
K(\pi_1(X),1) \arrow[r] \arrow[d] & K(\pi_1(X) \otimes R,1) \arrow[d, "\simeq"] \\
K(\pi_1(Y),1) \arrow[r] & K(\pi_1(X) \otimes R,1)
\end{tikzcd}$$

whose right vertical map is an equivalence by assumption and hence induces an isomorphism on R-homology. In Lemma 3.5, we have argued that the horizontal maps also induce isomorphisms on $H_*(-;R)$, hence so does the left vertical map. □

3.7. **Proposition** *Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ and let X be a connected simple space. Then there exists a map $X \to X'$ such that X' is simple, whose homotopy groups are R-modules and such that the map f is an R-localisation.*

Proof. We will construct X' and f so that f induces an isomorphism $\pi_*(X) \otimes R \to \pi_*(X') \otimes R \cong \pi_*(X')$ and hence is an R-local equivalence by Lemma 3.6. Since X' is R-local by Corollary 3.4, the map f factors as $X \to X_R \to X'$ for an essentially unique map $X_R \to X'$ which is again a R-local equivalence, and hence an equivalence as X_R is also R-local. To show the existence of X' and f, we now induct over the Postnikov tower. First, we recall from Example 3.3 that $K(\pi_1(X),1) \to K(\pi_1(X) \otimes R,1)$ is an R-localisation. Since X is simple, its Postnikov tower consists of principal fibrations, so that we have fibre sequences

$$\tau_{\leq n}X \to \tau_{\leq n-1}X \to K(\pi_n(X), n+1).$$

Consider the composite of the latter map with the R-localisation map $K(\pi_n(X), n+1) \to K(\pi_n(X) \otimes R, n+1)$. Assume inductively that there exists a map as claimed for $\tau_{\leq n-1}X$. Since such a map is an R-localisation, we obtain a commutative diagram

$$\begin{tikzcd}
\tau_{\leq n}X \arrow[r] \arrow[d, dashed] & \tau_{\leq n-1}X \arrow[r] \arrow[d] & K(\pi_n(X), n+1) \arrow[d] \\
(\tau_{\leq n}X)' \arrow[r] & (\tau_{\leq n-1}X)' \arrow[r, dashed] & K(\pi_n(X) \otimes R, n+1)
\end{tikzcd}$$

in which the dashed arrow exists by the universal property of rationalisation, the lower left most term is defined as the fibre of the horizontal dashed arrow and the vertical dashed arrow is the induced map on fibres. The vertical dashed arrow then, by construction, again induces an isomorphism on $\pi_*(-) \otimes R$. Inductively, we have then constructed commutative squares as in the left part of the above diagram, allowing to define $X' = \lim_n(\tau_{\leq n}X)'$. The resulting map $X \to X'$ again induces an isomorphism on $\pi_*(-) \otimes R$ and hence does the job. ☐

3.8. **Remark** In particular, this proposition shows that any R-localisation map on a connected simple space induces an isomorphism on $\pi_*(-) \otimes R$, not only on $H_*(-; R)$ (which is true by definition) and that R-local simple spaces have $\pi_*(-)$ being R-modules.

3.9. **Proposition** *Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ and let X be a connected anima.*

1. *If $\pi_*(X)$ is an R-module, then so is $\widetilde{H}_*(X; \mathbb{Z})$.*
2. *If X is simple and $\widetilde{H}_*(X; \mathbb{Z})$ is an R-module, then $\pi_*(X)$ is an R-module*

Proof. To see (1), it suffices to argue that $\widetilde{H}_*(X; \mathbb{F}_q) = 0$ for suitable primes q, namely all q in case $R = \mathbb{Q}$, all q different from p for $R = \mathbb{Z}_{(p)}$ and $q = p$ for $R = \mathbb{Z}[\frac{1}{p}]$. To that end, we first prove show that $\widetilde{H}_*(K(A, n); \mathbb{F}_q) = 0$ for all R-modules A and $n \geq 1$. For $n = 1$ we have already argued this earlier. Then consider the fibre sequence

$$K(A, n-1) \to * \to K(A, n)$$

and use Leray–Hirsch (or the Serre spectral sequence) to deduce inductively that $\widetilde{H}(*; \mathbb{F}_q) \to \widetilde{H}_*(K(A, n); \mathbb{F}_q)$ is an isomorphism. Having established this case, consider then the fibre sequence

$$K(\pi_n(X), n) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$$

and use again Leray–Hirsch and induction over n to deduce the claim for $\tau_{\leq n}X$. The result then follows since for $n > k$, we have $\widetilde{H}_k(X; \mathbb{F}_q) \cong \widetilde{H}_k(\tau_{\leq n}X; \mathbb{F}_q)$.

To see (2), the Hurewicz theorem implies that $\pi_1(X)$ is an R-module. Now we induct over n in the fibre sequence

$$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X$$

as follows. Inductively, we know that $\pi_*(\tau_{\leq n-1}X)$ is an R-module. Again, By (1) and assumption, consider again the Serre spectral sequence in $\mathbb{F}_q$-homology for the relevant q. We see that only the spots on the y-axes are possibly non-trivial, and so from convergence we deduce that $\widetilde{H}_*(\tau_{\geq n}X; \mathbb{Z})$ is an R-module, and the from Hurewicz that also $\pi_n(X)$ is an R-module, establishing the inductive step. ☐

Putting the above together we conclude:

3.10. **Corollary** *Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ and let X be a connected simple space. Then*

1. *X is R-local if and only if $\pi_*(X)$ is an R-module if and only if $\tilde{H}_*(X; \mathbb{Z})$ is an R-module.*
2. *A map $f \colon X \to Y$ between connected simple spaces is an R-equivalence if and only if it induces an isomorphism on $\pi_*(-) \otimes R$.*

Proof. (1) follows from Proposition 3.9 and one direction of (2) is Lemma 3.6. To see the converse, assume that f is an R-homology equivalence. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X_R \arrow[r] & Y_R
\end{tikzcd}$$

in which the the vertical maps are $\pi_*(-) \otimes R$ isomorphisms and the map $X_R \to Y_R$ is a R-local equivalence equivalence between simple R-local spaces, and hence an equivalence and consequently induces an isomorphism on $\pi_*(-) \otimes R$. It follows that the same is true for the top horizontal map as needed. □

3.11. **Corollary** *Let $R = \mathbb{Q}, \mathbb{Z}_{(p)}$, or $\mathbb{Z}[\frac{1}{p}]$ and let X be a connected simple space and $n \geq 2$. Suppose $\pi_*(X) \otimes R = 0$ for $1 \leq * \leq n-1$. Then $H_*(X;R) = 0$ for $1 \leq * \leq n-1$ and $\pi_n(X) \otimes R \to H_n(X;R)$ is an isomorphism and $\pi_{n+1}(X) \otimes R \to H_{n+1}(X;R)$ is surjective.*

Proof. Consider the R-localisation map $X \to X_R$ and note that $\pi_*(X_R) \cong \pi_*(X) \otimes R$ as well as $H_*(X_R;\mathbb{Z}) \cong H_*(X;R)$. The claim then follows from the usual Hurewicz theorem applied to X_R. □

Let us now work out an application of the above: In particular, that away from 2, homotopy groups of even dimensional spheres are understood once homotopy groups of odd dimensional spheres are understood.

3.12. **Proposition** *Away from 2 the EHP fibre sequence from Lemma 2.17 splits. That is, we have an equivalence*

$$\Omega S^{2n}[\tfrac{1}{2}] \simeq S^{2n-1}[\tfrac{1}{2}] \times \Omega S^{4n-1}[\tfrac{1}{2}]$$

and consequently, for all $i \geq 1$, we have isomorphisms

$$\pi_{i+1}(S^{2n})[\tfrac{1}{2}] \cong \pi_i(S^{2n-1})[\tfrac{1}{2}] \oplus \pi_{i+1}(S^{4n-1})[\tfrac{1}{2}].$$

Moreover, for odd k, there is a 2-local fibre sequence

$$S^{k-1}_{(2)} \to \Omega S^k_{(2)} \to \Omega S^{2k-1}_{(2)}.$$

Proof. Consider the map $f\colon S^{4n-1} \to S^{2n} \vee S^{2n} \to S^{2n}$ of which the first map is the attaching map for the top cell of $S^{2n} \times S^{2n}$. One computes that the Hopf invariant of this map is 2 (exercise). Then consider the composite

$$S^{2n-1} \times \Omega S^{4n-1} \xrightarrow{\eta \times \Omega f} \Omega S^{2n} \times \Omega S^{2n} \xrightarrow{\mu} \Omega S^{2n}$$

where the latter map is the multiplication map of the loop space (i.e. the concatenation of loops). This composite is a $\mathbb{Z}[\frac{1}{2}]$-homology isomorphism, and hence an equivalence away from 2. To see this, it is most convenient to first compute that $H_*(\Omega S^m;\mathbb{Z}) \cong \mathbb{Z}[x_{m-1}]$ as rings. Using this, the claim then follows from the computation that Ωf induces the multiplication by ± 2 map on H_{4n-2}; this in turn follows from the fact that f has Hopf invariant 2; see Remark 3.15 below for the argument.

For the moreover, we run the same argument as in Lemma 2.17. The difference is that now, the Hopf map $\Omega S^k \to \Omega S^{2k-1}$ induces on cohomology the map $\Gamma_{\mathbb{Z}}[x_{2k-2}] \to \Gamma_{\mathbb{Z}}[x_{k-1}]$, sending x_{2k-2} to the divided square of x_{k-1}, that is, the unique element y with $2y = x_{k-1}^2$. Exercise: After localising at 2, this map is an isomorphism in all degrees where the source is non-trivial; the same argument as in Lemma 2.17 therefore applies and shows that the fibre of the Hopf map is, 2-locally, equivalent to S^{k-1}. □

3.13. **Remark** We argue here why $H_*(\Omega S^m;\mathbb{Z}) = \mathbb{Z}[x_{m-1}]$. In fact, we prove something more general. First we note that $\Omega\Sigma X$ is the free $\mathbb{E}_1$-group on the pointed anima X. Indeed, this follows from the equivalence between $\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An}) \simeq \mathrm{An}_*^{\geq 1}$ given by $\mathrm{B}(-)$ and Ω as discussed in [Lan24]; we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{Grp}}(\Omega\Sigma X, G) &\simeq \mathrm{Map}_{\mathrm{Grp}}(\Omega\Sigma X, \Omega \mathrm{B}G)\\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(\Sigma X, \mathrm{B}G)\\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(X, \Omega\mathrm{B}G)\\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(X, G)\end{aligned}$$

as needed.

Furthermore, this free functor factors as $\mathrm{An}_* \to \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{An}) \to \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An})$ where the first functor is the free $\mathbb{E}_1$-monoid functor F. Now we claim that if $X \in \mathrm{An}_*$ is *connected*, then $F(X)$ is also connected and hence grouplike, and therefore $F(X) = \Omega\Sigma X$. Indeed, to see this, we claim that the diagram

$$\begin{tikzcd}\mathrm{An}_* \arrow[r, "F"] \arrow[d, "\pi_0"] & \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{An}) \arrow[d, "\pi_0"] \\ \mathrm{Set}_* \arrow[r] & \mathrm{Mon}(\mathrm{Set})\end{tikzcd}$$

in which the bottom horizontal map is the free monoid on a pointed set functor commutes; this is a simple matter of comparing universal properties using that π_0 is left adjoint to the inclusion $\mathrm{Set} \subseteq \mathrm{An}$. Exercise: This functor takes a pointed set S to the quotient of the free monoid on the unpointed set S by the submonoid generated by the basepoint. In particular, it sends a point to a point.

As a consequence, we find that for a connected pointed space X, we have that $\mathbb{S}[\Omega\Sigma X]$ is the image of X under the down-right composite of the following commutative square

$$\begin{tikzcd}\mathrm{An}_* \arrow[r, "{\mathbb{S}[-,*]}"] \arrow[d] & \mathrm{Sp} \arrow[d] \\ \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{An}) \arrow[r, "{\mathbb{S}[-]}"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})\end{tikzcd}$$

in which both left vertical maps are the free functors. This diagram indeed commutes as follows from inspecting the corresponding diagram of right adjoints. Hence, we see that $\mathbb{S}[\Omega\Sigma X]$ is the free $\mathbb{E}_1$-algebra in Sp on $\mathbb{S}[X, *] = \Sigma^\infty X$:

$$\mathbb{S}[\Omega\Sigma X] = \mathrm{free}_{\mathbb{E}_1}(\Sigma^\infty X).$$

Now, in general, if $\mathcal{C}$ is a cocomplete symmetric monoidal category in which the tensor product commutes with colimits in each variable, the forgetful functor $\mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $\mathrm{free}\colon \mathcal{C} \to \mathrm{Alg}(\mathcal{C})$, which takes an object $V \in \mathcal{C}$ to a commutative algebra $\mathrm{free}(V)$ whose underlying object is given by

$$\coprod_{k \geq 0} V^{\otimes k}$$

and whose multiplication map is induced by the evident equivalences

$$V^{\otimes k} \otimes V^{\otimes k'} \to V^{\otimes k+k'}.$$

As a consequence, we find

$$\mathbb{S}[\Omega\Sigma X] = \bigoplus_{n\geq 0} [\Sigma^\infty(X)]^{\otimes n} = \bigoplus_{n \geq 0} \Sigma^\infty(X^{\wedge n}).$$

To compute the homology, we apply the functor $\mathrm{Sp} \to \mathrm{Mod}(\mathbb{Z})$ given by $\mathbb{Z} \otimes -$. Since this functor is a symmetric monoidal left adjoint, it preserves free algebras, and we obtain

$$\mathbb{Z}[\Omega\Sigma X] = \bigoplus_{n\geq 0} \mathbb{Z}[X^{\wedge n}, *]$$

with evident multiplication map. Finally, for $X = S^{m-1}$, we then find

$$\mathbb{Z}[\Omega S^m] = \bigoplus_{n\geq 0} \Sigma^{n(m-1)}\mathbb{Z} = \mathbb{Z}[x_{m-1}]$$

as claimed.

3.14. **Remark** The same strategy as the one given above gives several stable splitting results; e.g. one also finds for X connected and pointed, and all $k \geq 1$ including $k = \infty$, that $\mathbb{S}[\Omega^k\Sigma^k X] \simeq \mathrm{free}_{\mathbb{E}_k}(\Sigma^\infty X)$ and therefore has underlying object given by

$$\bigoplus_{n\geq 0} [\mathbb{E}_k(n) \otimes \Sigma^\infty(X^{\wedge k})]_{h\Sigma_k}$$

For $k = 1$, we have $\mathbb{E}_1(n) = \Sigma_n$, while for $k = \infty$, we have $\mathbb{E}_\infty(n) = *$; we will make use of this case below, see Remark 3.17.

3.15. **Remark** let $n > 1$ and $f\colon S^{2n-1} \to S^n$ be a map whose cofibre we denote by X. We define the Hopf invariant $h(f)$ of f as follows. Note that there is a cofibre sequence

$$S^n \to X \to S^{2n}$$

so that the map $H^{2n}(S^{2n};\mathbb{Z}) \to H^{2n}(X;\mathbb{Z})$ is an isomorphism; denote by β the image of the cohomological fundamental class of S^{2n} (which we recall we have fixed a long times ago), and let α be a generator of $H^n(X;\mathbb{Z}) \cong \mathbb{Z}$. Then $h(f)$ is the unique integer satisfying $\alpha^2 = h(f)\cdot\beta$. Exercise: for $f\colon S^{4n-1} \to S^{2n} \vee S^{2n} \to S^{2n}$ as in the proof of Proposition 3.12, we have $h(f) = 2$.

We now note that the Hopf invariant of $f\colon S^{2n-1} \to S^n$ can also be computed as follows. Consider the map $(\Omega f)_*\colon H_{2n-2}(\Omega S^{2n-1};\mathbb{Z}) \to H_{2n-2}(\Omega S^n)$, and recall that both source and target are isomorphic to $\mathbb{Z}$. Then we have that this map is given by multiplication by $\pm h(f)$. To that end, consider the pushout square

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "f"] \arrow[d] & S^n \arrow[d, "p"] \\
* \arrow[r] & X
\end{tikzcd}$$

in which the map f is $(n-1)$-connected and the map p is $(2n-1)$-connected. We deduce from Blakers–Massey, see e.g. [Lan24, Theorem 2.23], that the map $F = \mathrm{fib}(f) \to \Omega X$ is $(3n-3)$-connected. Hence, in the Serre spectral sequence for the fibre sequence

$$\Omega S^{2n-1} \xrightarrow{\Omega f} \Omega S^n \to F$$

in homological degrees $< 3n-3$, we may replace F by ΩX. Note that $2n-2 < 3n-3$ if and only if $n > 1$, which we have assumed. So let us first compute the (co)homology of ΩX using the Serre spectral sequence. From the multiplicativity of the this spectral sequence, we find that the non-trivial cohomology groups in in degrees $\leq 2n-1$ are $H^{2n-1}(\Omega X;\mathbb{Z}) \cong \mathbb{Z}/h(f)$

and $H^{n-1}(\Omega X;\mathbb{Z}) \cong \mathbb{Z}$. Hence from UCT and the above comparison argument, we find that for $k \leq 2n-2$, we have

$$H_k(F;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n-1 \\ \mathbb{Z}/h(f) & \text{if } k = 2n-2 \\ 0 & \text{otherwise} \end{cases}$$

Hence, considering the Serre spectral sequence for

$$\Omega S^{2n-1} \xrightarrow{\Omega f} \Omega S^n \xrightarrow{q} F$$

we obtain a short exact sequence

$$0 \to H_{2n-2}(\Omega S^{2n-1};\mathbb{Z}) \xrightarrow{(\Omega f)_*} H_{2n-2}(\Omega S^n) \xrightarrow{q_*} H_{2n-2}(F;\mathbb{Z}) \to 0$$

showing the claim.

We now move towards studying the category $\mathrm{An}_{\mathbb{Q}}^{\geq 1,\mathrm{simple}}$ of connected simple rational spaces more algebraically. To that end, we first compute:

3.16. **Proposition** *We have the following isomorphism of $\mathbb{Q}$-algebras:*

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) \cong H^*(K(\mathbb{Q},n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_n] & \textit{if } n \textit{ is even} \\ \Lambda_{\mathbb{Q}}[e_n] & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. We run an induction over n, using the fibre sequence

$$K(\mathbb{Q},n) \to * \to K(\mathbb{Q},n+1).$$

The base cases $n = 1, 2$ have in fact already been computed. The Serre spectral sequence implies the claim. ☐

3.17. **Remark** In the category $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$, $\mathbb{Q}[x_n]$ and $\Lambda_{\mathbb{Q}}[e_n]$ are the free commutative algebras on an even and odd degree generator, respectively. We will make use of this momentarily. Similarly to before, if $\mathcal{C}$ is a cocomplete symmetric monoidal (∞)-category in which the tensor product commutes with colimits in each variable, the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ also has a left adjoint $\mathrm{free}\colon \mathcal{C} \to \mathrm{CAlg}(\mathcal{C})$, which takes an object $X \in \mathcal{C}$ to a commutative algebra $\mathrm{free}(X)$ whose underlying object now is given by

$$\coprod_{k\geq 0} [X^{\otimes k}]_{h\Sigma_k}$$

and whose multiplication map is induced by the evident maps

$$X^{\otimes k}_{h\Sigma_k} \otimes X^{\otimes l}_{h\Sigma_l} \simeq [X^{\otimes k+l}]_{h(\Sigma_k \times \Sigma_l)} \to [X^{\otimes k+l}]_{h\Sigma_{k+l}}$$

induced by the inclusion $\Sigma_k \times \Sigma_l \subseteq \Sigma_{k+l}$ as block permutations. Now, for $\mathcal{C} = \mathcal{D}(\mathbb{Q})$ and $X = \mathbb{Q}[n]$ for some $n \in \mathbb{Z}$, we find $X^{\otimes k} = \mathbb{Q}[nk]$ with trivial Σ_k-action if n is even, and with sign action (via the abelianisation map $\Sigma_k \to C_2$) if n is odd. Since $\mathbb{Q}$ is a projective $\mathbb{Q}[G]$-module for all finite groups G, we have $H_*(\Sigma_k;\mathbb{Q}) = \mathbb{Q}$, while $H_*(\Sigma_k;\mathbb{Q}^-) = 0$. As a consequence, we find that the map of commutative algebras in $\mathcal{D}(\mathbb{Q})$

$$\mathrm{free}(\mathbb{Q}[n]) \to C^*(K(\mathbb{Q},n);\mathbb{Q}))$$

is an equivalence.

We will need the following further result for which we consider the following situation

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} C^*(W;R) \arrow[r] \arrow[d] & C^*(Y;R) \arrow[d] \\ C^*(Z;R) \arrow[r] & C^*(X;R) \end{tikzcd}$$

where R is a commutative ring, and the left hand square is a commutative diagram in anima with W connected. The right hand square is then a commutative diagram in $\mathcal{D}(R) \simeq \mathrm{Mod}_{\mathrm{Sp}}(R)$ simply given by applying $C^*(-;R)$ to the left hand diagram. Since pushouts in this category are given by the relative tensor product (just as we are used to from ordinary commutative algebra), we obtain a canonical map

$$\Phi\colon C^*(Z;R) \otimes_{C^*(W;R)} C^*(Y;R) \to C^*(X;R).$$

If the square of anima is a pullback square, then one may wonder whether this map is an equivalence. This is not always the case, but the following gives a sufficient condition which is often true in practice. As last term, let us denote by $r\colon W \to *$ the unique map and by $h\colon X \to W$ the composite in the square. For any anima T, in $\mathrm{Fun}(T, \mathcal{D}(R))$ we will always write R for the monoidal unit, that is, for the constant diagram with value R.

The following proposition is not aiming for maximal generality. The proof will reveal what refined statements in fact hold true, we will comment on this later.

3.18. **Theorem** (Eilenberg–Moore) *In the above situation, assume that $R = k$ is a field, that $H^n(\mathrm{fib}(f);k)$ is finite dimensional for all n,*[8] *and that the $\pi_1(W)$-action on $H^n(\mathrm{fib}(f);k)$ is nilpotent for all n. Then the map*

$$C^*(Z;k) \otimes_{C^*(W;k)} C^*(Y;k) \to C^*(X;k)$$

is an isomorphism.

Proof. For ease of notation, we write $C^*(-)$ for $C^*(-;k)$. For $\mathcal{L} \in \mathrm{Fun}(W, \mathcal{D}(k))$, we may consider the object $g_*g^*(k) \otimes_k \mathcal{L}$ of $\mathrm{Fun}(W, \mathcal{D}(k))$. Since $r_*\colon \mathrm{Fun}(W, \mathcal{D}(k)) \to \mathcal{D}(k)$ is lax symmetric monoidal (it is the right adjoint of a symmetric monoidal functor), we obtain a canonical map

$$\Phi_{\mathcal{L}}\colon r_*g_*g^*(k) \otimes_{r_*(k)} r_*(\mathcal{L}) \to r_*(g_*g^*(k) \otimes_k \mathcal{L})$$

in $\mathcal{D}(k)$. Note that $r_*(k) = C^*(W)$ and $r_*g_*g^*(k) = C^*(Z)$ and that the above map is the component of a natural transformation between exact functors $\mathrm{Fun}(W; \mathcal{D}(k)) \to \mathcal{D}(k)$. Let us consider the special case $\mathcal{L} = f_*f^*(k)$. Then the left hand side becomes $C^*(Z) \otimes_{C^*(W)} C^*(Y)$, and we claim that the right hand term becomes $C^*(X)$. To see, this, we observe that there is a canonical map $\varphi\colon g_*g^*(k) \otimes_k f_*f^*(k) \to h_*h^*(k)$, adjoint to the following composite (here we use $gb = h = fa$):

$$b^*g^*g_*g^*(k) \otimes_k a^*f^*f_*f^*(k) \to b^*g^*(k) \otimes_k a^*f^*(k) \to k$$

where the first map is the counit in each tensor factor and the second map is the multiplication map of the unit k (which is of course an equivalence). We claim that the map φ is an equivalence. To do this, it suffices to show that it is an equivalence pointwise, i.e. after applying i^* where $i\colon \{w\} \to W$ is a point. Doing so, the resulting map becomes (exercise) the map

$$C^*(\mathrm{fib}(g)) \otimes C^*(\mathrm{fib}(f)) \to C^*(\mathrm{fib}(g) \times \mathrm{fib}(f))$$

---

[8] We also say that $\mathrm{fib}(f)$ is of finite cohomological k-type.

which is an isomorphism by Künneth and the assumption that $C^*(\mathrm{fib}(f))$ is of finite cohomological k-type. The map $\Phi_{f_*f^*(k)}$ therefore becomes a map

$$C^*(Z) \otimes_{C^*(W)} C^*(Y) \to C^*(X)$$

which turns out to be the map Φ we are trying to show is an equivalence.

Now, we observe that $\Phi_{\mathcal{L}}$ is evidently an equivalence in the case $\mathcal{L} = k$, and hence also in the case where $\mathcal{L}$ lies in the thick subcategory generated by $\mathcal{L}$. Note that for $\mathcal{L}$ in $\mathrm{Fun}(W, \mathcal{D}(k))$, we may consider its pointwise Whitehead tower $\tau_{\geq \bullet}\mathcal{L}$, obtained by the functor $\mathrm{Fun}(W, \mathcal{D}(k)) \to \mathrm{Fun}(W, \mathrm{Fil}(\mathcal{D}(k))$ induced by $\mathcal{D}(k) \to \mathrm{Fil}(\mathcal{D}(k))$ given by sending $A \in \mathcal{D}(k)$ to its Whitehead tower $\tau_{\geq \bullet}A$.[9] Inductively over m, we now show that $\tau_{\geq -m}f_*f^*(k)$ lies in the thick subcategory generated by the unit k. Indeed, note that the pointwise formula for the Kan extension gives that $f_*f^*(k) = C^*(\mathrm{fib}(f); k)$ viewed as a functor on W. Inducting through the Whitehead tower, we see that it suffices to show that $H^n(\mathrm{fib}(f); k)$ lies in the thick subcategory generated by k. But this is exactly what the nilpotency of the action gives: The the functor on W taking a point to $H^n(\mathrm{fib}(f); k)$ factors through the map $W \to B\pi_1(W)$, the nilpotency then precisley means that there is a finite filtration on $H^n(\mathrm{fib}(f); k)$ with associated graded which has trivial $\pi_1(W)$-action. Hence, $H^n(\mathrm{fib}(f); k)$ indeed lies in the thick subcategory generated by k. Inductively, we then deduce that for all $m \leq 0$, the functor on W given by $\tau_{\geq -m}f_*f^*(k)$ lies in the thick subcategory generated by k, and hence that for $\mathcal{L} = \tau_{\geq m}f_*f^*(k)$, the map $\Phi_{\mathcal{L}}$ is an equivalence.

We now claim that both source and target of the map $\Phi_{\mathcal{L}}$ commute with the filtered colimit $\operatorname*{colim}_{m \to -\infty} f_*f^*(k) \simeq f_*f^*(k)$, more precisely, we consider the following diagram

$$\begin{tikzcd}
\operatorname*{colim}_{m\to-\infty} C^*(Z) \otimes_{C^*(W)} r_*[\tau_{\geq m}f_*f^*(k)] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \operatorname*{colim}_{m\to-\infty} r_*(g_*g^*(k) \otimes \tau_{\geq m}f_*f^*(k)) \arrow[d] \\
C^*(Z) \otimes_{C^*(W)} \operatorname*{colim}_{m\to-\infty} r_*[\tau_{\geq m}f_*f^*(k)] \arrow[d] & r_*[\operatorname*{colim}_{m\to-\infty} (g_*g^*(k) \otimes \tau_{\geq m}f_*f^*(k))] \arrow[d, "\simeq"] \\
C^*(Z) \otimes_{C^*(W)} r_*[f_*f^*(k)] \arrow[r] & r_*[(g_*g^*(k) \otimes f_*f^*(k)]
\end{tikzcd}$$

where the upper horizontal equivalence comes form the previously established case and the two vertical equivalences come from commutative of tensor products with colimits. We want to see that the lower horizontal map is an equivalence, so it suffices to argue that the two remaining vertical ones are. For the left hand side, this is the same argument we have already seen in the construction of th Serre spectral sequence: the maps in the colimit system have more and more coconnective cofibres, so the claim follows from the fact that r_*, being the limit over W preserves coconnectivity. For the right vertical map we appeal to the same argument and have to use: the maps in the colimit system for $g_*g^*(k) \otimes \tau_{\geq \bullet}f_*f^*(k)$ still have more and more coconnected cofibres: Indeed, the cofibres are obtained from those of $\tau_{\geq \bullet}f_*f^*(k)$ upon tensoring with $g_*g^*(k)$, which is coconnective since it is $C^*(\mathrm{fib}(g); k)$. Now since k is a field, this tensor product is still as coconnective as the cofibre of $\tau_{\geq \bullet}f_*f^*(k)$, whose coconnectivity tends to $-\infty$. This proves the theorem. □

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[9] We have recalled in the beginning of this course that this works for spectra, but it works equally for $\mathcal{D}(k)$, in fact, for any stable ∞-category equipped with a t-structure.

3.19. **Remark** Inspecting the proof of Theorem 3.18 we find that the argument applies to more general commutative rings R whenever

(1) the R-cohomological Künneth theorem holds for $\mathrm{fib}(f) \times \mathrm{fib}(g)$, and
(2) the tensor product of a coconnective R-module with an m-coconnective R-module is $\varphi(m)$-coconnective for some function $m \mapsto \varphi(m)$ tending to $-\infty$ if m tends to $-\infty$.

If we assume that $\mathrm{fib}(f)$ has R-cohomologically finite type (i.e. $H^n(\mathrm{fib}(f); R)$ is a finitely generated R-module for all n) and that the $\pi_1(W)$-action is still nilpotent, this can be shown to be true for noetherian commutative rings of finite global dimension. In particular it is true for fields and PIDs like the integers.

The following is then a version of Sullivan's famous result on rational homotopy theory.

3.20. **Theorem** *The functor* $\mathrm{An}_{\mathbb{Q}} \to \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))^{\mathrm{op}}$*, given by* $X \mapsto C^*(X; \mathbb{Q})$*, is fully faithful when restricted to connected, simple spaces of finite rational type.*[10]

Proof. We will show that for any space T and rational, simple space of finite rational type X, the map

$$\mathrm{Map}(T, X) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))}(C^*(X; \mathbb{Q}), C^*(T; \mathbb{Q}))$$

is an equivalence. To that end, write $X = \lim_n \tau_{\leq n} X$, then we find $\mathrm{colim}_n\, C^*(\tau_{\leq n}; \mathbb{Q}) \to C^*(X; \mathbb{Q})$ is an equivalence in $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ simply because for each cohomological degree, the colimit stabilizes. Hence commuting out limits/colimits, we reduce to the case where $X = \tau_{\leq n} X$. Then we may induct on n. For the inductive step, we consider the fibration

$$K(\pi_n(X), n) \to \tau_{\leq n} X \to \tau_{\leq n-1} X$$

and obtain the diagram

$$\begin{tikzcd}
\mathrm{Map}(T, K(\pi_n(X), n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))}(C^*(K(\pi_n(X), n); \mathbb{Q}), C^*(T; \mathbb{Q})) \arrow[d] \\
\mathrm{Map}(T, \tau_{\leq n} X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))}(C^*(\tau_{\leq n} X; \mathbb{Q}), C^*(T; \mathbb{Q})) \arrow[d] \\
\mathrm{Map}(T, \tau_{\leq n-1} X) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))}(C^*(\tau_{\leq n-1} X; \mathbb{Q}), C^*(T; \mathbb{Q}))
\end{tikzcd}$$

in which we claim both vertical sequences to be fibre sequences: For the left hand one, this follows from the fact that $\mathrm{Map}(T, -)$ preserves fibre sequences, for the right hand one, we note that the square

$$\begin{tikzcd}
C^*(\tau_{\leq n-1} X; \mathbb{Q}) \arrow[r] \arrow[d] & C^*(\tau_{\leq n} X; \mathbb{Q}) \arrow[d] \\
\mathbb{Q} \arrow[r] & C^*(K(\pi_n(X), n); \mathbb{Q})
\end{tikzcd}$$

is a pushout in $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ by Theorem 3.18, and then use that the functor

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))}(-, C^*(T; \mathbb{Q}))\colon \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))^{\mathrm{op}} \to \mathrm{An}$$

---

[10] In fact, similar arguments as we present here also show the same claim for nilpotent spaces of finite rational type.

preserves limits. Hence, the inductive step as well as the base case follow once we show the claim for $X = K(A, n)$ with A a finite dimensional $\mathbb{Q}$-vector space, in other words where X is a finite product of $K(\mathbb{Q}, n)$. Using the Künneth theorem, and the fact that the tensor product in $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ is the coproduct, we finally reduce to the case $X = K(\mathbb{Q}, n)$ (exercise). In other words we need to investigate the map

$$\mathrm{Map}(T, K(\mathbb{Q}, n)) \to \mathrm{Map}(C^*(K(\mathbb{Q}, n); \mathbb{Q}), C^*(T; \mathbb{Q})).$$

Similarly as previously, using Yoneda in the homotopy category of anima, it suffices to show that for all further anima U, the map

$$\mathrm{Map}(U, \mathrm{Map}(T, K(\mathbb{Q}, n)) \to \mathrm{Map}(U, \mathrm{Map}(C^*(K(\mathbb{Q}, n); \mathbb{Q}), C^*(T; \mathbb{Q})))$$

induces a bijection on π_0. Exercise: there is a canonical equivalence

$$\mathrm{Map}(U, \mathrm{Map}(C^*(K(\mathbb{Q}, n); \mathbb{Q}), C^*(T; \mathbb{Q}))) \simeq \mathrm{Map}(C^*(K(\mathbb{Q}, n); \mathbb{Q}), C^*(U \times T; \mathbb{Q}))$$

under which the above map becomes equivalent to the map

$$\mathrm{Map}(U \times T, K(\mathbb{Q}, n)) \to \mathrm{Map}(C^*(K(\mathbb{Q}, n); \mathbb{Q}), C^*(U \times T; \mathbb{Q})).$$

Using that $C^*(K(\mathbb{Q}, n); \mathbb{Q})$ is free on a class of (cohomological) degree n, see Remark 3.17, we deduce that on π_0, this map induces the map

$$[U \times T, K(\mathbb{Q}, n)] \to H^n(U \times T; \mathbb{Q})$$

sending f to $f^*(\iota_n)$ which we have shown to be an isomorphism in [Lan24, Theorem 3.7]. $\square$

3.21. **Remark** The functor $X \mapsto C^*(X; \mathbb{Q})$ is left adjoint to the functor $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))^{\mathrm{op}} \to \mathrm{An}$ given by $A \mapsto \mathrm{Map}_{\mathrm{CAlg}(\mathbb{Q})}(A, \mathbb{Q})$. In particular, this functor, restricted to underlying coconnective objects with are degreewise finite dimensional homotopy groups, is the inverse of the above fully faithful functor.

3.22. **Remark** The essential image of the above established fully faithful functor, when restricted to simply connected rational spaces of finite rational type is the full subcategory of $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ on those objects whose underlying object of $\mathcal{D}(\mathbb{Q})$ is coconnective with $\pi_0(-)$ isomorphic to $\mathbb{Q}$, $\pi_{-1}(-)$ trivial, and which have degreewise finite dimensional homotopy groups over $\mathbb{Q}$ as we show now:

Indeed, suppose given such an $A \in \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$. We will inductively construct a rational, simply connected anima X_n with a map $f_n \colon C^*(X_n; \mathbb{Q}) \to A$ in $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ which induces an isomorphism on π_k for $k > -n$ and a surjection on π_{-n} and maps $\alpha_n \colon X_{n+1} \to X_n$ together with an identification

$$f_{n+1} \circ \alpha_n^* \sim f_n \colon C^*(X_n; \mathbb{Q}) \to A.$$

For the induction start, we note that a similar argument as in Remark 3.17 shows that for V a finite dimensional $\mathbb{Q}$-vector space, we have that $C^*(K(V^\vee, n); \mathbb{Q})$ identifies with the free commutative $\mathbb{Q}$-algebra on $V[-n]$; here $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ denotes the dual vector space. The induction start is then $X_2 = K(\pi_{-2}(A)^\vee, 2)$ which by what we have just argued, is equipped with a tautological map $C^*(K(\pi_{-2}(A)^\vee, 2); \mathbb{Q}) \to A$ which induces an isomorphism on π_i for $i = 0, -1, -2$; this uses the finite type hypothesis on A. Now assume inductively that X_n and f_n have been constructed. Let

$$V = \ker(H^n(X_n; \mathbb{Q}) \xrightarrow{f_n} \pi_{-n}(A)).$$

Using again that $C^*(K(V^\vee, n); \mathbb{Q})$ is free, we obtain a canonical map $C^*(K(V^\vee, n); \mathbb{Q}) \to C^*(X_n; \mathbb{Q})$ classifying the inclusion $V \subseteq H^n(X_n; \mathbb{Q})$. By the established fully faithfulness of

$C^*(-;\mathbb{Q})$ this map is the map induced on rational cochains of a map of anima $X_n \to K(V^\vee, n)$. Let X'_{n+1} be the fibre of this map, so that we have a squares

$$\begin{tikzcd}
X'_{n+1} \arrow[r, "\alpha'_n"] \arrow[d, "\phi_n"] & X_n \arrow[d] \\
* \arrow[r] & K(V^\vee, n)
\end{tikzcd}
\qquad
\begin{tikzcd}
C^*(K(V^\vee, n)) \arrow[r] \arrow[d] & C^*(X_n) \arrow[d] \\
\mathbb{Q} \arrow[r] & C^*(X'_{n+1})
\end{tikzcd}$$

of which the left is a pullback square of simple rational anima and the right is a pushout in $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ by Theorem 3.18. Note that in case $n = 2$ we have $V = 0$ and this part of the argument only appears for $n \geq 3$ which we assume from here on. In particular, $\pi_1(X'_{n+1}) \to \pi_1(X_n) = 0$ is an isomorphism so that X'_{n+1} is indeed simply connected.

It then follows from the universal properties that the map $f_n \colon C^*(X_n;\mathbb{Q}) \to A$ extends to a map $f'_{n+1} \colon C^*(X_{n+1};\mathbb{Q}) \to A$, i.e. such that $f_n \sim f^*_{n+1} \circ \alpha'_n$ as needed. We now claim that f'_{n+1} induces an isomorphism on π_{-i} for $i \leq n$. To see this, it suffices to argue that the map $\alpha^* \colon H^k(X_n;\mathbb{Q}) \to H^k(X'_{n+1};\mathbb{Q})$ is an isomorphism for $k < n$ and is surjective with kernel equal to V for $k = n$. To see this, consider the cohomological Serre spectral sequence for the fibre sequence

$$K(V^\vee, n-1) \to X'_{n+1} \to X_n.$$

We have $H^k(K(V^\vee, n-1);\mathbb{Q}) = 0$ for $k < n-1$ and that the differential

$$V \cong H^{n-1}(K(V^\vee, n);\mathbb{Q}) \to H^n(X_n;\mathbb{Q})$$

identifies naturally with the map induced by $X_n \to K(V^\vee, n)$ as follows from naturality of the Serre spectral sequence applied to the map of fibre sequences

$$\begin{tikzcd}
K(V^\vee, n-1) \arrow[r] \arrow[d] & X'_{n+1} \arrow[r] \arrow[d] & X_n \arrow[d, "\phi_n"] \\
K(V^\vee, n-1) \arrow[r] & * \arrow[r] & K(V^\vee, n)
\end{tikzcd}$$

and the fact that the corresponding differential in the spectral sequence of the lower fibre sequence is an isomorphism for formal reasons. Moreover, $H^n(K(V^\vee, n-1);\mathbb{Q}) = 0$ since $n \geq 3$ and so that n is not a multiple of $n-1$. Since X_n is simply connected, we also have that $E_2^{1,n-1} = H^1(X_n; V) = 0$. Hence, from Lemma 2.14 we find $H^n(X'_{n+1};\mathbb{Q}) \cong H^n(X_n;\mathbb{Q})/V$ as needed.

Define then

$$X_{n+1} = X'_{n+1} \times K(\pi_{-(n+1)}(A)^\vee, n+1).$$

By Künneth we then have

$$C^*(X_{n+1};\mathbb{Q}) = C^*(X'_{n+1};\mathbb{Q}) \otimes_{\mathbb{Q}} C^*(K(\pi_{-(n+1)}(A)^\vee, n+1))$$

which is a coproduct in $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ of the two terms. Hence, the map f'_{n+1} extends to a map $f_{n+1} \colon C^*(X_{n+1};\mathbb{Q}) \to A$ which is then by construction also surjective on $\pi_{-(n+1)}$. Moreover, by Künneth, the inclusion $X'_{n+1} \to X_{n+1}$ induces an isomorphism on $H^k(-;\mathbb{Q})$ for $k < n+1$, so the pair (X_{n+1}, f_{n+1}) proves the wanted inductive step.

We then finally define $X = \lim_n X_n$ and $f = \mathrm{colim}_n f_n \colon \mathrm{colim}_n C^*(X_n;\mathbb{Q}) \to A$. Using then that the canonical map $\mathrm{colim}_n C^*(X_n;\mathbb{Q}) \to C^*(X;\mathbb{Q})$ is an isomorphism, we have finally shown that $A \simeq C^*(X;\mathbb{Q})$ as needed.

3.23. **Remark** One may wonder what the image of $C^*(-;\mathbb{Q})$ is when restricted to connected, simple or nilpotant rational spaces of finite rational type. One could (and in fact, I did) think that it is given by those $A \in \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ whose underlying object in $\mathcal{D}(\mathbb{Q})$ is coconnective with π_0 isomorphic to $\mathbb{Q}$ and all homotopy groups finite dimensional over $\mathbb{Q}$.

To prove this, one could try to run the same argument as in Remark 3.22 to build, for fixed such $A \in \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ a simple space X with $C^*(X;\mathbb{Q}) \simeq A$. Trying to do this, one runs into a problem in the inductive step where we argued that the map $C^*(X'_{n+1};\mathbb{Q}) \to A$ is injective on π_{-n}. Indeed, in that situation, the term $H^1(X_n;V)$ a priori contributes to the cokernel of the map $H^n(X_n;\mathbb{Q}) \to H^n(X_{n+1};\mathbb{Q})$; in fact, a closer analysis shows that $E_\infty^{1,n-1}$ consists of those $x \in H^1(X_n;V) \cong H^1(X_n;\mathbb{Q}) \otimes_\mathbb{Q} V \subseteq H^1(X_n;\mathbb{Q}) \otimes H^n(X_n;\mathbb{Q})$ which lie in the kernel of the multiplication map to $H^{n+1}(X_n;\mathbb{Q})$. But a priori, there is no reason that this term vanishes in general.

Have we just not been clever enough? It turns out that no; there really is a problem here: Assume that we have proven the more general claim about fully faithfulness of $C^*(-;\mathbb{Q})$ when restricted to connected, nilpotent, rational spaces of finite rational type (the argument we gave for simple spaces really extends readily to the case of nilpotent spaces). If the above argument were to work, we find that every connected, nilpotent, rational space of finite rational type has a rational homology equivalence to a connected, simple, rational space of finite rational type. We now show that this is not the case:

Consider the central extension

$$1 \to \mathbb{Q} \to N \to \mathbb{Q}^2 \to 1$$

classified by a generator of $H^2(\mathbb{Q}^2;\mathbb{Q}) \cong H^2(\mathbb{Z}^2;\mathbb{Q}) \cong H^2(T^2;\mathbb{Q}) \cong \mathbb{Q}$ and consider the connected, nilpotent, rational space of finite type BN. If BN has a rational homology isomorphism to a connected, simple, rational space X, we conclude $\mathbb{Q}^2 = N_{\mathrm{ab}} \otimes_\mathbb{Z} \mathbb{Q} \cong H_1(BN;\mathbb{Q}) \cong H_1(X;\mathbb{Q}) \cong \pi_1(X)$: Indeed, N is a rational version of the classical Heisenberg group, which is generated by two elements x, y with relations that x and y both commute with the commutator $z = xyx^{-1}y^{-1}$. Exercise: $H^*(BN;\mathbb{Q})$ is $\mathbb{Q}$ in degree $0, 3$ and $\mathbb{Q}^2$ in degree $1, 2$. In fact, constructing the "same" central extension with $\mathbb{Z}$'s in place of $\mathbb{Q}$'s, taking the classifying space one obtains an aspherical 3-manifold M whose rationalisation is equivalent to BN.

Assuming now that X is simple, we have a fibre sequence

$$\tau_{\geq 2}X \to X \to B\mathbb{Q}^2$$

so we find that $H^2(\tau_{\geq 2}X;\mathbb{Q}) \cong \mathbb{Q}$ to obtain $H^2(X;\mathbb{Q}) \cong H^2(BN;\mathbb{Q}) \cong \mathbb{Q}^2$. But then we find that $E_2^{1,2} = H^1(B\mathbb{Q}^2; H^2(\tau_{\geq 2}X;\mathbb{Q})) \cong \mathbb{Q}^2$ and that this term does not admit any non-trivial differentials, showing that $\dim_\mathbb{Q} H^3(X;\mathbb{Q}) \geq 2$, contradicting the assumption that $H^3(X;\mathbb{Q}) \cong H^3(BN;\mathbb{Q}) \cong \mathbb{Q}$.

As discussed above, this argument implies that the image of $C^*(-;\mathbb{Q})$ when restricted to connected, simple, rational spaces of finite rational type is *not* merely those $A \in \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))$ which are coconnective, with π_0 equal to $\mathbb{Q}$ and all homotopy groups finite dimensional over $\mathbb{Q}$. Unfortunately, I don't know at this time how to describe the essential image when restricted to simple, and also not when restricted to nilpotent, rational spaces of finite rational type.

3.24. **Remark** I have now understood more than before, thanks to discussions with Jonas Stelzig. The starting point is the following observation: Instead of dealing with connected anima, let us work with connected pointed anima. Since the point is nilpotent, what we have (almost) shown earlier implies that the functor $C^*(-;\mathbb{Q})$ is again fully faithful when restricted to pointed connected rational nilpotent anima of rational finite type, when viewed as taking values in augmented coconnected commutative $\mathbb{Q}$-algebras.. Now, there is a functor taking indecomposables of an augmented commutative $\mathbb{Q}$-algebra; it gives a functor $Q\colon \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}} \to \mathcal{D}(\mathbb{Q})$, which sends $\mathrm{free}(V)$ to V for all $V \in \mathcal{D}(\mathbb{Q})$. Indeed, the indecomposables functor is left adjoint to a "trivial square zero extension" functor $\mathcal{D}(\mathbb{Q}) \to \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}}$, which informally sends V to $\mathbb{Q} \oplus V$ with multiplication determined by the zero multplication on V; Formally, this functor can be obtained by showing that $\mathrm{Sp}(\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}}) \simeq \mathcal{D}(\mathbb{Q})$, so that Ω^∞ serves as this trivial square zero functor (of course, it has to be shown that it sends C to some augmented algebra whose underlying object is $\mathbb{Q} \oplus V$); the functor Q can then be defined as the functor $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}} \to \mathcal{D}(\mathbb{Q})$ left adjoint to Ω^∞ (often denoted Σ^∞). Exercise: Show from this that $Q(\mathrm{free}(V)) = V$ using that $\mathrm{free}\colon \mathcal{D}(\mathbb{Q}) \to \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}}$ is left adjoint to the underlying object of the augmentation ideal functor $\epsilon\colon \mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}} \to \mathcal{D}(\mathbb{Q})$, $A \to \mathbb{Q} \mapsto \mathrm{fib}(A \to \mathbb{Q})$.

Now, applying $C^*(-;\mathbb{Q})$ to pointed connected anima gives a map

$$\pi_n(X) = \pi_0(\mathrm{Map}_{\mathrm{An}_*}(S^n, X)) \to \pi_0(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}}}(C^*(X), C^*(S^n)).$$

Moreover, it can be shown that $C^*(S^n) = \mathbb{Q} \oplus \mathbb{Q}[-n]$ is the trivial square zero extension on $\mathbb{Q}[-n]$; this is very classical. To see this, note that for n odd this is immediate since both sides in addition agree with the free algebra on $\mathbb{Q}[-n]$ (exercise); Now for $n \geq 2$ even, we have

$$C^*(S^n;\mathbb{Q}) \simeq C^*(\Sigma S^{n-1};\mathbb{Q}) \simeq \Omega C^*(S^{n-1};\mathbb{Q}) \simeq \Omega(\mathbb{Q} \oplus \mathbb{Q}[-n+1]) \simeq \mathbb{Q} \oplus \mathbb{Q}[-n]$$

where Ω is the loop functor of the pointed category $\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}}$. The last equality holds since the square zero extension functor, as a right adjoint, commutes with limits and hence preserves loop objects. Hence, we obtain by adjunction the equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(\mathbb{Q}))_{/\mathbb{Q}}}(C^*(X), C^*(S^n) \simeq \mathrm{Map}_{\mathcal{D}(\mathbb{Q})}(QC^*(X), \mathbb{Q}[-n])$$

from which we then obtain a canonical map

$$\pi_n(X) \to \pi_{-n}(QC^*(X))^\vee.$$

This map is an isomorphism if X is pointed, connected, nilpotent, rational of finite rational type, by fully faitfhfulness of the functor $C^*(-;\mathbb{Q})$. As a consequence, we find that the image of $C^*(-;\mathbb{Q})$, when restricted to connected, nilpotent, rational anima of finite rational type, is contained in coconnected, i.e. coconnective with $\pi_0(-) = \mathbb{Q}$, commutative $\mathbb{Q}$-algebras A of finite type whose indecomposables QA are also of finite type.

We obtain the following computation of rational homotopy groups of spheres:

3.25. **Proposition** *For n odd, the map $S^n \to K(\mathbb{Z}, n)$ is a rational equivalence. For n even, the map $S^n \to \mathrm{fib}(K(\mathbb{Z}, n) \xrightarrow{\iota_n^2} K(\mathbb{Z}, 2n))$ is a rational equivalence. In particular,*

$$\pi_k(S^n)_{\mathbb{Q}} \cong \begin{cases} \mathbb{Q} & \text{if } k = n \text{ or } k = 2n-1 \text{ and } n \text{ is even} \\ 0 & \text{else.} \end{cases}$$

Consequently, the map $\mathbb{S} \to \mathbb{Q}$ is a rational equivalence, so that $\pi_0(\mathbb{S})_{\mathbb{Q}} \cong \mathbb{Q}$ and $\pi_k(\mathbb{S})_{\mathbb{Q}} = 0$ if $k \neq 0$.

Proof. The case where n is odd is an immediate consequence of Proposition 3.16. In case n is even, denote by F the fibre of the map $K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ classifying ι_n^2. The Serre spectral sequence with $\mathbb{Q}$-coefficients reveals that

$$H_k(F;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0, n \\ 0 & \text{else} \end{cases}$$

so that the canonical map $S^n \to F$ induces an isomorphism on rational homology, and therefore also on rational homotopy by Corollary 3.10. □

3.26. **Remark** As noted before, it is not difficult to construct a generator of the $\mathbb{Q}$-vector space $\pi_{4n-1}(S^{2n})_{\mathbb{Q}}$. Indeed, the composite

$$S^{4n-1} \to S^{2n} \vee S^{2n} \to S^{2n}$$

where the first map is the attaching map for the top cell of $S^{2n} \times S^{2n}$ and the second map is the fold map. This map is not null homotopic as it has Hopf invariant 2, and hence generates the 1-dimensional $\mathbb{Q}$-vector space $\pi_{4n-1}(S^{2n})_{\mathbb{Q}}$.

A bit more explicitly, we find $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus A$ where A is a finite abelian group, by the classification of finitely generated abelian groups of rank 1 and Corollary 2.24. Now, the cup product in the cone of an element of $\pi_{4n-1}(S^{2n})$ leads to a map to a homomorphism to $\mathbb{Z}$ called the Hopf invariant. As noted earlier, the element we have constructed above has Hopf invariant 2. Most of the times, that is, unless $n = 1, 3, 7$ there is no element of odd Hopf invariant. This is, however, still a non-trivial result which we have not yet proven. In particular, though true, we do not know just yet that outside of these 3 special cases, the element we have written above in fact gives rise to a generator of the maximal torsion free quotient of $\pi_{4n-1}(S^{2n})$.

Let us now turn to p-local homotopy theory. To begin, we need to explicate the failure of Proposition 3.16 to hold true integrally.

3.27. **Lemma** *Let $n \geq 3$ and p be a prime. If n is odd, then the canonical map $\Lambda_{\mathbb{Z}}[\iota_n] \to H^*(K(\mathbb{Z}, n);\mathbb{Z})/\mathrm{tors}$ is an isomorphism. If n is even, the canonical map $\mathbb{Z}[\iota_n] \to H^*(K(\mathbb{Z}, n);\mathbb{Z})/\mathrm{tors}$ is an isomorphism, and the canonical map $\mathbb{Z}_{(p)}[\iota_n] \to H^*(K(\mathbb{Z}, n);\mathbb{Z}_{(p)})$ is an isomorphism in degree $* < 2p - 1 + n$. Moreover, for all n, we have*

$$H^{2p-1+n}(K(\mathbb{Z}, n);\mathbb{Z}_{(p)}) \cong \mathbb{F}_p$$

and this is the lowest degree in which non-trivial p-torsion appears.

Proof. First note that by Theorem 2.23, $H_*(K(\mathbb{Z}, n);\mathbb{Z})$ is degreewise finitely generated, so the same is true for $H^*(K(\mathbb{Z}, n);\mathbb{Z})$ and we have $H^*(K(\mathbb{Z}, n);\mathbb{Z}_{(p)}) \cong H^*(K(\mathbb{Z}, n);\mathbb{Z})_{(p)}$. If n is odd, it then follows immediately from Proposition 3.16, that $\Lambda_{\mathbb{Z}}[\iota_n] \to H^*(K(\mathbb{Z}, n);\mathbb{Z})/\mathrm{tors}$ is an isomorphism. Similarly, it follows that for n even, the map $\mathbb{Z}[\iota_n] \to H^*(K(\mathbb{Z}, n);\mathbb{Z})/\mathrm{tors}$ is injective and the target is infinite cyclic or trivial, so it suffices to show that ι_n^k is not divisible. Consider the map $\mathbb{CP}^\infty \to K(\mathbb{Z}, n)$ classifying $x^{\frac{n}{2}}$, where $x \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ is a generator. Then we obtain a map $\mathbb{Z}[\iota_n] \to H^*(K(\mathbb{Z}, n);\mathbb{Z})/\mathrm{tors} \to H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$, sending ι_n^k to $x^{\frac{nk}{2}}$ which is not divisible, so neither is ι_n^k.

Now we aim to prove the remaining cases inductively over n. To begin, we need to consider the case $n = 3$ for which we analyze the Serre spectral sequence for $K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3)$

and cohomology with $\mathbb{Z}_{(p)}$-coefficients. Since $H^*(K(\mathbb{Z},2);\mathbb{Z}) = \mathbb{Z}[\iota_2]$, we find that $E_2^{3,2k} \cong \mathbb{Z}_{(p)}$, and that the differential

$$H^{2k}(K(\mathbb{Z},2);\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)} \cong H^3(K(\mathbb{Z},3);H^{2k-2}(K(\mathbb{Z},2);\mathbb{Z}_{(p)}))$$

is given by multiplication by k. In particular, p-locally, the first instance in which this map is not an isomorphism is when $k = p$, so that we have $E_4^{3,2p-2} \cong \mathbb{F}_p$. Again inductively, we deduce that this $\mathbb{F}_p$ supports a differential with target necessarily in the $q = 0$ line of the spectral sequence; the target is then given by $H^{2p+2}(K(\mathbb{Z},3);\mathbb{Z}_{(p)})$. This shows that the first p-torsion appears in the degree $2p+2$ as claimed. Now let us prove the inductive step, and consider the fibration $K(\mathbb{Z},n-1) \to * \to K(\mathbb{Z},n)$ with n even and again the Serre spectral sequence with p-local coefficients. In this case, we find that the differential

$$H^{n-1}(K(\mathbb{Z},n-1);\mathbb{Z}_{(p)}) \to H^n(K(\mathbb{Z},n);\mathbb{Z}_{(p)})$$

is an isomorphism. From this, and multiplicativity of the spectral sequence, we find that $H^{kn}(K(\mathbb{Z},n);\mathbb{Z}_{(p)})$ is isomorphic to $\mathbb{Z}_{(p)}$ as long as $kn < 2p-1+n$. Note that we have already shown that this group is generated by ι_n^k. Now, by induction, we have an isomorphism $H^{2p-1+n-1}(K(\mathbb{Z},n-1);\mathbb{Z}_{(p)}) \cong \mathbb{F}_p$, so this term also has to support a differential for the spectral sequence to converge to the cohomology of a point. The only option is that it is the longest differential, going from the y-axes to the x-axes. This results in $H^{2p-1+n}(K(\mathbb{Z},n);\mathbb{Z}_{(p)}) \cong \mathbb{F}_p$; note that $2p-1+n$ is odd since we are in the situation where n is even.

Finally, we need to prove the inductive step using the fibration $K(\mathbb{Z},n-1) \to * \to K(\mathbb{Z},n)$ in case n is odd. But this is similar to the analysis in the computation of the inductive start, so we shall leave this part as an exercise. □

3.28. **Proposition** *Let $n \geq 3$ be odd. Then we have*

$$\pi_k(S^n)_{(p)} = \begin{cases} \mathbb{Z}_{(p)} & \text{if } k = n \\ \mathbb{F}_p & \text{if } k = 2p-3+n \\ 0 & \text{if } k < 2p-3+n \text{ and } k \neq n \end{cases}$$

Proof. Consider the map $S^n_{(p)} \to K(\mathbb{Z}_{(p)};n)$ which is an isomorphism on π_n and let F be its fibre. By Lemma 3.27, we find from the universal coefficient theorem:

$$H_k(K(\mathbb{Z};n);\mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{if } k = n \\ \mathbb{F}_p & \text{if } k = 2p-2+n \\ 0 & \text{if } k < 2p-2+n \text{ and } k \neq n \end{cases}$$

Then, from the Serre spectral sequence for the fibration $F \to S^n_{(p)} \to K(\mathbb{Z}_{(p)};n)$ or the p-local relative Hurewicz theorem, we find for $k \leq 2p-3+n$ that

$$\pi_k(F)_{(p)} \cong H_k(F;\mathbb{Z}_{(p)}) \cong \begin{cases} 0 & \text{if } 0 < k < 2p-2-n \\ \mathbb{F}_p & \text{if } k = 2p-3+n \end{cases}$$

From the long exact sequence associated to the defining fibre sequence and the observation that $2p-2 > 2p-3 > 0$ for all primes p, we also find that the canonical map

$$\pi_k(F)_{(p)} \to \pi_k(S^n)_{(p)}$$

is an isomorphism for $n < k \leq 2p-3+n$, giving the not yet proven part of the proposition. □

3.29. **Corollary** *For all primes p and all $n \geq 3$, we have that the first torsion which appears in $\pi_k(S^n)_{(p)}$ is an $\mathbb{F}_p$ in degree $k = 2p - 3 + n$.*

Proof. Only the case of even spheres remains S^{2n} to be argued. If $p = 2$, then Freudenthal implies that the map $\pi_4(S^3) \to \pi_{2n+1}(S^{2n})$ is an isomorphism, so we may focus on the case of odd primes p. There, by Proposition 3.12 we have

$$\pi_k(S^{2n}) \cong \pi_{k-1}(S^{2n-1}) \oplus \pi_k(S^{4n-1})$$

from which the claim follows immediately. ☐

3.30. **Remark** Note that we have in particular shown that $\pi_*(S^3)$ contains p-torsion for every prime p and is non-trivial for infinitely many degrees: Indeed, $\pi_{2p}(S^3)_{(p)} \cong \mathbb{F}_p$ for all p.

3.31. **Proposition** *For all $n \geq 3$ and all primes p, the suspension maps*

$$\pi_{2p-3+n}(S^n)_{(p)} \to \pi_{2p-3+n+1}(S^{n+1})_{(p)}$$

are isomorphisms. In particular,

$$\pi_k(\mathbb{S})_{(p)} \cong \pi_k(\mathbb{S}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & \textit{if } k = 0 \\ \mathbb{F}_p & \textit{if } k = 2p-3 \\ 0 & \textit{if } 0 < k < 2p-3 \end{cases}$$

Proof. For $p = 2$, this is a consequence of Freudenthal, see e.g. [Win24, Theorem 5.4.5], so we may assume p is odd. In case n is odd, it follows from Proposition 3.12 that the map in question is injective, and hence an isomorphism since both source and target are isomorphic to $\mathbb{F}_p$ by Proposition 3.28. Now still assuming n is odd, we consider the composite

$$S^n \to \Omega S^{n+1} \to \Omega^2 S^{n+2}.$$

We will argue below that

$$H_k(\Omega^2 S^{n+2}; \mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{if } k = 0, n \\ \mathbb{F}_p & \text{if } k = p(n+1) - 2 \\ 0 & \text{if } k < p(n+1) - 2 \text{ and } k \neq 0, n \end{cases}$$

It then follows from either Blakers–Massey or the Serre spectral sequence that the above map $S^n \to \Omega^2 S^{n+2}$ is about $p(n+1) - 2$ connected. Since $2p - 3 + n < p(n+1) - 2$, we deduce that the composite

$$\pi_{2p-3+n}(S^n)_{(p)} \to \pi_{2p-3+n+1}(S^{n+1})_{(p)} \to \pi_{2p-3+n+2}(S^{n+2})_{(p)}$$

is an isomorphism. Since all groups appearing in this composite are isomorphic to $\mathbb{F}_p$, we find that also the second map is an isomorphism, treating now also the case of the proposition in which n is even. ☐

3.32. **Lemma** *For odd $n \geq 3$, we have*

$$H_k(\Omega^2 S^{n+2}; \mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & \textit{if } k = 0, n \\ \mathbb{F}_p & \textit{if } k = p(n+1) - 2 \\ 0 & \textit{if } k < p(n+1) - 2 \textit{ and } k \neq 0, n \end{cases}$$

Proof. We consider the Serre spectral sequence for the fibre sequence

$$\Omega^2 S^{n+2} \to * \to \Omega S^{n+2}.$$

Recall that $H^*(\Omega S^{n+2};\mathbb{Z}) = \Gamma_{\mathbb{Z}}[x_{n+1}]$, the free divided power algebra on a generator of degree $n+1$. In the cohomological Serre spectral sequence, we then first deduce that the differential

$$\mathbb{Z} \cong H^n(\Omega^2 S^{n+2};\mathbb{Z}) \to H^{n+1}(\Omega S^{n+2};\mathbb{Z})$$

is an isomorphism. Consequently, we inductively find that

$$E_2^{(k-1)(n+1),n} = H^{(k-1)(n+1)}(\Omega S^{n+2};\mathbb{Z}) \to H^{k(n+1)}(\Omega S^{n+2};\mathbb{Z}) = E_2^{k(n+1),0}$$

is given by multiplication by k, at least as long as $k \leq p$. In particular, p-locally all these maps are isomorphisms, so with $\mathbb{Z}_{(p)}$-coefficients, we deduce that the first next non-trivial entry on the y-axes of the spectral sequence is at row $p(n+1)-1$:

$$H^{n(p+1)-1}(\Omega^2 S^{n+2};\mathbb{Z}_{(p)}) \cong \mathbb{F}_p$$

The claim then follows from UCT. □

3.33. **Remark** One can show that for odd primes p, the next p-torsion in $\pi_*(\mathbb{S})$ appears in degree $4p-5$ and is again isomorphic to $\mathbb{F}_p$. The proof is similar to the one above, but more complicated. We leave the details to the reader for now, and perhaps add the argument later.

4. COHOMOLOGY OF EILENBERG–MAC LANE SPACES

We begin with an important property of the (cohomological) Serre spectral sequence often referred to as the transgression theorem. So let $F \to E \to B$ be a simple fibre sequence with F and B connected and A an abelian group. First, we need a definition.

4.1. **Definition** A pair $(x,y) \in H^n(F;A) \times H^{n+1}(B;A)$ is called transgressive[11] if $d_r(x) = 0$ for all $2 \leq r \leq n$ and $d_{n+1}(x) = [y] \in E_{n+1}^{n+1,0}$.

We now aim to prove the following theorem.

4.2. **Theorem** (Transgression Theorem) *In the above situation, let* $\theta\colon K(A,n+1) \to K(B,m)$ *be a cohomology operation and* (x,y) *a transgressive pair. Then* $(\Omega\theta(x),\theta(y))$ *is again a transgressive pair.*

The proof of the above theorem is essentially about characterising transgressive pairs in the following way. Thanks to Achim Krause for reminding me of this characterisation.

4.3. **Lemma** *In the above situation, a pair* (x,y) *is transgressive if and only if it participates in a map of fibre sequences*

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d, "x"] & E \arrow[r, "\pi"] \arrow[d] & B \arrow[d, "y"] \\
K(A,n) \arrow[r] & * \arrow[r] & K(A,n+1)
\end{tikzcd}$$

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[11] Sometimes, just x is called transgressive.

Proof. First suppose given a map of fibre sequences as displayed. By Remark 2.12, there is a map from the cohomological Serre spectral sequence of the bottom fibre sequence to that of the top fibre sequence. Since $(\iota_n^A, \iota_{n+1}^A) \in H^n(K(A,n);A) \times H^{n+1}(K(A,n+1);A)$ is a transgressive pair for formal reasons, the same then holds true for its image under the just indicated map of spectral sequences, showing that (x,y) is transgressive.

Conversely, suppose that (x,y) is transgressive. We deduce that $[y] = 0 \in E_\infty^{n+1,0}$ which by Lemma 2.14 implies that $\pi^*(y) = 0$. We may therefore choose a null-homotopy γ of the composite $E \to B \to K(A,n+1)$, giving rise to a map of fibre sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d, dashed, "\tilde{x}"] & E \arrow[r] \arrow[d] & B \arrow[d, "y"] \\
K(A,n) \arrow[r] & * \arrow[r] & K(A,n+1)
\end{tikzcd}$$

where $\tilde{x}$ is the map induced on fibres. By what we have just discussed, we find that $(\tilde{x}, y)$ is also a transgressive pair, and in particular, that $d_{n+1}(x - \tilde{x}) = 0$ and hence $x - \tilde{x} \in E_\infty^{0,n}$. By Lemma 2.14, there exists $z \in H^n(E;A)$ with $i^*(z) = x - \tilde{x}$. We may use z to change the null-homotopy γ, and doing so, the induced map on fibres $F \to K(A,n)$ becomes x. This shows the lemma. □

Proof of Theorem 4.2. By Lemma 4.3, the following diagram consists of maps of fibre sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "x"] & E \arrow[r] \arrow[d] & B \arrow[d, "y"] \\
K(A,n) \arrow[r] \arrow[d, "\Omega\theta"] & * \arrow[r] \arrow[d] & K(A,n+1) \arrow[d, "\theta"] \\
K(B,m-1) \arrow[r] & * \arrow[r] & K(B,m)
\end{tikzcd}$$

so another application of Lemma 4.3 gives the claim. □

4.4. **Example** Recall from [Lan24, Remark 6.6] that, in somewhat abusive notation, $\Omega \mathrm{Sq}^i = \mathrm{Sq}^i$. Hence, for $A = \mathbb{F}_2$ and (x,y) a transgressive pair, $(\mathrm{Sq}^i(x), \mathrm{Sq}^i(y))$ is also a transgressive pair for all $i \geq 0$. This fact is extremely helpful for computations, as we shall see next.

The next goal is to compute $H^*(K(\mathbb{F}_p,n);\mathbb{F}_p)$. Somewhat surprisingly, this can actually be done. Following what I have done in [Lan24], we work this out for $p = 2$, the odd primary case will at most be indicated. Recall that a multiindex $I = (i_1, \dots, i_k)$ determines $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_k}$ and that I is called admissible if $i_j \geq 2i_{j+1}$ for all j. As a consequence of the Adem relations, we have seen in [Lan24] that every element in the Steenrod algebra $\mathcal{A}^*$ is a sum of admissible monomials Sq^I, that is, monomial Sq^I with I admissible.

4.5. **Definition** Let $I = (i_1, \dots, i_k)$ be an admissible multiindex. We define its excess $e(I)$ as

$$e(I) = \sum_{j \geq 0} i_j - 2i_{j+1} = i_1 - i_2 - i_3 - \cdots - i_k$$

so that $i_1 = e(I) + i_2 + \cdots + i_k$.

4.6. **Example** The admissible monomials of excess ≤ 0 is precisely Sq^0. The admissible monomials of excess ≤ 1 are precisely $\mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^1, \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1, \dots$ etc.

4.7. **Remark** Let $x \in H^*(X;\mathbb{F}_2)$ and I is admissible with $e(I) > |x|$. Then $\mathrm{Sq}^I(x) = 0$ in $H^*(X;\mathbb{F}_2)$. Indeed, in this case we have $i_1 > |x| + i_2 + \dots + i_k = |\mathrm{Sq}^{i_2} \dots \mathrm{Sq}^{i_k}(x)|$ so that

$$Sq^I(x) = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \dots \mathrm{Sq}^{i_k}(x) = 0$$

for degree reasons. In particular, $\mathrm{Sq}^I(\iota_n) = 0$ in $H^*(K(\mathbb{F}_2, n);\mathbb{F}_2)$ if $e(I) > n$.

We then aim to prove the following theorem.

4.8. **Theorem** *The canonical maps*

(1) $\mathbb{F}_2[\mathrm{Sq}^I(\iota_n) \mid I \textit{ adm. with } e(I) < n] \to H^*(K(\mathbb{F}_2, n);\mathbb{F}_2)$*, for* $n \geq 1$*,*
(2) $\mathbb{F}_2[\mathrm{Sq}^I(\iota_n) \mid I \textit{ adm. not containing } 1 \textit{ with } e(I) < n] \to H^*(K(\mathbb{Z}, n);\mathbb{F}_2)$*, for* $n \geq 2$*,*
(3) $\mathbb{F}_2[\mathrm{Sq}^I(\iota_n), \mathrm{Sq}^J(\kappa_{n+1}) \mid I, J \textit{ adm. not containing } 1 \textit{ with } e(I) < n, e(J) \leq n] \to H^*(K(\mathbb{Z}/2^k, n);\mathbb{F}_2)$ *where* $k > 1$ *and* κ_{n+1} *is a generator of* $H^{n+1}(K(\mathbb{Z}/2^k, n);\mathbb{F}_2) \cong \mathbb{F}_2$*.*

are isomorphisms of $\mathbb{F}_2$*-algebras.*

The proof of Theorem 4.8 will be by induction over n. For the case (1) with $n = 1$, recall that only Sq^0 is an admissible monomial of excess 0. Hence, the statement translates to the fact that $\mathbb{F}_2[\iota_1] \to H^*(K(\mathbb{F}_2, 1);\mathbb{F}_2)$ is an isomorphism, which we have proven a long time ago. For the proof, we will make use of the following definition.

4.9. **Definition** Let R be a graded commutative $\mathbb{F}_2$-algebra. A subset $I \subseteq R$ consisting of homogenous elements is called a simple system of generators if the set

$$\{x_J = \prod_{x \in J} x \mid J \subseteq I \text{ finite subset } \}$$

forms a basis of the underlying $\mathbb{F}_2$-vector space of R.

4.10. **Example** (1) The exterior algebra $\Lambda_{\mathbb{F}_2}[x]$ has $\{x\}$ as simple system of generators.
(2) The polynomial algebra $\mathbb{F}_2[x]$ has $\{x^{2^i}\}_{i \geq 0}$ as simple system of generators.
(3) More generally, $\mathbb{F}_2[x_1, x_2, \dots]$ has $\{x_k^{2^i}\}_{k \geq 1, i \geq 0}$ as simple system of generators.
(4) Even more generally, $\Lambda_{\mathbb{F}_2}[y_1, y_2, \dots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_1, x_2, \dots]$ has $\{y_j \otimes x_k^{2^i}\}_{i,j,k}$ as simple system of generators.

The computation of the cohomology of Eilenberg–Mac Lane spaces then rests on the following theorem.

4.11. **Theorem** (Borel) *Let* B *be a simply connected anima. Assume that* $H^*(\Omega B;\mathbb{F}_2)$ *has a simple system of generators* $(x_1, x_2, \dots)$ *of positive degrees with only finitely many* x_i*'s of fixed degree. Assume further that we can choose for each* $i \geq 1$ *an element* $y_i \in H^*(B;\mathbb{F}_2)$ *such that* (x_i, y_i) *is a transgressive pair. Then the map* $\mathbb{F}_2[y_1, y_2, \dots] \to H^*(B;\mathbb{F}_2)$ *is an isomorphism.*

Proof. To ease notation, we write $H^*(-)$ for $H^*(-;\mathbb{F}_2)$. First, we claim that there is a unique multiplicative spectral sequence with second page given by

$$F_2^{p,q} = \mathbb{F}_2[y_i \mid i \in I]_p \otimes_{\mathbb{F}_2} H^q(\Omega B)$$

and such that the pairs (x_i, y_i) are transgressive and abutment given by $\mathbb{F}_2$ concentrated in degree 0. Here, $\mathbb{F}_2[y_i \mid i \in I]_p$ denotes the degree p part of the graded ring $\mathbb{F}_2[y_i \mid i \in I]$ where $|y_i| = |x_i| + 1$. Indeed, we need to convince ourselves that all differentials determined by the Leibniz rule and $d(x_i) = y_i$ lead to the infinite page of the spectral sequence being concentrated in bidegree $(0, 0)$. Furthermore, there is a unique map of spectral sequences θ from the just described spectral sequence to the cohomological Serre spectral sequence of the fibration $\Omega B \to * \to B$ sending x_i to x_i and y_i to y_i. By construction, θ then induces an isomorphism

$$F_2^{0,q} \xrightarrow{\cong} E_2^{0,q} \text{ and } F_\infty^{p,q} \xrightarrow{\cong} E_\infty^{p,q} \text{ for all } p, q \geq 0$$

We will now show that this implies that

$$\mathbb{F}_2[y_i \mid i \in I]_p \cong F_2^{p,0} \xrightarrow{\theta_2^{p,0}} E_2^{p,0} = H^p(B),$$

is also an isomorphism, giving the theorem. This is a tedious but not really complicated argument in homological algebra. We aim to show that the map

$$\theta_2^{k,0} \colon F_2^{k,0} \to E_2^{k,0}$$

is an isomorphism, via induction over k. The induction start $k = 0$ is true by assumption. So let us fix k and assume that we know that $\theta_2^{p,0}$ is an isomorphism for $p \leq k$. Observe that this implies, again by assumption, that $\theta_2^{p,q}$ is an isomorphism for all $p \leq k$ and all $q \geq 0$. First, we show that this implies that for all $r \geq 2$,

1. $\theta_r^{p,q}$ is an isomorphism for all $p \leq k - r + 2$, and
2. $\theta_r^{p,q}$ is injective for $p \leq k$.

This claim will be proven by induction over r, the induction start $r = 2$ having just been observed. For the induction step and part (1), consider the diagram

$$(*) \qquad \begin{tikzcd} F_r^{p-r,q+r-1} \arrow[r, "d_r^F"] \arrow[d, "\theta_r^{p-r,q-r+1}"'] & ZF_r^{p,q} \arrow[r] \arrow[d, "\theta_r^{p,q}"] & F_{r+1}^{p,q} \arrow[r] \arrow[d, "\theta_{r+1}^{p,q}"] & 0 \\ E_r^{p-r,q+r-1} \arrow[r, "d_r^E"] & ZE_r^{p,q} \arrow[r] & E_{r+1}^{p,q} \arrow[r] & 0 \end{tikzcd}$$

where ZF and ZE stand for the cycles, that is, the kernel of the corresponding differentials in the respective spectral sequences. Now, if $p \leq k - (r + 1) + 2$, then $p - r \leq k - r + 2$, so inductively, the left most vertical map is an isomorphism. To see that the right vertical map is an isomorphism, it therefore suffices to show that the middle vertical map is one. To see this, consider the diagram

$$(**) \qquad \begin{tikzcd} 0 \arrow[r] & ZF_r^{p,q} \arrow[r] \arrow[d, "\theta_r^{p,q}"] & F_r^{p,q} \arrow[r, "d_r^F"] \arrow[d, "\theta_r^{p,q}"] & F_r^{p+r,q-r+1} \arrow[d, "\theta_r^{p+r,q-r+1}"] \\ 0 \arrow[r] & ZF_r^{p,q} \arrow[r] & E_r^{p,q} \arrow[r, "d_r^E"] & E_r^{p+r,q-r+1} \end{tikzcd}$$

Now, since $p \leq k - (r + 1) + 2$, we find $p \leq k + 1$ so the inductive hypothesis gives that both the middle and the right vertical maps are isomorphisms, and hence so is the middle vertical map.

To see (2), we argue again by induction over r. Considering again diagram $(*)$, note that for $p \leq k$, we find that by the just proven part (1), the left most vertical map is an isomorphism.

Then note that, by induction over r and using diagram $(**)$, the middle vertical map is injective. It then follows from the 5-Lemma that (2) holds as claimed.

To establish the inductive step that $\theta_2^{k,0}\colon F_2^{k,0} \to E_2^{k,0}$ is an isomorphism, consider the following diagram.

$$\begin{tikzcd}
ZF_r^{k-r+1,r-1} \arrow[r] \arrow[d] & F_r^{k-r+1,r-1} \arrow[r, "d_r^F"] \arrow[d, "\theta_r^{k-r+1,r-1}"] & F_r^{k+1,0} \arrow[r] \arrow[d, "\theta_r^{k+1,0}"] & F_{r+1}^{k+1,0} \arrow[r] \arrow[d, "\theta_{r+1}^{k+1,0}"] & 0 \\
ZE_r^{k-r+1,r-1} \arrow[r] & E_r^{k-r+1,r-1} \arrow[r, "d_r^E"] & E_r^{k+1,0} \arrow[r] & E_{r+1}^{k+1,0} \arrow[r] & 0
\end{tikzcd}$$

By the assumption that θ_∞ is an isomorphism and that the spectral sequence is concentrated in the first quadrant, we find that the right most vertical map is an isomorphism for sufficiently large r. We then run a downwards induction over r. Now, by the already established (1) above, and since $k-r+1 \leq k-r+2$, we find that the second left most map is an isomorphism for all r. Hence, to establish the inductive step and hence the theorem, by the 5-Lemma, it suffices to argue that the left most vertical map is surjective. To see this, consider the diagram

$$\begin{tikzcd}
F_s^{k-r-s+1,r+s-2} \arrow[r, "d_s^F"] \arrow[d, "\theta_s^{k-r-s+1,r+s-2}"'] & ZF_s^{k-r+1,r-1} \arrow[r] \arrow[d] & F_{s+1}^{k-r+1,r-1} \arrow[r] \arrow[d, "\theta_{s+1}^{k-r+1,r-1}"] & 0 \\
E_s^{k-r-s+1,r+s-2} \arrow[r, "d_s^E"] & ZE_s^{k-r-1,r-1} \arrow[r] & E_{s+1}^{k-r+1,r-1} \arrow[r] & 0
\end{tikzcd}$$

Again, we run a downwards induction over s until $s = r$ is reached. Similarly to before, the right most vertical map is an isomorphism for s sufficiently large, so we may assume inductively that it is an isomorphism. Using again the proven (1) above, we also find that the left most vertical map is an isomorphism; indeed $k-r-s+1 \leq k-s+2$, and hence by the 4-Lemma we find that the middle vertical map is an isomorphism as needed. □

Proof of Theorem 4.8. We begin with (1) and prove the result by induction, the case $n = 1$ already having been noted. For the induction step, assume that $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Sq}^I(\iota_n) \mid e(I) < n]$ which has a simple system of generators given by $\{[\mathrm{Sq}^I(\iota_n)]^{2^k}\}$ for $k \geq 0$ and I admissible with $e(I) < n$. We claim that this set is precisely the set $\{\mathrm{Sq}^J(\iota_n)\}$ for J admissible with $e(J) \leq n$.

Indeed, the case $k = 0$ just gives the monomials of excess less than n, so it suffices to argue that if J is admissible with $e(J) = n$, then $\mathrm{Sq}^J(\iota_n)$ identifies uniquely with $\mathrm{Sq}^I(\iota_n)^{2^k}$. Indeed, we have $i_1 = n + i_2 + \cdots + i_k$, so that $\mathrm{Sq}^J(\iota_n) = \mathrm{Sq}^I(\iota_n)^2$ where $I = (i_2, \ldots, i_k)$. Note that $e(I) \leq e(J)$, so either $e(I) < n$ in which case we are done, or $e(I) = n$ in which case we argue in the same manner with I instead of J. Since the length of I is strictly smaller than that of J, this process terminates at a finite step.

Hence, we find that $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ has a simple system of generators given by $\mathrm{Sq}^J(\iota_n)$ with $e(J) \leq n$. By Theorem 4.2 the pairs $(\mathrm{Sq}^J(\iota_n), \mathrm{Sq}^J(\iota_{n+1}))$ are transgressive pairs, so Borel's theorem gives the claim.

The details of (2) and (3) are left as an exercise, but here are some hints. To see (2), we start the induction at $n = 2$: Here $H^*(K(\mathbb{Z}, 2); \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 2$ so has a simple system of generators $\{x^{2^i}\}_{i \geq 0}$. A similar argument as in the previous case then gives the result. For (3), we note that $H^*(K(\mathbb{Z}/2^k, 1); \mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}[e] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u]$ with $|e| = 1$ and $|u| = 2$ and again use induction via Borel's theorem. □

Recall now that $\widetilde{H}^*(X)$ is an $\mathcal{A}^*$-module for all anima X.

4.12. **Proposition** *The canonical map* $\mathcal{A}^* \to \widetilde{H}^*(K(\mathbb{F}_2, n))[-n]$ *is an isomorphism in degrees* $\leq n$.

Proof. Note that the elements of degree $< n$ in $\widetilde{H}^*(K(\mathbb{F}_2, n)[-n]$, viewed as elements of degree $< 2n$ in $\widetilde{H}^*(K(\mathbb{F}_2, n))$ are indecomposable for degree reasons. Hence, as a consequence of Theorem 4.8, the degree $< 2n$ elements of $\widetilde{H}^*(K(\mathbb{F}_2, n))[-n]$ have an additive basis consisting of the admissible monomials Sq^J with $|J| < n$, where $|J| = j_1 + \dots j_k$ is the total degree of J; note that $e(J) \leq |J|$. Moreover, the only decomposable element in degree $2n$ is given by $\iota_n^2 = \mathrm{Sq}^n(\iota_n)$. Together, this implies that the map $\mathcal{A}^* \to \widetilde{H}^*(K(\mathbb{F}_2, n))[-n]$ is surjective in degrees $\leq n$. Now since the set $\{\mathrm{Sq}^J(\iota_n)\}$, with J running through admissible multiindices of degree $< n$ is linearly independent, we deduce that the also the set $\{\mathrm{Sq}^J\} \subseteq \mathcal{A}^*$, with same J's, is linearly independent in $\mathcal{A}^*$, and by the Adem relations, we find that they therefore form a basis of the degree $< n$ elements of $\mathcal{A}^*$. It then follows that an additive basis of the degree $\leq n$ elements of both $\mathcal{A}^*$ and $\widetilde{H}^*(K(\mathbb{F}_2, n))[-n]$ is given by admissible monomials associated to an admissible mulitiindex J of degree $< n$ and Sq^n (which has degree equal to excess equal to n), and the proposition follows. ☐

4.13. **Corollary** *The admissible monomials form an additive basis of* $\mathcal{A}^*$.

Proof. Pick a finite set S of admissible monomials in $\mathcal{A}^*$. Then they lie in the degree $\leq n$ part of $\mathcal{A}^*$ for some n, so the claim follows from Proposition 4.12, as S is then linearly independent in $H^*(K(\mathbb{F}_2, n))$. ☐

4.14. **Corollary** *The canonical maps*

$$\mathbb{F}_2^*(\mathbb{F}_2) = \pi_{-*}\mathrm{map}(\mathbb{F}_2, \mathbb{F}_2) \to \lim_n \widetilde{H}^*(K(\mathbb{F}_2, n))[-n]$$

as well as

$$\mathcal{A}^* \to \lim_n \widetilde{H}^*(K(\mathbb{F}_2, n)[-n]$$

are isomorphisms. In particular, $\mathcal{A}^* \cong {\mathbb{F}_2}^*(\mathbb{F}_2)$ *and the admissible monomials form an additive basis of* $\mathcal{A}^*$.

Proof. Recall that the canonical map $\mathrm{colim}_n\, \Omega^n\Sigma^\infty K(\mathbb{F}_2, n) \to \mathbb{F}_2$ is an equivalence of spectra (exercise). Hence, we find

$$\mathrm{map}(\mathbb{F}_2, \mathbb{F}_2) = \lim_n \mathrm{map}_*(K(\mathbb{F}_2, n), \Sigma^n\mathbb{F}_2).$$

Note that the inverse limit system of homotopy groups stabilizes in each homotopical degree as a consequence of Proposition 4.12 and representability of cohomology. In particular, Milnors lim-$\lim^1$-sequence has no $\lim^1$-term and we obtain the first claim. The second claim follows also directly from Proposition 4.12. To see the final claim, it remains to show that the admissible monomials are linearly independent in $\mathcal{A}^*$. So pick a finite set S of such admissible monomials. It suffices to show that the image of S under the canonical map to $H^*(K(\mathbb{F}_2, n))$ is linearly independent for some n. Choosing n large enough, the set S then even consist of polynomial generators of a polynomial algebra again by Proposition 4.12, and so is in particular additively linearly independent. ☐

4.15. **Corollary** *The canonical map* $\mathbb{Z} \to \mathbb{F}_2$ *induces an isomorphism* $\mathcal{A}^*/\mathrm{Sq}^1 \cong \mathbb{F}_2^*(\mathbb{Z})$.[12] *Moreover,* $\mathbb{F}_2^*(\mathbb{Z})$ *has an additive basis consisting of those admissible monomials* Sq^I *where* $1 \notin I$.

Proof. Consider the Bockstein fibre sequence $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{F}_2$. It induces a fibre sequence

$$\mathrm{map}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{map}(\mathbb{Z}, \mathbb{F}_2) \xrightarrow{\cdot 2} \mathrm{map}(\mathbb{Z}, \mathbb{F}_2)$$

in which the second map is canonically trivial, as $2 = 0$ in $\mathbb{F}_2$. Note that the connecting map $\mathbb{F}_2 \to \Sigma\mathbb{Z}$ in the Bockstein fibre sequence is the Bockstein operator β (hence the name). Therefore, the long exact sequence of the displayed fibre sequence breaks up into short exact sequences

$$0 \to \mathbb{F}_2^{k-1}(\mathbb{Z}) \xrightarrow{\beta^*} \mathbb{F}_2^k(\mathbb{F}_2) \xrightarrow{p^*} \mathbb{F}_2^k(\mathbb{Z}) \to 0$$

Since the latter map is surjective for all k, we also have an exact sequence

$$\mathbb{F}_2^{k-1}(\mathbb{F}_2) \xrightarrow{(p\beta)^*} \mathbb{F}_2^k(\mathbb{F}_2) \xrightarrow{p^*} \mathbb{F}_2^k(\mathbb{Z}) \to 0$$

Since $p\beta$ is equal to Sq^1 and using that $\mathcal{A}^* = \mathbb{F}_2^*(\mathbb{F}_2)$, we find an exact sequence

$$\mathcal{A}^{*-1} \xrightarrow{-\cdot \mathrm{Sq}^1} \mathcal{A}^* \to \mathbb{F}_2^*(\mathbb{Z}) \to 0$$

showing the first part of the corollary. For the latter, simply note that right multiplication by Sq^1 preserves admissible monomials in the following sense: For an admissible sequence $I = (i_1, \dots, i_k)$, there are two options: Either $i_k = 1$, in which case $\mathrm{Sq}^I\mathrm{Sq}^1 = 0$, or $i_k > 1$, in which case $\mathrm{Sq}^I\mathrm{Sq}^1 = \mathrm{Sq}^J$ with $J = (I, 1) = (i_1, \dots, i_k, 1)$ again admissible. Therefore, we see that $\mathcal{A}^*/\mathrm{Sq}^1$ has the claimed additive basis. ☐

4.16. **Proposition** *For each* $k > 0$*, the group* $\mathbb{Z}^k\mathbb{Z}$ *is finite. Moreover, all elements of 2-power order are in fact of order 2.*

Proof. First, as earlier, we have that $\mathbb{Z}^*(\mathbb{Z}) = \pi_{-*}\mathrm{map}(\mathbb{Z}, \mathbb{Z})$ and

$$\mathrm{map}(\mathbb{Z}, \mathbb{Z}) = \lim_n \mathrm{map}_*(K(\mathbb{Z}, n), \Sigma^n\mathbb{Z})$$

and this limit stabilises on a fixed homotopy group, all of which are then finitely generated. It follows that the map $\mathrm{map}(\mathbb{Z}, \mathbb{Z}) \otimes \mathbb{Q} \to \mathrm{map}(\mathbb{Z}, \mathbb{Q}) = \mathrm{map}(\mathbb{S}, \mathbb{Q}) = \mathbb{Q}$ is an equivalence, and hence that for each $k > 0$, $\mathbb{Z}^k\mathbb{Z}$ is a finite group as claimed.

The second claim is then equivalent to the statement that the multiplication by 2 map on $\mathbb{Z}^k\mathbb{Z}$ is the zero map for $k > 0$. To that end, consider the long exact sequence

$$\dots \to \mathbb{Z}^{k-1}(\mathbb{Z}) \xrightarrow{\cdot 2} \mathbb{Z}^{k-1}(\mathbb{Z}) \xrightarrow{p_*} \mathbb{F}_2^{k-1}(\mathbb{Z}) \xrightarrow{\beta_*} \mathbb{Z}^k(\mathbb{Z}) \xrightarrow{\cdot 2}\to \mathbb{Z}^k(\mathbb{Z}) \to \dots$$

where $p\colon \mathbb{Z} \to \mathbb{F}_2$ is the projection and $\beta\colon \mathbb{F}_2 \to \Sigma\mathbb{Z}$ is the Bockstein.

So pick for $k > 0$ an element $y \in \mathbb{Z}^k\mathbb{Z}$. We aim to show that $2y = 0$. First we note that $0 = \mathrm{Sq}^1 \cdot p_*(y) \in \mathbb{F}_2^{k+1}(\mathbb{Z})$, since is is represented by the composite

$$\Omega^k\mathbb{Z} \xrightarrow{y} \mathbb{Z} \xrightarrow{p} \mathbb{F}_2 \xrightarrow{\mathrm{Sq}^1} \Sigma\mathbb{F}_2$$

of which the latter two maps canonically compose to the zero map. We now claim that

$$\ker[\mathrm{Sq}^1 \cdot -\colon \mathbb{F}_2^k(\mathbb{Z}) \to \mathbb{F}_2^{k+1}(\mathbb{Z})] = \mathrm{im}[\mathrm{Sq}^1 \cdot -\colon \mathbb{F}_2^{k-1}(\mathbb{Z}) \to \mathbb{F}_2^k(\mathbb{Z})].$$

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[12] Here, $\mathcal{A}^*/\mathrm{Sq}^1$ denotes the quotient of $\mathcal{A}^*$ by the left ideal generated by Sq^1, that is, the image of right multiplication by Sq^1 on $\mathcal{A}^*$.

Hence, we find an x in $\mathbb{F}_2^{k-1}(\mathbb{Z})$ such that $\mathrm{Sq}^1 \cdot x = p_*(y)$ and since $\mathrm{Sq}^1 = p_*\beta$, we find that $y - \beta(x)$ lies in the kernel of p_*, which by exactness of the Bockstein sequence gives a $y_2 \in \mathbb{Z}^k(\mathbb{Z})$ such that

$$y - \beta(x) = 2y_2.$$

Running the same argument again, we can then inductively find for y_i (with $y_1 = y$ and $y_2 = y_2$) elements x_i and y_{i+1} such that for all $i \geq 1$ we have

$$y_i - \beta(x_i) = 2y_{i+1}.$$

This gives:

$$2y = 2y_2 + 2\beta(x) = 2y_2 = \cdots = 2^i y_i$$

for all $i \geq 1$. But since $\mathbb{Z}^k\mathbb{Z}$ is a finite group by the first part of the proposition, its 2-exponent is some finite number e, so once $i > e$, we deduce

$$2y = 2^i y_i = 0.$$

It therefore remains to prove the above claim about the left multiplication by Sq^1 on $\mathbb{F}_2^*(\mathbb{Z}) = \mathcal{A}^*/\mathrm{Sq}^1$. We leave this verification as an exercise. Hint: Use the basis of $\mathcal{A}^*/\mathrm{Sq}^1$ established in Corollary 4.15. □

4.17. **Remark** It is also true that for p an odd prime, all elements of p-power order in $\mathbb{Z}^*(\mathbb{Z})$ are of order p; the proof is similar to the one given above, but of course relies on $\mathbb{F}_p^*(\mathbb{F}_p) = \mathcal{A}_p^*$ and the computation of $\mathbb{F}_p^*(\mathbb{Z})$. This allows an inductive computation of $\mathbb{Z}^*(\mathbb{Z})$ from $\mathbb{F}_p^*(\mathbb{F}_p)$ for all primes p. Exercise: Work out the 2-torsion in $\mathbb{Z}^*(\mathbb{Z})$ for $* \leq k$ for k as large as you want.

5. Some homotopy groups of spheres

We reprove what we already know:

5.1. **Proposition** *We have* $\pi_4(S^3) \cong \mathbb{Z}/2$.

Proof. Consider the 3-truncation map $S^3 \to K(\mathbb{Z}, 3)$ and denote by F_4 its fibre. Then we find

$$\pi_4(S^3) \cong \pi_4(F_4) \cong H_4(F_4; \mathbb{Z})$$

and therefore want to compute the homology of F_4 using the Serre spectral sequence for the fibre sequence $K(\mathbb{Z}, 2) \to F_4 \to S^3$. Again, it is convenient to first consider the cohomological spectral sequence, which using the multiplicativity reveals

$$\widetilde{H}^*(F_4; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{for } * = 2k+1 \\ 0 & \text{else} \end{cases}$$

It then follows from the universal coefficient theorem that

$$\widetilde{H}_*(F_4; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{for } * = 2k \\ 0 & \text{else} \end{cases}$$

and in particular that $H_4(F_4; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ as claimed. □

We now aim to compute $\pi_5(S^3)$ with a similar method; later we will present a slightly different, second approach. Since $\pi_4(S^3) \cong \mathbb{Z}/2$, it may not be a surprise that we will need to calculate something about the (integral) homology of mod 2 Eilenberg–Mac Lane spaces.

5.2. **Lemma** *We have*

$$\widetilde{H}^k(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \begin{cases} 0 & \text{for } k = 0,1,2,4 \\ \mathbb{Z}/2 & \text{for } k = 3,6 \\ \mathbb{Z}/4 & \text{for } k = 5 \end{cases}$$

and the non-zero element of degree 3 squares to the non-zero element of degree 6.

Proof. We consider the Serre spectral sequence in integral cohomology for the fibre sequence $K(\mathbb{Z}/2,1) \to * \to K(\mathbb{Z}/2,2)$. We recall that $H^*(K(\mathbb{Z}/2,1);\mathbb{Z}) = \mathbb{Z}[u]/2u$ with $|u| = 2$. By Hurewicz, we know the claim for $k = 0,1,2$. Note also that by the universal coefficient theorem and the already established fact that $\widetilde{H}^k(K(\mathbb{Z}/2,2);\mathbb{Z})$ is a finite abelian 2-group, we have isomorphisms $\widetilde{H}^k(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \widetilde{H}_{k-1}(K(\mathbb{Z}/2,2);\mathbb{Z})$ for $k \geq 1$. In particular, we have $H^3(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \mathbb{Z}/2$ and $H^4(K(\mathbb{Z}/2,2);\mathbb{Z}) = 0$ andthe differential $d_3\colon H^2(K(\mathbb{Z}/2,1);\mathbb{Z}) \to H^3(K(\mathbb{Z}/2,2);\mathbb{Z})$ is an isomorphism for formal reasons. Now, to compute $H^5(K(\mathbb{Z}/2,2;\mathbb{Z})$, we first see that the differential

$$d_3\colon \mathbb{Z}/2 \cong H^2(K(\mathbb{Z}/2,2);H^2(K(\mathbb{Z}/2,1);\mathbb{Z})) \to H^5(K(\mathbb{Z}/2,2);\mathbb{Z})$$

has to be injective, as the source cannot be hit by a differential as the only possible source of such a differential is $H^3(K(\mathbb{Z}/2,1);\mathbb{Z}) = 0$. Furthermore, the only other possible differential with target $H^5(K(\mathbb{Z}/2,2);\mathbb{Z})$ has source $H^4(K(\mathbb{Z}/2,1);\mathbb{Z}) \cong \mathbb{Z}/2$. The map from $\mathbb{Z}$-coefficients to $\mathbb{F}_2$-coefficients induces a map of spectral sequences for the same fibre sequence $K(\mathbb{Z}/2,1) \to * \to K(\mathbb{Z}/2,2)$. Note that the map $\widetilde{H}^*(K(\mathbb{Z}/2,1);\mathbb{Z}) \to \widetilde{H}^*(K(\mathbb{Z}/2,1);\mathbb{F}_2)$ induces an isomorphism whenever the source is non-trivial. We deduce that the map of spectral sequences induces an isomorphism on the rows corresponding to even q's. In particular, we deduce that the d_3 differential eminating from spot $(0,4)$ is determined by the $\mathbb{F}_2$-coefficient spectral sequence. In this, we have that $H^4(K(\mathbb{Z}/2,1);\mathbb{F}_2)$ is generated by $\iota_1^4 = \mathrm{Sq}^2\mathrm{Sq}^1(\iota_1)$ which is transgressive by the transgression theorem. In particular, we deduce that in the $\mathbb{Z}$-coefficient spectral sequence, the differential

$$d_5\colon H^4(K(\mathbb{Z}/2,1);\mathbb{Z}) \to H^5(K(\mathbb{Z}/2,2);\mathbb{Z})$$

is also injective. This results in the conclusion that $H^5(K(\mathbb{Z}/2,2);\mathbb{Z})$ is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. To decide which one it is, we note that for formal reasons, the differential

$$d_3\colon \mathbb{Z}/2 = H^3(K(\mathbb{Z}/2,2);H^2(K(\mathbb{Z}/2,1)) \to H^6(K(\mathbb{Z}/2,2);\mathbb{Z})$$

is an isomorphism. Exercise: deduce from the homological universal coefficient theorem (using that $C_*^{\mathrm{sing}}(K(\mathbb{Z}/2,2);\mathbb{Z})$ is homologically of finite type) applied the chain complex $C^*_{\mathrm{sing}}(K(\mathbb{Z}/2,2);\mathbb{Z})$ that

$$H^5(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong H^5(K(\mathbb{Z}/2,2);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^6(K(\mathbb{Z}/2,2);\mathbb{Z}),\mathbb{Z}/2).$$

Now, as a consequence of Theorem 4.8 the left hand side is 2-dimensional over $\mathbb{F}_2$. From $H^6(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \mathbb{Z}/2$, we then deduce that $H^5(K(\mathbb{Z}/2,2);\mathbb{Z}) = \mathbb{Z}/4$.

For the claim about the multiplication, note that the degree 3 generator is given by $\beta(\iota_2)$ where $\beta\colon H^2(K(\mathbb{Z}/2,2);\mathbb{F}_2) \to H^3(K(\mathbb{Z}/2,2);\mathbb{Z})$ is the Bockstein. Hence,

$$\mathrm{red}_2(\beta(\iota_2)^2) = \mathrm{Sq}^1(\iota_2)^2 \neq 0$$

by Theorem 4.8, so $\beta(\iota_2)^2$ is also non-zero as claimed. □

Exercise. Compute $H^7(K(\mathbb{Z}/2,2);\mathbb{Z})$. Spectral sequence shows: either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Now, the mod 2-reduction of this group injects into $H^7(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ with image contained in the kernel of Sq^1. One computes that this is a 1-dimensional $\mathbb{F}_2$-vector space generated by $\iota_2^2 \cdot \mathrm{Sq}^1(\iota_2)$ since $\mathrm{Sq}^1(\iota_2 \cdot \mathrm{Sq}^2\mathrm{Sq}^1(\iota_2)) \neq 0$. It follows that $H^7(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \mathbb{Z}/4$.

5.3. **Lemma** *We have*

$$\widetilde{H}^k(K(\mathbb{Z}/2,3);\mathbb{Z}) \cong \begin{cases} 0 & \text{for } k = 0,1,2,3,5 \\ \mathbb{Z}/2 & \text{for } k = 4,6,7 \end{cases}$$

Proof. Consider the Serre spectral sequence for the fibre sequence $K(\mathbb{Z}/2,2) \to * \to K(\mathbb{Z}/2,3)$. We claim that the differential

$$d_3\colon E_3^{0,5} = H^5(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \mathbb{Z}/4 \to H^3(K(\mathbb{Z}/2,3);H^3(K(\mathbb{Z}/2,2);\mathbb{Z})) \cong \mathbb{Z}/2$$

is non-trivial. To see this, consider the comparison map from the $\mathbb{Z}$-coefficients to $\mathbb{F}_2$-coefficients. It induces an isomorphism $H^3(K(\mathbb{Z}/2,2);\mathbb{Z}) \to H^3(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ and sends the generator of the source to $\mathrm{Sq}^1(\iota_2)$. Since the change-of-coefficients induces a map of spectral sequences, we obtain a commutative diagram

$$\begin{tikzcd} H^5(K(\mathbb{Z}/2,2);\mathbb{Z}) \arrow[r, "d_3"] \arrow[d] & H^3(K(\mathbb{Z}/3,3);H^3(K(\mathbb{Z}/2,2);\mathbb{Z})) \arrow[d] \\ H^5(K(\mathbb{Z}/2,2);\mathbb{F}_2) \arrow[r, "d_3"] & H^3(K(\mathbb{Z}/2,3);H^3(K(\mathbb{Z}/2,2);\mathbb{F}_2) \end{tikzcd}$$

of which we want to show that the top horizontal map is non-trivial. Since the right vertical map is an isomorphism, it suffices to show that the composite through the lower left corner is non-trivial. Now, first recall that a basis of $H^5(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ is given by $\mathrm{Sq}^2(\mathrm{Sq}^1(\iota_2))$ and $\iota_2 \cdot \mathrm{Sq}^1(\iota_2)$. Now, the left vertical map induces an injection

$$H^5(K(\mathbb{Z}/2,2);\mathbb{Z})/2 \to H^5(K(\mathbb{Z}/2,2);\mathbb{F}_2)$$

which image contained in the kernel of $\mathrm{Sq}^1 \cdot -$. Using the above basis, we find that this kernel is spanned by $\mathrm{Sq}^2\mathrm{Sq}^1(\iota_2) + \iota_2 \cdot \mathrm{Sq}^1(\iota_2)$. Now by the transgression theorem, we have $d_3(\mathrm{Sq}^2\mathrm{Sq}^1(\iota_2)) = 0 = d_3(\mathrm{Sq}^1(\iota_2))$, and by multiplicativity of the spectral sequence we have

$$d_3(\iota_2 \cdot \mathrm{Sq}^1(\iota_2)) = d_3(\iota_2) \cdot \mathrm{Sq}^1(\iota_2) = \iota_3 \cdot \mathrm{Sq}^1(\iota_2) \neq 0$$

as needed.

Hence we find $E_4^{5,0} = \ker(d_3) = \mathbb{Z}/2 \subseteq \mathbb{Z}/4$. We then find $d_4 = d_5 = 0$ on that $\mathbb{Z}/2$, since the respective targets of these differentials vanish. Since the spectral sequence converges to $\mathbb{F}_2$, we then find that the differential

$$d_6\colon \mathbb{Z}/2 \to H^6(K(\mathbb{Z}/2,3);\mathbb{Z})$$

is an isomorphism.

Next we show that $d_4\colon E_4^{6,0} = H^6(K(\mathbb{Z}/2,2);\mathbb{Z}) \to H^4(K(\mathbb{Z}/2,3);H^3(K(\mathbb{Z}/2,2);\mathbb{Z})) = E_4^{4,3}$ vanishes. Indeed, by Lemma 5.2, the source is generated by the square of $\beta(\iota_2)$, the non-trivial element in degree 3. By the Leibniz rule, we then obtain

$$d_4(\beta(\iota_2)^2) = 2\beta(\iota_2) \cdot \beta(\iota_3) = 0$$

since the group in which this element lives is $\mathbb{F}_2$. Again, we then find that the differential

$$d_7\colon E_7^{6,0} = H^6(K(\mathbb{Z}/2,2);\mathbb{Z}) \cong \mathbb{Z}/2 \to H^7(K(\mathbb{Z}/2,3);\mathbb{Z})$$

is an isomorphism. □

5.4. **Proposition** *We have* $\pi_5(S^3) \cong \mathbb{Z}/2$.

Proof. Consider the 4-truncation map $F_4 \to K(\mathbb{Z}/2, 4)$ and denote by F_5 its fibre. Then we find

$$\pi_5(S^3) \cong \pi_5(F_4) \cong \pi_5(F_5) \cong H_5(F_5;\mathbb{Z})$$

and therefore want to compute the (co)homology of F_5 using the Sere spectral sequence for the fibre sequence $K(\mathbb{Z}/2, 3) \to F_5 \to F_4$. From this, we deduce

$$\widetilde{H}^k(F_5;\mathbb{Z}) \cong \begin{cases} 0 & \text{for } k = 0,1,2,3,4,5 \\ \mathbb{Z}/2 & \text{for } k = 6 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \text{for } k = 7 \end{cases}$$

In particular, we deduce $H_5(F_5;\mathbb{Z}) \cong \mathbb{Z}/2$ as claimed. □

As promised, we now demonstrate a different way of doing these computations. The main observation is:

5.5. **Lemma** *For all* $n \geq 3$ *we have isomorphisms*

$$H_k(\tau_{\leq n-1}S^3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{for } k = 0,3 \\ 0 & \textit{for } k = 1,2 \textit{ and } 3 < k \leq n \\ \pi_n(S^3) & \textit{for } k = n+1 \end{cases}$$

Proof. Indeed, the map $S^3 \to \tau_{\leq n}S^3$ induces a isomorphism on $H_k(-;\mathbb{Z})$ for $k \leq n$ and a surjection for $k = n+1$ giving the computations for $k \leq n$. Then consider the fibre sequence

$$K(\pi_n(S^3), n) \to \tau_{\leq n}S^3 \to \tau_{\leq n-1}S^3$$

and the associated Serre spectral sequence. By what we have just argued, the differential

$$H_{n+1}(\tau_{\leq n-1}S^3;\mathbb{Z}) \to H_n(K(\pi_n(S^3), n);\mathbb{Z}) \cong \pi_n(S^3)$$

is an isomorphism, as its kernel contributes to $H_{n+1}(\tau_{\leq n}S^3;\mathbb{Z}) = 0$ and its cokernel contributes to $H_n(\tau_{\leq n}S^3;\mathbb{Z}) = 0$, giving the remaining claim. □

Then we consider the fibre sequence

$$K(\pi_4(S^3), 4) \to \tau_{\leq 4}S^3 \to \tau_{\leq 3}S^3 = K(\mathbb{Z}, 3).$$

Exercise. Show that

$$H^k(K(\mathbb{Z},3);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0,3 \\ \mathbb{Z}/2 & \text{for } k = 6,9,10 \\ \mathbb{Z}/3 & \text{for } k = 8 \\ 0 & \text{for } k = 1,2,4,5,7 \end{cases}$$

and that ι_3^2 and ι_3^3 are non-zero.

Hence $H_5(K(\mathbb{Z},3);\mathbb{Z}) \cong H^6(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}/2$ so that by Lemma 5.5 we find, yet again, $\pi_4(S^3) \cong \mathbb{Z}/2$. Then we want to use the Serre spectral sequence to compute $H^k(\tau_{\leq 4}S^3;\mathbb{Z})$ for $k \leq 9$. To begin, we first record:

5.6. **Lemma** *We have*

$$\widetilde{H}^k(K(\mathbb{Z}/2,4);\mathbb{Z}) \cong \begin{cases} 0 & \text{for } k = 0,1,2,3,4,6 \\ \mathbb{Z}/2 & \text{for } k = 5,7,8 \end{cases}$$

Proof. We consider the Serre spectral sequence for $K(\mathbb{Z}/2,3) \to * \to K(\mathbb{Z}/2,4)$. All claims with $k \leq 7$ are then formal. To also see the claim for $k = 8$, we need to show that the differential

$$d_4\colon H^7(K(\mathbb{Z}/2,3);\mathbb{Z}) \to H^4(K(\mathbb{Z}/2,4);H^4(K(\mathbb{Z}/2,3);\mathbb{Z}))$$

is trivial, as then the longest differential must be an isomorphism giving the case $k = 8$ as well. To do so, we again compare with the mod 2 spectral sequence, which results in a commutative square

$$\begin{tikzcd} H^7(K(\mathbb{Z}/2,3);\mathbb{Z}) \arrow[r] \arrow[d] & H^4(K(\mathbb{Z}/2,4);H^4(K(\mathbb{Z}/2,3);\mathbb{Z})) \arrow[d] \\ H^7(K(\mathbb{Z}/2,3);\mathbb{F}_2) \arrow[r] & H^4(K(\mathbb{Z}/2,4);H^4(K(\mathbb{Z}/2,3);\mathbb{F}_2)) \end{tikzcd}$$

in which the right vertical map is an isomorphism and the left vertical map is injective with image contained in the kernel of Sq^1. Similarly to before, $H^7(K(\mathbb{Z}/2,3);\mathbb{F}_2)$ is 2-dimensional with basis $\iota_3\mathrm{Sq}^1(\iota_3)$ and $\mathrm{Sq}^3\mathrm{Sq}^1(\iota_2)$ since the admissible sequence $(3,1)$ has excess 2. We deduce that the kernel of $\mathrm{Sq}^1\colon H^7 \to H^8$ is generated by $\mathrm{Sq}^3\mathrm{Sq}^1(\iota_3)$ which lies in the kernel of d_4 by the transgression theorem. ☐

We obtain again:

5.7. **Proposition** *We have* $\pi_5(S^3) \cong \mathbb{Z}/2$.

Proof. By Lemma 5.5, we have $\pi_5(S^3) \cong H_6(\tau_{\leq 4}S^3;\mathbb{Z})$ which by UCT is in turn isomorphic to $H^7(\tau_{\leq 4}S^3;\mathbb{Z})$. Then we investigate the cohomological Serre spectral sequence for

$$K(\mathbb{Z}/2,4) \to \tau_{\leq 4}S^3 \to K(\mathbb{Z},3).$$

As observed earlier, the differential

$$d_6\colon H^5(K(\mathbb{Z}/2,4);\mathbb{Z}) \to H^6(K(\mathbb{Z},3);\mathbb{Z})$$

is an isomorphism. The only non-trivial term with total degree 7 is then $H^7(K(\mathbb{Z}/2,4);\mathbb{Z})$ which has a possible differential to $H^3(K(\mathbb{Z},3);H^5(K(\mathbb{Z}/2,4);\mathbb{Z}))$. Considering the commutative square

$$\begin{tikzcd} H^7(K(\mathbb{Z}/2,4);\mathbb{Z}) \arrow[r] \arrow[d] & H^3(K(\mathbb{Z},3);H^5(K(\mathbb{Z}/2,4);\mathbb{Z})) \arrow[d] \\ H^7(K(\mathbb{Z}/2,4);\mathbb{F}_2) \arrow[r] & H^3(K(\mathbb{Z},3);H^5(K(\mathbb{Z}/2,4);\mathbb{F}_2)) \end{tikzcd}$$

in which the right vertical map is an isomorphism and the horizontal maps are the differentials in the $\mathbb{Z}$- and $\mathbb{F}_2$-coefficients Serre spectral sequence; by the same reasoning as in the proof of Lemma 5.6, we deduce that the top horizontal map vanishes. The only other possible differential is the

$$d_8\colon \mathbb{Z}/2 = H^7(K(\mathbb{Z}/2,4);\mathbb{Z}) \to H^8(K(\mathbb{Z},3);\mathbb{Z}) = \mathbb{Z}/3$$

which vanishes since there is no non-trivial 2-torsion in $\mathbb{Z}/3$. Therefore, we deduce from the spectral sequence that $H^7(\tau_{\leq 4}S^3;\mathbb{Z}) \cong H^7(K(\mathbb{Z}/2,4);\mathbb{Z}) \cong \mathbb{Z}/2$ as claimed.

Since we will use it momentarily, we also record that $H^8(\tau_{\leq 4}S^3;\mathbb{Z}) \cong \mathbb{Z}/6$; Indeed, what we have already shown implies that neither $H^8(K(\mathbb{Z}/2,4);\mathbb{Z}) \cong \mathbb{Z}/2$ nor $H^8(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}/3$ participate in some non-trivial differential and hence both contribute to $H^8(\tau_{\leq 4}S^3;\mathbb{Z})$. The spectral sequence also implies that

$$H^9(\tau_{\leq 4}S^4;\mathbb{Z}) \cong \ker(d_{10}\colon H^9(K(\mathbb{Z}/2,4);\mathbb{Z}) \to H^{10}(K(\mathbb{Z},3);\mathbb{Z}) = \mathbb{Z}/2).$$

□

5.8. **Proposition** *We have $\pi_6(S^3) \cong \mathbb{Z}/12$.*

Proof. We again use Lemma 5.5 and compute $H^8(\tau_{\leq 5}S^3;\mathbb{Z})$ using the Serre spectral sequence for $K(\mathbb{Z}/2,5) \to \tau_{\leq 5}S^3 \to \tau_{\leq 4}S^3$. For $k \leq 9$, we have that $\widetilde{H}^k(K(\mathbb{Z}/2,5);\mathbb{Z})$ vanishes unless $k = 6,8,9$ in which case the group is $\mathbb{Z}/2$ and we have computed $H^k(\tau_{\leq 4}S^3;\mathbb{Z})$ for $k \leq 8$ above and is given by $\mathbb{Z}$ in degree 3, $\mathbb{Z}/2$ in degree 7 and $\mathbb{Z}/6$ in degree 8. Consider the change-of-coefficient commutative diagram

$$\begin{tikzcd}
H^8(K_5;\mathbb{Z}) \arrow[r] \arrow[d] & H^3(\tau_{\leq 4}S^3; H^6(K(\mathbb{Z}/2,5);\mathbb{Z})) \arrow[d] \\
H^8(K_5;\mathbb{F}_2) \arrow[r] & H^3(\tau_{\leq 4}S^3; H^6(K(\mathbb{Z}/2,5);\mathbb{F}_2))
\end{tikzcd}$$

of which again the right vertical map is an isomorphism. The image of the left vertical map is again contained in the kernel of Sq^1; this kernel is then generated by $\mathrm{Sq}^3\mathrm{Sq}^1(\iota_5)$ – the degree 8 part is generated by $\mathrm{Sq}^2\mathrm{Sq}^1(\iota_5)$ and $\mathrm{Sq}^3(\iota_5)$ and the former is not in the kernel of Sq^1. But $\mathrm{Sq}^3(\iota_5)$ is transgressive by the transgression theorem since $H^5(\tau_{\leq 5}S^3;\mathbb{F}_2) = 0$ as a consequence of Lemma 5.5. Then we claim that also the differential

$$d_9\colon H^8(K_5;\mathbb{Z}) \to H^9(\tau_{\leq 4}S^3;\mathbb{Z})$$

vanishes (this still has to be done; for now, we record it as an exercise). If this is so, we find that $H^8(\tau_{\leq 5}S^3;\mathbb{Z})$ is an extension of $\mathbb{Z}/2$ by $\mathbb{Z}/6$. To decide which one it is we claim that $H^7(\tau_{\leq 5}S^3;\mathbb{F}_2) = \mathbb{F}_2$. Since this group surjects onto the 2-torsion of $H^8(\tau_{\leq 5}S^3;\mathbb{Z})$ by the Bockstein long exact sequence, we find that the latter has to be $\mathbb{Z}/12$.

Now, to show that $H^7(\tau_{\leq 5}S^3;\mathbb{F}_2) = \mathbb{F}_2$, we consider the cohomological Serre spectral sequence for $K(\mathbb{Z}/2,5) \to \tau_{\leq 5}S^3 \to \tau_{\leq 4}S^3$. From our earlier computations on $H^*(\tau_{\leq 4}S^3;\mathbb{Z})$, we find that

$$H^*(\tau_{\leq 4}S^3;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } k = 0,3,6 \\ \mathbb{F}_2^{\oplus 2} & \text{for } k = 7 \\ 0 & \text{for } k = 1,2,4,5 \end{cases}$$

and that $\mathrm{Sq}^1\colon H^6 \to H^7$ is non-trivial. In the Serre spectral sequence we then have $d_6(\iota_5)$ equal to the non-trivial class in degree 6, and hence by the transgression theorem that $d_7(\mathrm{Sq}^1(\iota_5))$ is also non-trivial. So to see that $H^7(\tau_{\leq 5}S^3;\mathbb{F}_2) = \mathbb{F}_2$ it remains to show that the differential $d_8\colon H^7(K(\mathbb{Z}/2,5);\mathbb{F}_2) \to H^8(\tau_{\leq 4}S^3;\mathbb{F}_2)$ is non-trivial. Since the source is generated by $\mathrm{Sq}^2(\iota_5)$, it suffices to show that $\mathrm{Sq}^2\colon H^6(\tau_{\leq 4}S^3;\mathbb{F}_2) \to H^8(\tau_{\leq 4}S^3;\mathbb{F}_2)$ is non-trivial. To see this, we claim that the map $K(\mathbb{Z}/2,4) \to \tau_{\leq 4}S^3$ induces an isomorphism on $H^6(-;\mathbb{F}_2)$; to see this, consider the Serre spectral sequence in $\mathbb{F}_2$-cohomology for $K(\mathbb{Z}/2,4) \to \tau_{\leq 4}S^3 \to K(\mathbb{Z},3)$. The terms contributing to $\mathbb{F}_2 \cong H^6(\tau_{\leq 4}S^3;\mathbb{F}_2)$ a priori are $H^6(K(\mathbb{Z}/2,4);\mathbb{F}_2)$ and

$H^6(K(\mathbb{Z},3);\mathbb{F}_2)$. Similarly to before, we have that $\mathrm{Sq}^1\colon H^5(K(\mathbb{Z},3);\mathbb{F}_2) \to H^6(K(\mathbb{Z},3);\mathbb{F}_2)$ is an isomorphism, and we have $d_5(\iota_4)$ is the non-trivial element in $H^5(K(\mathbb{Z},3);\mathbb{F}_2)$. This shows that $d_6(\mathrm{Sq}^1(\iota_4)) \neq 0$, and as a consequence, we have $H^6(\tau_{\leq 4}S^3;\mathbb{F}_2) \to H^6(K(\mathbb{Z}/2,4);\mathbb{F}_2)$ is an isomorphism. Finally, note that $H^6(K(\mathbb{Z}/2,4);\mathbb{F}_2)$ is generated by $\mathrm{Sq}^2(\iota_4)$. The Adem relation give $\mathrm{Sq}^2(\mathrm{Sq}^2(\iota_4)) = \mathrm{Sq}^3\mathrm{Sq}^1(\iota_4) \neq 0$, so we deduce that $\mathrm{Sq}^2\colon H^6(\tau_{\leq 4}S^3;\mathbb{F}_2) \to H^8(\tau_{\leq 4}S^3;\mathbb{F}_2)$ is non-trivial as needed. □

As another exercise, we have:

Exercise. For A, B abelian groups, we have that the map

$$\pi_{-i}\mathrm{map}(A,B) = H^i(A;B) \to H^{i+n}(K(A,n);B)$$

induced by $\Sigma^\infty K(A,n) \to \Sigma^n A$, is an isomorphism for $i < n$ and injective for $i = n$.

6. BORDISM

In this section, we aim to compute the *rationalized* oriented bordism groups. Recall that these groups are defined as follows:

6.1. **Definition** An (oriented) bordism between two (oriented) closed n-manifolds M_1 and M_2 is an oriented compact manifold $(n+1)$-W together with an isomorphism[13] $\partial W \cong M_1 \amalg -M_2$. Here $-M_2$ denotes the manifold M_2 with its reversed orientation.

Exercise: The relation $M_1 \sim M_2$ if and only if there exists an (oriented) bordism induces an equivalence relation on the set of isomorphism classes of (oriented) isomorphism classes of (oriented) closed n-manifolds. We say M_1 and M_2 are (oriented) *bordant* if there exists an oriented bordism between them.

We then define Ω_n^{SO} as the set of bordism classes of oriented closed n-manifolds; classically this is often denoted by Ω_*. Similarly, we define Ω_n^{O} as the set of bordism classes of n-manifolds; classically this often denoted by $\mathcal{N}_*$.

6.2. **Remark** It is perhaps more illuminating to define the following generalised version of the above which takes as input a map $\theta\colon B \to \mathrm{BO}$ of anima – the above cases are the ones where θ is either the map $\mathrm{BSO} \to \mathrm{BO}$ (the oriented case) or the identity of BO. A θ-manifold is a manifold M together with a map $f\colon M \to B$ and a homotopy $h\colon \theta f \simeq \nu_M$ between the pullback of θ along f and the stable normal bundle ν_M of M. Recall that the stable normal bundle is simply the inverse of the stable tangent bundle $T^s(M)$; both viewed as maps to BO.

Observe that since BO is a group, a self-homotopy of ν_M is equivalently given by a self-homotopy of the constant map $M \to * \to \mathrm{BO}$; this in turn is given by a map $M \to \Omega\mathrm{BO} \simeq \mathrm{O}$. In particular, given a θ-structure (f,h) on M and a map $\alpha\colon M \to \mathrm{O}$, we obtain a new θ-structure $(f, \alpha \star h)$, where $\alpha \star h$ is the homotopy $\theta f \overset{h}{\sim} \nu_M \overset{\alpha}{\sim} \nu_M$. Recall that $\pi_0(\mathrm{O}) = \{\pm 1\}$; we denote by m the constant map at $-1 \in \mathrm{O}$. For a θ-structure (f,h) on M, we define the opposite θ-structure by $(f, m \star h)$; we write $(M,f,h)^{\mathrm{op}}$ for $(M,f,m\star h)$.

Given a manifold with boundary $(W, \partial W)$, there is an isomorphism $TW_{|\partial W} \cong T(\partial W) \oplus \mathbb{R}$ obtained by trivialising the normal bundle of the embedding $\partial W \subseteq W$ via the inward point normal frame (the fact that the normal bundle is trivialisable is essentially equivalent to the existence of collars of the boundary, that is, of an embedding $\partial W \times [0,1] \to W$). It follows that if (W,F,H) is a θ-manifold, then $(\partial W, F_{|\partial W}, H_{|\partial W})$ is also a θ-manifold.

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[13] That is, orientation preserving homeomorphism for topological manifolds and orientation preserving diffeomorphism for smooth manifolds.

A θ-bordism between two θ-manifolds (M_0, f_0, h_0) and (M_1, f_1, h_1) is a compact θ-manifold (W, F) together with an isomorphism $(\partial W, \partial F, \partial H) \cong (M_0, f_0, h_0) \amalg (M_1, f_1, h_1)^{\mathrm{op}}$.

Exercise: The notion of θ-cobordism gives an equivalence relation on θ-manifolds, so for each $d \geq 0$, we obtain a set Ω_d^θ of bordism classes of d-dimensional θ-manifolds. This set is canonically an abelian group under disjoint union of θ-manifolds. Consequently, Ω_*^θ is a graded abelian group. There are exterior products

$$\Omega_d^\theta \times \Omega_{d'}^{\theta'} \to \Omega_{d+d'}^{\theta\times\theta'}$$

which make Ω_*^θ into a graded (commutative) ring if $\theta\colon B \to \mathrm{BO}$ is a map of ($\mathbb{E}_2$-)groups in anima. Moreover, given a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\theta"] \arrow[d, "f"'] & \mathrm{BO} \\ B' \arrow[ur, "\theta'"'] & \end{tikzcd}$$

one obtains a canonical map $\Omega_*^\theta \to \Omega_*^{\theta'}$.

6.3. **Remark** The same arguments work for $\xi\colon B \to \mathrm{BTop}$ a stable euclidean bundle and topological manifolds throughout.

6.4. **Example** (The θ-structure BO) When θ is the identity of BO, one finds that Ω_*^θ is canonically isomorphic to $\Omega_*^{\mathrm{O}} = \mathcal{N}_*$ as described earlier. In particular, $\pi_*(\mathrm{MO})$ is canonically isomorphic to the unoriented bordism ring Ω_*^{O}.

6.5. **Example** (The θ-structure BSO) We work through the example where θ is the map $\mathrm{BSO} \to \mathrm{BO}$. A θ-structure can then equivalently be described by equipping the stable normal bundle ν_M with an orientation, hence a θ-structure is often referred to as an orientation. Exercise: Given a θ-structure (M, f, h) correspoding to an orientation o on ν_M, show that $(M, f, h)^{\mathrm{op}}$ corresponds to the reversed orientation $\bar{o}$ on ν_M. Hint: changing the orientation gives rise to a self-map $\mathrm{rev}\colon \mathrm{BSO} \to \mathrm{BSO}$. This map is homotopic to the identity such that the following two triangles are equivalent

$$\begin{tikzcd} \mathrm{BSO} \arrow[r, "\mathrm{rev}"] \arrow[d, "\theta"'] & \mathrm{BSO} \arrow[dl, "\theta"] \\ \mathrm{BO} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{BSO} \arrow[r, equal] \arrow[d, "\theta"'] & \mathrm{BSO} \arrow[dl, "\theta"] \\ \mathrm{BO} & \end{tikzcd}$$

where the left triangle is filled with the trivial homotopy h_{triv} and the right square is filled with the homotopy $m \star h_{\mathrm{triv}}$. Deduce then the exercise. In particular, $\pi_*(\mathrm{MSO}) \cong \Omega_*^\theta$ is canonically isomorphic to Ω_*^{SO}, the oriented bordism ring.

6.6. **Theorem** (Pontryagin–Thom construction) *Extracting from a θ-manifold its Pontryagin–Thom collapse map induces an isomorphism of graded abelian groups*

$$\Omega_*^\theta \to \pi_*(M\theta).$$

Proof sketch. Given a d-dimensional θ-manifold (M, f, h), the pair (f, h) induces a map $M\nu_M \to M\theta$; recall here that $M\colon \mathrm{An}_{/\mathrm{BO}} \to \mathrm{Sp}$ is the Thom spectrum functor. Moreover, as indicated in [Lan25, Remark 5.29], the Pontryagin–Thom collapse map is a map $\mathbb{S}^d \to M\nu_M$

which witnesses that the underlying spherical fibration of ν_M to be the dualizing spectrum of the Poincaré duality complex underlying M, see [Lan25, Theorem 5.28]. Composing it with the afore mentioned map $M\nu_M \to M\theta$, we obtain a map $\mathbb{S}^d \to M\nu_M \to M\theta$. The construction sending a θ-manifold (M, f, h) to the map $\mathbb{S}^d \to M\theta$ can be shown to induce the isomorphism of the theorem. Perhaps we will add some details here at some later point. □

Exercise. Show that $\pi_0(\mathrm{MSO}) \cong \mathbb{Z}$ and that $\pi_1(\mathrm{MSO}) = \pi_2(\mathrm{MSO}) = 0$. Similarly, show that $\pi_0(\mathrm{MO}) = \mathbb{Z}/2$, $\pi_1(\mathrm{MO}) = 0$ and $\pi_2(\mathrm{MO}) \cong \mathbb{Z}/2$. If you are eager, think about $\pi_3(\mathrm{MSO})$ and $\pi_3(\mathrm{MO})$.

6.7. **Remark** The Pontryagin–Thom construction is compatible with exterior products, that is, the diagram

$$\begin{tikzcd}
\Omega^{\theta}_{d} \times \Omega^{\theta'}_{d'} \arrow[r] \arrow[d] & \pi_d(M\theta) \times \pi_d(M\theta') \arrow[d] \\
\Omega^{\theta\times\theta'}_{d+d'} \arrow[r] & \pi_{d+d'}(M\theta \otimes M\theta')
\end{tikzcd}$$

commutes. In particular, the isomorphism of Theorem 6.6 is one of (graded commutative) rings if θ is a map of ($\mathbb{E}_2$-)groups.

6.8. **Definition** Let R be a commutative ring. Given an element $x \in H^d(B; R^\theta)$, there is associated an R-valued *characteristic number* for d-dimensional θ-manifolds (M, f) given by $\langle f^*(x), [M]\rangle \in R$. Here R^θ refers to the local system of R-modules on B induced by $w_1(\theta)$. Note that R^θ is an invertible object of $\mathrm{Fun}(B, \mathrm{Mod}(R))$ with inverse equivalent to R^θ (essentially since $(-1)^2 = 1$).

We say that θ is oriented if it is equipped with a lift $B \to \mathrm{BSO} \to \mathrm{BO}$. In that case, $w_1(\theta) = 0$ and no local coefficients appear in the above construction. Similarly, we say that θ is R-oriented if the local system of R-modules determined by $w_1(\theta)$ vanishes.[14]

6.9. **Lemma** *The formation of R-valued characteristic numbers is invariant under bordisms and additive. In other words, it leads to an R-module map*

$$\chi_R \colon \Omega^{\theta}_{*} \otimes_{\mathbb{Z}} R \to \mathrm{Hom}_R(H^*(B; R^\theta), R).$$

Proof. The formation of characteristic numbers is readily checked to send disjoint unions to sums. To get a map $\chi\colon \Omega^\theta_* \to \mathrm{Hom}_R(H^*(B; R^\theta), R)$ we then need to show that if (W, F) is a θ-manifold with d-dimensional boundary $(\partial W, \partial F) = (M, f)$, then $\chi(M, f) = 0$. To see this, recall that W has a relative fundamental class $[W] \in H_{d+1}(W, M; R^\theta)$ whose image under the boundary map $H_{d+1}(W, M; R^\theta) \to H_d(M; R^\theta)$ is the fundamental class of M. In particular, denoting by $i\colon M \to W$ the inclusion, we find that $i_*[M] = 0$. As a consequence, for $x \in H^d(B; R^\theta)$, we find

$$\langle f^*(x), [M]\rangle = \langle i^*F^*(x), [M]\rangle = \langle F^*(x), i_*[M]\rangle = 0.$$

Finally, the desired maps exists simply because the target is naturally an R-module. □

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[14] For instance, this is automatically the case if R is an $\mathbb{F}_2$-algebra.

6.10. **Remark** In fact, the above group homomorphism fits into the following commutative diagram:

$$\begin{tikzcd}
\Omega_*^\theta \otimes_{\mathbb{Z}} R \arrow[rr, "\chi_R"] \arrow[d, "\cong"'] & & \operatorname{Hom}_R(H^*(B;R^\theta),R) \\
\pi_*(M\theta)\otimes_{\mathbb{Z}} R \arrow[r, "h"] & H_*(M\theta;R) \arrow[r, "\cong"] & H_*(B;R^\theta) \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

where the left vertical map is induced by the Pontryagin–Thom isomorphism, the bottom left map is the Hurewicz homomorphism, the bottom right map is the Thom isomorphism, and the right vertical map is the Kronecker evaluation map (that this is well-defined as written uses that R^θ is its own inverse). Exercise: Prove this.

6.11. **Corollary** *If B is rationally of finite type, the formation of rational characteristic numbers gives an isomorphism*

$$\chi_{\mathbb{Q}} \colon \Omega_*^\theta \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \operatorname{Hom}(H^*(B;\mathbb{Q}^\theta),\mathbb{Q}).$$

This map is multiplicative for the ring structure on the target coming from the coalgebra structure on $H^*(B;\mathbb{Q}^\theta)$ induced by the group structure on B.

Proof. This follows from Remark 6.10. Indeed, the Hurewicz homomorphism is an isomorphism since $\mathbb{S}_{\mathbb{Q}} \simeq \mathbb{Q}$ by Serre's finiteness theorem for the stable homotopy groups of spheres, Proposition 3.25, and the Kronecker evaluation homomorphism is an isomorphism as follows from the finite type hypotheses. Now, by construction the composite $\Omega_*^\theta \otimes \mathbb{Q} \to H_*(B;\mathbb{Q}^\theta)$ is a ring homomorphism for the ring structure coming from the assumption that B is a group. By the finite type hypothesis, this ring is, under the Kronecker evaluation map, isomorphic to the ring obtained as the dual of the coalgebra $H^*(B;\mathbb{Q}^\theta)$. □

6.12. **Remark** The proof above reveals that $\Omega_*^\theta \otimes \mathbb{Q}$ is isomorphic to $H_*(B;\mathbb{Q}^\theta)$ as rings also without the finite type hypothesis on R. In this case, the characteristic number map χ_R is still injective as it and identifies with the canonical map from $H_*(B;\mathbb{Q}^\theta)$ to its double dual (which is injective for any $\mathbb{Q}$-vector space).

6.13. **Example** We consider the case where θ is the map $\mathrm{BSO} \to \mathrm{BO}$. Recall that $H^*(\mathrm{BSO};\mathbb{Q}) \cong \mathbb{Q}[p_i \mid i \geq 1]$. Then we obtain that the map

$$\Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Hom}(\mathbb{Q}[p_i \mid i \geq 1],\mathbb{Q}), \quad [M] \mapsto [p_I \mapsto \langle p_I(\nu_M),[M]\rangle]$$

is an isomorphism. Here, $I = (i_1,\dots,i_k)$ is a multi-index and $p_I = p_{i_1}\cdots p_{i_k}$ the associated monomial in the Pontryagin classes p_{i_j}.

Note that the inversion map $(-)^{-1} \colon \mathrm{BSO} \to \mathrm{BSO}$ is an equivalence and hence induces an isomorphism on cohomology, that is, an automorphism inv of $\mathbb{Q}[p_i \mid i \geq 1]$. Hence, the composite induced by this isomorphism

$$\Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Hom}(\mathbb{Q}[p_i \mid i \geq 1],\mathbb{Q}) \xrightarrow{\mathrm{inv}^*} \operatorname{Hom}(\mathbb{Q}[p_i \mid i \geq 1],\mathbb{Q})$$

is also an isomorphism. Exercise: This composite is given by sending the bordism class of an oriented manifold M to the function sending a monomial p_I to $\langle p_I(TM),[M]\rangle$.

Consequently, extracting from an oriented manifold its normal or tangential Pontryagin numbers results in isomorphisms $\chi_{\mathbb{Q}}, \chi_{\mathbb{Q}}^t \colon \Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \operatorname{Hom}(\mathbb{Q}[p_i \mid i \geq 1],\mathbb{Q})$.

6.14. **Proposition** *Ω_*^{SO} is degreewise finitely generated. In particular, Ω_k^{SO} is finite unless $k \equiv 0(4)$.*

Proof. We find that $H_*(\mathrm{MSO};\mathbb{Z}) \cong H_*(\mathrm{BSO};\mathbb{Z})$ is finitely generated in each degree. Similarly to the argument in Serre class theory, one then shows that a bounded below spectrum such as MSO has degreewise finitely generated homotopy if and only if it has degreewise finitely generated homology. □

Recall that for $4n$-dimensional closed oriented manifolds M, the cup product determines a unimodular symmetric form on $H^{2n}(M;\mathbb{Z})/\mathrm{tors}$, and in particular on $H^{2n}(M;\mathbb{R})$. Such a symmetric unimodular form is represented by a symmetric matrix over $\mathbb{R}$, which is therefore diagonalisable. The number of positive minus the number of negative eigenvalues is called the signature of such a form. In this way, we obtain the signature $\mathrm{sign}(M)$ of M as the signature of the symmetric unimodular form determined by the cup product on $H^{2n}(M;\mathbb{R})$.

Exercise. Show that the signature is an oriented bordism invariant, i.e. that if W is an oriented compact $(4n+1)$-manifold, then $\mathrm{sign}(\partial W) = 0$. Hint: Show that the signature of a unimodular form (V, b) over $\mathbb{R}$ vanishes if there is a half-dimensional subspace $L \subseteq V$ on which the form vanishes, that is, where $b(v, w) = 0$ for all $v, w \in L$. Then find such a subspace in $H^{2n}(\partial W;\mathbb{R})$.

6.15. **Corollary** *The map $H^*(\mathrm{BSO};\mathbb{Q}) \to \mathrm{Hom}(\Omega_*^{\mathrm{SO}},\mathbb{Q})$ again extracting tangential (or normal) characteristic numbers, is an isomorphism. In particular, for all $n \geq 1$, there exists unique polynomials $L_n(p_1,\ldots,p_n)$ such that for all oriented closed $4n$-dimensional manifolds M, we have the signature formula*

$$\mathrm{sign}(M) = \langle L_n(p_1,\ldots,p_n)(TM), [M]\rangle.$$

Proof. The first part is immediate from what we have seen earlier using again that BSO is of finite type. The in particular follows since sending an oriented manifold to its signature induces a homomorphism $\Omega_{4*}^{\mathrm{SO}} \to \mathbb{Z}$ by the exercise above. □

Hirzebruch found a concrete way to determine the polynomials $L_n(p_1,\ldots,p_n)$. This is a beautiful story, that we unfortunately will not have the time to go into. We will content ourselves with determining L_1 and L_2 in this course. To that end, we will make use of the following.

6.16. **Proposition** *The tautological map $\mathbb{Q}[\mathbb{CP}^n \mid n \geq 1] \to \Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ of graded commutative $\mathbb{Q}$-algebras is an isomorphism. As a consequence, the map $\mathbb{Z}[\mathbb{CP}^{2n} \mid n \geq 1] \to \Omega_*^{\mathrm{SO}}$ is injective.*

Proof. By the previous result, it suffices to show that the composite

$$\mathbb{Q}[\mathbb{CP}^{2n} \mid n \geq 1] \to \Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \xrightarrow{\chi_{\mathbb{Q}}^t} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[p_i \mid i \geq 0],\mathbb{Q})$$

where the second map is the formation of tangential characteristic numbers, is an isomorphism of $\mathbb{Q}$-vector spaces. Concretely, this amounts to proving the following: Fix $k \geq 1$ and consider a partition of the number $4k$, that is, a set $\{i_1,\ldots,i_\ell\}$ such that $\sum_{j=1}^{\ell} i_j = 4k$. We may think of such a set as an unordered multiindex I. Denote then by $p(k)$ the set of partitions of $4k$. Then one can form the $p(k) \times p(k)$ matrix whose entry at a pair of partitions (I, I') is given by $\chi(\mathbb{CP}^{2I'})(p_I)$. Here, $p_I = p_1^{i_1} \cdots p_\ell^{i_\ell}$ and $\mathbb{CP}^{2I'} = \mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_{\ell'}}$. It is an algebraic lemma that this matrix has non-trivial determinant, see e.g. [**?**, 16.8]. It follows that the map under investigation is injective, so since both domain and codomain have degreewise the same dimension, so the map is in fact an isomorphism. For example, for $p(1) = 1$, and we obtain the 1×1 matrix $p_1(\mathbb{CP}^2)$. Similarly, $p(2) = 2$, with partitions $2 = 2$ and $2 = 1 + 1$; We then obtain the 2×2 matrix

$$\begin{pmatrix} p_1^2(\mathbb{CP}^2 \times \mathbb{CP}^2) & p_1^2(\mathbb{CP}^4) \\ p_2(\mathbb{CP}^2 \times \mathbb{CP}_2) & p_2(\mathbb{CP}^4) \end{pmatrix}$$

To compute these matrices explicitly, we recall or note that $T\mathbb{CP}^n$ is stably isomorphic to $\gamma_{\mathbb{C}}^{\oplus n+1}$ where $\gamma_{\mathbb{C}}$ is the universal line bundle on $\mathbb{CP}^n$. It follows then that the total Pontryagin class of $\mathbb{CP}^n$ satisfies

$$p(\mathbb{CP}^n) = (1 + x^2)^{n+1} = \sum_{i \geq 0} \binom{n+1}{i} x^{2i}$$

In other words, $p_i(\mathbb{CP}^n) = \binom{n+1}{i} x^{2i}$, where, as always $x \in H^2(\mathbb{CP}^n; \mathbb{Z})$ is a generator. Using then that $p(\mathbb{CP}^2 \times \mathbb{CP}^2) = p(\mathbb{CP}^2) \cdot p(\mathbb{CP}^2)$, we obtain that the above matrices are given by the 1×1 matrix (3) and the 2×2 matrix

$$\begin{pmatrix} 18 & 9 \\ 25 & 10 \end{pmatrix}$$

whose determinant is $-45 \neq 0$. □

6.17. **Example** Without proof, we note here some low dimension oriented bordism groups.

1. $\Omega_n^{\mathrm{SO}} \cong \mathbb{Z}$, generated by the oriented manifold $\{*\}$.
2. $\Omega_n^{\mathrm{SO}} = 0$ for $n = 1, 2, 3$.
3. $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$, generated by $\mathbb{CP}^2$ and detected by the signature.
4. $\Omega_5^{\mathrm{SO}} \cong \mathbb{Z}/2$, generated by the Wu manifold $W = \mathrm{SU}(3)/\mathrm{SO}(3)$ or the mapping torus of complex conjugation on $\mathbb{CP}^2$ (they are not diffeomorphic, but oriented bordant), and detected by the $\mathbb{F}_2$-characteristic number associated to $w_2 w_3$.
5. $\Omega_n^{\mathrm{SO}} = 0$ for $n = 6, 7$
6. $\Omega_8^{\mathrm{SO}} \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$; under this isomorphism, the signature map is the fold map.
7. $\Omega_9^{\mathrm{SO}} \cong \mathbb{Z}/2$, generated by some hypersurface in $\mathbb{RP}^2 \times \mathbb{RP}^8$ and detected by ..?
8. $\Omega_{10}^{\mathrm{SO}} \cong \mathbb{Z}/2$, generated by $W \times W$, detected by $w_2^2 w_3^2$.
9. $\Omega_{11}^{\mathrm{SO}} \cong \mathbb{Z}/2$, generated by some hypersurface in $\mathbb{RP}^4 \times \mathbb{RP}^8$ and detected by ..?

Taking suitable products of elements as described above, one then finds that $\Omega_n^{\mathrm{SO}} \geq 0$ for all $n \geq 12$.

Exercise. Suppose that M is an oriented closed manifold which admits an orientation reversing diffeomorphism $f \colon M \to M$. Show that M represents a torsion element in Ω_*^{SO}. Hint: You may use (or show) that $f^*(TM) \cong TM$.

Exercise. Suppose that M is an oriented closed manifold and that $p \colon \hat{M} \to M$ is an n-fold covering map. Show that $[\hat{M}] = n[M]$ in $\Omega_{\dim(M)}^{\mathrm{SO}} \otimes \mathbb{Q}$. Hint: You may use (or show) that $p^*(TM) \cong T\hat{M}$.

Exercise. Show that the two manifolds described in part (4) above are indeed non-trivial elements by showing that they have non-trivial characteristic number associated to $w_2 w_3$.

Exercise. Work out a formula for a characteristic number c such that $(\mathrm{sign}, c)\colon \Omega_8^{\mathrm{SO}} \to \mathbb{Z}\oplus\mathbb{Z}$ is an isomorphism.

6.18. **Remark** Recall that we have noted (but not proven) in [Lan25, Remark 3.19] that the map BO → BTop is a rational equivalence; the same applies for the map BSO → BSTop. It then follows from the above that the map MSO → MSTop is also a rational equivalence. In particular, $\mathbb{Q}[\mathbb{CP}^{2n} \mid n \geq 1]$ is also isomorphic to $\pi_*(\mathrm{MSTop})\otimes\mathbb{Q} \cong \Omega_*^{\mathrm{STop}}\otimes\mathbb{Q}$, the rationalised topological oriented bordism ring.

Note, however, that since the Pontryagin classes are elements in the integral cohomology of BSO, we obtain a canonical map

$$\Omega_*^{\mathrm{SO}} \to \mathrm{Hom}(H^*(\mathrm{BSO};\mathbb{Z}),\mathbb{Z})$$

so that $\chi(M)(p_I) = \langle p_I(TM), [M]\rangle$ is an integer, if I is a multiindex of total degree equal to the dimension of M. For topological manifolds, this is not necessarily the case. In particular, the (non-) integrality of Pontryagin numbers is something that distinguishes smooth from topological manifolds.

6.19. **Corollary** *We have* $L_1(p_1) = \frac{1}{3}p_1$ *and* $L_2(p_1,p_2) = \frac{1}{45}(7p_2 - p_1^2)$.

Proof. We have to show that for all oriented 4-manifolds M and all oriented 8-manifolds N, we have

$$\frac{1}{3}p_1(M) = \mathrm{sign}(M)$$

and

$$\frac{1}{45}(7p_2(N) - p_1^2(N)) = \mathrm{sign}(N).$$

By Proposition 6.16, it suffices to show this for generators of the rational bordism ring in these degrees, i.e. for $\mathbb{CP}^2$, $\mathbb{CP}^2\times\mathbb{CP}^2$ and $\mathbb{CP}^4$. To that end, as noted earlier, we have find $p_1(\mathbb{CP}^2) = 3x^2$, giving the formula for the L-polynomial in degree 4. Moreover, we have $p_1(\mathbb{CP}^4) = 5x^2$ and $p_2(\mathbb{CP}^4) = 10x^4$. With that we compute

$$\frac{1}{45}(7p_2(\mathbb{CP}^4) - p_1^2(\mathbb{CP}^4)) = \frac{1}{45}(70x^4 - 25x^4)$$

which evaluates to 1 against $[\mathbb{CP}^4]$. To do the relevant computation for $\mathbb{CP}^2\times\mathbb{CP}^2$ we recall that

$$p(\mathbb{CP}^2\times\mathbb{CP}^2) = p(\mathbb{CP}^2)\times p(\mathbb{CP}^2) = (1+3x^2)\cdot(1+3y^2) = 1+3x^2+3y^2+9x^2y^2$$

Hence, $p_1(\mathbb{CP}^2\times\mathbb{CP}^2) = 3x^2+3y^2$ and $p_2(\mathbb{CP}^2\times\mathbb{CP}^2) = 9x^2y^2$. Consequently,

$$\frac{1}{45}(7p_2(\mathbb{CP}^2\times\mathbb{CP}^2) - p_1^2(\mathbb{CP}^2\times\mathbb{CP}^2)) = \frac{1}{45}(63x^2y^2 - 18x^2y^2) = x^2y^2$$

which again evaluates to $1 = \mathrm{sign}(\mathbb{CP}^2\times\mathbb{CP}^2)$ □

Exercise. Show that $[\mathbb{HP}^2] = 3[\mathbb{CP}^2\times\mathbb{CP}^2] - 2[\mathbb{CP}^4] \in \pi_8(\mathrm{MSO})\otimes\mathbb{Q}$.

We will use the signature formula to show that some manifold we construct is not diffeomorphic, but homotopy equivalent to a S^7. To perform the relevant computations showing that the manifold in question is homotopy equivalent to S^7, or a *homotopy sphere* for short, we will use some geometric ways to compute cup products in manifolds.

6.20. **Construction** Let M be a connected smooth manifold and $i\colon N \to M$ a smooth embedding of a connected manifold N. In [Lan25, ...], we have indicated that this embedding comes with an essentially unique with normal bundle $\nu_{N,M}$, in addition it comes with an essentially unique tubular neighbourhood, i.e. an embedding $D(\nu_{N,M}) \subseteq M$ where $D(\nu_{N,M})$ is the disk-bundle of $\nu_{N,M}$. There is therefore associated a tautological collapse map $c_{N,M}\colon M \to \mathrm{Th}(\nu_{N,M})$ depending only on the embedding i. If M and N are oriented, then there is an induced orientation on $\nu_{N,M}$ so that there is a Thom class $u \in H^{m-n}(\mathrm{Th}(\nu_{N,M}))$ with integral coefficients in the oriented case or $\mathbb{F}_2$-coefficients in the non-orientable case; here $m = \dim(M)$ and $n = \dim(N)$. In particular, in the above situation, we obtain an associated element $c^*_{M,N}(u) \in H^{m-n}(M)$.

We then aim to prove the following theorems which explain why the cohomological pairing on a manifold induced by Poincaré duality and the cup product is called the *intersection pairing*. To that end, let us denote the Poincaré duality isomorphism $- \cap [M]\colon H^*(M) \to H_{m-*}(M)$ by $\mathrm{PD}(-)$.

6.21. **Theorem** *Assume in addition to the above that M and N are closed oriented manifolds. Then we have*

$$\mathrm{PD}(i_*[N]) = c^*_{M,N}(u)$$

6.22. **Theorem** *Suppose that $i\colon N \to M$ and $i'\colon N' \to M$ are the embeddings of smooth submanifolds. Assume that N and N' are transversal; in particular that $K = N \cap N'$ is also a smoothly embedded submanifold of N, N' and M, respectively Dnote the embedding of K to M by k. Then*

$$\mathrm{PD}(i_*[N]) \cup \mathrm{PD}(i'_*[N']) = \mathrm{PD}(k_*[K]).$$

Both theorems are, from the point of view presented here, about the geometry of various collapse maps. We begin with the following observations:

6.23. **Remark** First, let us explain how the cap product in singular (co)homology is given when using the description of singular (co)homology as the (co)homology theory associated to the spectrum $\mathbb{Z}$. To that end, let $x \in H^k(X;\mathbb{Z})$ and $y \in H_l(X;\mathbb{Z})$. Then the cap product $x \cap y$ is an element of $H_{l-k}(X;\mathbb{Z})$. If x is represented by a map $x\colon X \to \Sigma^k\mathbb{Z}$ and y by a map $\mathbb{S}^l \to X \otimes \mathbb{Z}$, the $x \cap y$ is represented by the composite

$$\mathbb{S}^l \xrightarrow{y} X \otimes \mathbb{Z} \xrightarrow{\Delta\otimes\mathrm{id}} X \otimes X \otimes \mathbb{Z} \xrightarrow{\mathrm{id}\otimes x\otimes\mathrm{id}} X \otimes \Sigma^k\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\mathrm{id}\otimes m} X \otimes \Sigma^k\mathbb{Z}$$

which equivalently is a map $\mathbb{S}^{l-k} \to X \otimes \mathbb{Z}$ as needed.

Next, we describe the Thom isomorphism in this picture. Recall that given an (oriented) spherical fibration $\xi\colon E \to X$ with typical fibre S^{d-1}, we may form the Thom space $\mathrm{Th}(\xi)$ and have then proved that cup product with the Thom class $u \in H^d(\mathrm{Th}(\xi);\mathbb{Z})$ induces an isomorphism $H^k(X;\mathbb{Z}) \cong H^{d+k}(\mathrm{Th}(\xi);\mathbb{Z})$. In what follows, by abuse of notation, we denote the suspension spectrum $\Sigma^\infty\mathrm{Th}(\xi)$ of $\mathrm{Th}(\xi)$ again by $\mathrm{Th}(\xi)$. If $x\colon X \to \Sigma^k\mathbb{Z}$ represents an element of $H^k(X;\mathbb{Z})$ the composite

$$\mathrm{Th}(\xi) \to X_+ \wedge \mathrm{Th}(\xi) \xrightarrow{x\otimes u} \Sigma^k\mathbb{Z} \otimes \Sigma^d\mathbb{Z} \xrightarrow{m} \Sigma^{d+k}\mathbb{Z}$$

represents the corresponding element $x \cdot u \in H^{d+k}(\mathrm{Th}(\xi);\mathbb{Z})$. Here, the first map is the Thom diagonal, obtained from the pullback diagram

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & 0 \times \xi \arrow[d] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

Similarly, the homological Thom isomorphism $H_{k+d}(\mathrm{Th}(\xi);\mathbb{Z}) \cong H_k(X;\mathbb{Z})$ sends a class represented by $\mathbb{S}^{k+d} \to \mathrm{Th}(\xi) \otimes \mathbb{Z}$ to the composite

$$\mathbb{S}^{k+d} \to \mathrm{Th}(\xi) \otimes \mathbb{Z} \to X \otimes \mathrm{Th}(\xi) \otimes \mathbb{Z} \to X \otimes \Sigma^d\mathbb{Z} \otimes \mathbb{Z} \to X \otimes \Sigma^d\mathbb{Z}$$

which in turn is equivalent to the desired map $\mathbb{S}^k \to X \otimes \mathbb{Z}$.

Finally, we combine this to make explicit the function $\mathrm{PD}(-)\colon H^k(M;\mathbb{Z}) \to H_{m-k}(M;\mathbb{Z})$ for M a closed oriented m-manifold. To that end, we first recall that the collapse map $c_M\colon S^r \to \mathrm{Th}(\nu_M)$ for ν_M the normal bundle of an embedding $M \to S^r$ determines the homological fundamental classes of M in the following sense: The composite

$$\pi_r(\mathrm{Th}(\nu_M)) \to H_r(\mathrm{Th}(\nu_M)) \cong H_m(M)$$

where the second map is the Thom isomorphism, sends $[c_M]$ to $[M]$. By the above, we find that $[M]$ is represented by the composite in the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^r \arrow[r, "c_M"] & \mathrm{Th}(\nu_M) \arrow[r] \arrow[d, "\mathrm{id}\otimes i"] & M \otimes \mathrm{Th}(\nu_M) \arrow[r, "\mathrm{id}\otimes u"] \arrow[d] & M \otimes \Sigma^{r-m}\mathbb{Z} \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes i"] \\
 & \mathrm{Th}(\nu_M) \otimes \mathbb{Z} \arrow[r] & M \otimes \mathrm{Th}(\nu_M) \otimes \mathbb{Z} \arrow[r, "\mathrm{id}\otimes u\otimes\mathrm{id}"] & M \otimes \Sigma^{r-m}\mathbb{Z} \otimes \mathbb{Z} \arrow[d, "\mathrm{id}\otimes m"] \\
 & & & M \otimes \Sigma^{r-m}\mathbb{Z}
\end{tikzcd}$$

where $i\colon \mathbb{S} \to \mathbb{Z}$ is the unit and $m\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the multiplication of the ring spectrum $\mathbb{Z}$ and the unlabelled maps are (induced by) the Thom diagonal. Since the right most vertical composite is the identity, the top horizontal composite represents $[M]$. We deduce the following description of the function $\mathrm{PD}(-)\colon H^k(M;\mathbb{Z}) \to H_{m-k}(M;\mathbb{Z})$. For $x\colon M \to \Sigma^k\mathbb{Z}$, we consider the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^r \arrow[r] \arrow[drr, bend right] & \mathrm{Th}(\nu_M) \arrow[r] & M \otimes \mathrm{Th}(\nu_M) \arrow[r] \arrow[d] & M \otimes \Sigma^{r-m}\mathbb{Z} \arrow[d] & \\
 & & M \otimes M \otimes \mathrm{Th}(\nu_M) \arrow[r] \arrow[dr, swap, "\mathrm{id}\otimes x\otimes u"] & M \otimes M \otimes \Sigma^{r-m}\mathbb{Z} \arrow[d] & \\
 & & & M \otimes \Sigma^k\mathbb{Z} \otimes \Sigma^{r-m}\mathbb{Z} \arrow[r, "\mathrm{id}\otimes m"] & M \otimes \Sigma^{r-m+k}
\end{tikzcd}$$

in which the composite now represents $\mathrm{PD}(x)$; here the first diagonal map is a "doubled" Thom diagonal.

Proof of Theorem 6.21. Choose a smooth embedding $h\colon M \to S^r$ for suitably large $r > \dim(M)$. Then $hi\colon N \to M \to S^r$ is again an embedding. Let us denote by ν_N and ν_M the normal bundles of these embeddings hi and h. Recall that the associated collapse map

$c_M\colon S^r \to \mathrm{Th}(\nu_M)$ and $c_N\colon S^r \to \mathrm{Th}(\nu_N)$ then determine the homological fundamental classes of M and N as described in Remark 6.23.

Exercise: we have a preferred isomorphism $\nu_N \cong \nu_{N,M} \oplus (\nu_M)_{|N}$. Hint: $TM_{|N} = TN \oplus \nu_{N,M}$ for any embedding $N \subseteq M$ with normal bundle $\nu_{N,M}$. In particular, there are pullback diagrams

$$\begin{tikzcd}
\nu_N \arrow[r] \arrow[d] & \nu_{N,M} \times (\nu_M)_{|N} \arrow[r] \arrow[d] & \nu_{N,M} \times \nu_M \arrow[d] \\
N \arrow[r] & N \times N \arrow[r] & N \times M
\end{tikzcd}$$

and we obtain an induced relative Thom diagonal map $\mathrm{Th}(\nu_N) \to \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M)$. The geometric fact to contemplate is then that the following diagram commutes

$$\begin{tikzcd}
S^r \arrow[rr, "c_N"] \arrow[d, "c_M"] & & \mathrm{Th}(\nu_N) \arrow[d] \\
\mathrm{Th}(\nu_M) \arrow[r] & M_+ \wedge \mathrm{Th}(\nu_M) \arrow[r, "c_{N,M}\wedge \mathrm{id}"] & \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M)
\end{tikzcd}$$

where the unlabelled maps are the just explained (relative) Thom diagonals. Exercise: adapt the above construction of the relative Thom diagonal to contruct a commutative diagram

$$\begin{tikzcd}
\mathrm{Th}(\nu_N) \arrow[r] \arrow[d] & N_+ \wedge \mathrm{Th}(\nu_N) \arrow[d] \\
\mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M) \arrow[r] & M_+ \wedge \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M)
\end{tikzcd}$$

Hint: Consider the commutative diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & N \times N \arrow[d] & n \arrow[r, mapsto] \arrow[d, mapsto] & (n,n) \arrow[d, mapsto] \\
N \times M \arrow[r] & N \times M \times M & (n,i(n)) \arrow[r, mapsto] & (n,i(n),i(n))
\end{tikzcd}$$

and consider the bundle $\nu_{N,M} \times \nu_M \times 0$ over $N \times M \times M$ and the pullbacks along all appearing maps. Glueing the two diagrams we obtain

$$\begin{tikzcd}
S^r \arrow[r] \arrow[d] & N_+ \wedge \mathrm{Th}(\nu_N) \arrow[d] \\
\mathrm{Th}(\nu_M) \arrow[r] & M_+ \wedge \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M)
\end{tikzcd}$$

Denote now by $u_{N,M}\colon \mathrm{Th}(\nu_{N,M}) \to \Sigma^{m-n}\mathbb{Z}$ and $u_M\colon \mathrm{Th}(\nu_M) \to \Sigma^{r-m}\mathbb{Z}$ the maps classifying the Thom classes. Since Thom classes are natural for pullbacks of bundles, we find that the composite

$$\mathrm{Th}(\nu_N) \to \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M) \to \Sigma^{m-n}\mathbb{Z} \wedge \Sigma^{r-m}\mathbb{Z} \to \Sigma^{r-n}\mathbb{Z}$$

classifies the Thom class u_N of ν_N. Therefore, also the following diagram commutes:

$$\begin{tikzcd}
S^r \arrow[r] \arrow[d] & N_+ \wedge \mathrm{Th}(\nu_N) \arrow[r, "u_N"] \arrow[d] & N \otimes \Sigma^{r-n}\mathbb{Z} \arrow[d, "i\otimes \mathrm{id}"] \\
\mathrm{Th}(\nu_M) \arrow[r] & M_+ \wedge \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_M) \arrow[r, "u_{N,M}\otimes u_M"] & M \otimes \Sigma^{r-n}\mathbb{Z}
\end{tikzcd}$$

The top horizontal then classifies an element in $H_n(N;\mathbb{Z})$; the above discussion about fundamental classes implies that this element in $[N]$. $\square$

Proof of Theorem 6.22. The fact that N and N' are transversal implies that there is a pullback diagram as follows:

$$\begin{tikzcd}
\nu_{K,M} \arrow[r] \arrow[d] & \nu_{N,M} \times \nu_{N',M} \arrow[d] \\
K \arrow[r] & N \times N'
\end{tikzcd}$$

In particular, there is associated a canonical relative Thom collapse map $\mathrm{Th}(\nu_{K,M}) \to \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_{N',M})$. The geometric fact to contemplate in this situation is that the following diagram involving the collapse map and the relative Thom diagonal commutes:

$$\begin{tikzcd}
M \arrow[r, "c_{K,M}"] \arrow[dr, bend right=10, "c_{N,M}\wedge c_{N'.M}"'] & \mathrm{Th}(\nu_{K,M}) \arrow[d] \\
 & \mathrm{Th}(\nu_{N,M}) \wedge \mathrm{Th}(\nu_{N',M})
\end{tikzcd}$$

The result then follows again from the naturality of Thom classes under pullback diagrams and Theorem 6.21. $\square$

6.24. **Remark** One can try to generalize the above arguments to topological manifolds; but note that for embeddings of topological manifolds, a normal bundle need not exists, and if it exist it need not be unique and/or admit a disk bundle representative, which even if it exists might not be part of a tubular neighbourhood of the embedding. In addition, for a topological version of Theorem 6.22, a correct notion of transversality needs to be used, as to ensure that the two geometric facts about the collapse maps hold true in case the statements make sense. To avoid discussing all these subtleties, we restrict attention to smooth manifolds here.

We now consider the following construction.

6.25. **Construction** We recall the E_8-graph, a tree with vertix set V of cardinality 8. Fix $n \geq 1$ and denote by $D(n)$ the disk-bundle of the tangent bundle of S^{2n}. For a point $x \in S^{2n}$, we may fix an embedding $D^{2n} \subseteq S^{2n}$ sending 0 to x, and denote by D^{2n}_x its image, viewed as closed subset of S^{2n}. Then $D(n)_{|D^{2n}_x}$ is canonically diffeomorphic to $D^{2n} \times D^{2n}$ which bundle projection corresponding to the projection onto the first factor together with the fixed homeomorphism $D^{2n}_x \cong D^{2n}$.

Now consider the disjoint union $\coprod_{v\in V} D(n)$ and for each $v \in V$ pick $x(v) \in S^{2n}$ such that D^{2n} and embeddings $D^{2n} \cong D^{2n}_{x(v)}$ such that the subsets $D^{2n}_{x(v)}$ of S^{2n} are pairwise disjoint. Define $E_8(n)$ to be the quotient space of $\coprod_{v\in V} D(n)$ given by the following relation. Whenever there is an edge between v and v', we inductively glue together the two copies of $D(n)$ indexed by v and v' according to the following pushout square

$$\begin{tikzcd}
D^{2n} \times D^{2n} \arrow[r, "\mathrm{sw}"] \arrow[d] & D^{2n} \times D^{2n} \arrow[r] & D(n) \arrow[d] \\
D(n) \arrow[rr] & & D(n)'
\end{tikzcd}$$

where sw is the diffeomorphism switching the two factors. This defines a smooth[15] manifold $E_8(n)$ of dimension $4n$ with boundary $\partial E_8(n)$ of dimension $4n-1$. This manifold is often called the Milnor manifold.

6.26. **Lemma** *The tangent bundle of $E_8(n)$ is trivial.*

Proof. By construction, for each vertex of the E_8-graph, we have an embedding $TS^{2n} \subseteq E_8(n)$. In particular, we have embeddings of S^{2n} into $E_8(n)$ whose normal bundle identifies with TS^{2n} and $TE_8(n)_{|S^{2n}}$ becomes isomorphic to $TS^{2n} \oplus TS^{2n}$. This is a bundle of rank $4n > 2n$, so is already a stable bundle: The map $\mathrm{BO}(4n) \to \mathrm{BO}$ induces an isomorphism on π_{2n}. Therefore, since $TS^{2n} \oplus \mathbb{R} \cong S^{2n} \times \mathbb{R}^{2n+1}$ is trivial, we deduce that $TE_8(n)_{|S^{2n}}$ is trivial for each canonical embedding $S^{2n} \subseteq E_8(n)$ corresponding to a vertex. The claim then follows from noting that the these maps combine to a homotopy equivalence $\bigvee_{x\in V} S^{2n} \to E_8(n)$. ☐

Exercise. If $n \geq 2$, then $\pi_1(\partial E_8(n))$ is trivial. Hint: Show that $\pi_1(\partial D(n))$ is trivial and apply Seifert van Kampen several times to control π_1 of each step of the glueing appearing in the definition of $E_8(n)$.

Next we aim to prove the following:

6.27. **Proposition** *For all $n \geq 1$, we have $H_*(\partial E_8(n); \mathbb{Z}) \cong H_*(S^{4n-1}; \mathbb{Z})$.*

To prove this, we first record the following general claim. Let W be a connected oriented compact $2n$-manifold with connected boundary ∂W. Then we may consider the cup product pairing

$$H^n(W, \partial W) \times H^n(W, \partial W) \to H^{2n}(W, \partial W) \xrightarrow{\mathrm{PD}} \mathbb{Z}.$$

Then Poincaré duality also identifies $H^n(W, \partial W)$ with $H_n(W)$. The above composite is then adjoint to a map

$$H^n(W, \partial W) \to \mathrm{Hom}(H_n(W), \mathbb{Z}).$$

6.28. **Lemma** *The just described map is itself given by the composite*

$$H^n(W, \partial W) \to H^n(W) \to \mathrm{Hom}(H_n(W), \mathbb{Z})$$

where the first map is the map part of the long exact sequence of the pair $(W, \partial W)$ and the second is the evaluation map appearing in the universal coeffcient theorem.

Proof. Fix $x \in H^n(W, \partial W)$. For $\alpha \in H_n(W)$ there exists a unique $y \in H^n(W, \partial W)$ such that $y \cap [W] = \alpha$. By definition, the map described just before the statement of the lemma sends x to the map sending α to $(x \cup y) \cap [W]$. But we have

$$(x \cup y) \cap [W] = x \cap (y \cap [W]) = x \cap \alpha = \langle x, \alpha \rangle.$$

This shows that the map depends only on the image of x under $H^n(W, \partial W) \to H^n(W)$ and is then given as claimed. ☐

6.29. **Remark** Similarly, the composite $H_n(W) \to H_n(W, \partial W) \cong H^n(W) \to \mathrm{Hom}(H_n(W); \mathbb{Z})$ is adjoint to a map

$$H_n(W) \times H_n(W) \to \mathbb{Z}$$

which, under the Poincaré duality isomorphism $H_n(W) \cong H^n(W, \partial W)$ is given by the cup product pairing

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[15] By a procedure called smoothening the corners.

Proof of Proposition 6.27. We compute the cohomology of $\partial E_8(n)$ by means of the long exact sequence

$$H_{k+1}(E_8(n)) \to H_{k+1}(E_b(n(, \partial E_8(n)) \to H_k(\partial E_8(n)) \to H_k(E_8(n)) \to H_k(E_8(n), \partial E_8(n))$$

Since $E_8(n) \simeq \bigvee S^{2n}$ and using Lemma 6.28 and Remark 6.29, the claim follows if we can show that the cup product pairing

$$H^{2n}(E_8(n), \partial E_8(n)) \times H^{2n}(E_8(n), \partial E_8(n)) \to \mathbb{Z}$$

is unimodular. Now, under the Poincaré duality isomorphism $H^{2n}(E_8(n), \partial E_8(n)) \cong H_{2n}(E_8(n))$, we find that a basis is represented by the various embedded submanifolds $S^{2n} \subseteq E_8(n)$. Hence, in order to compute the cup product pairing, by Theorem 6.22, we may compute geometric intersection numbers of these embedded submanifolds. Then note that two embeddings of S^{2n} corresponding to vertices $v \neq v'$ in the E_8-graph are disjoint if v and v' are not connected by an edge; in this case, the cup product is therefore 0. When $v \neq v'$ are connected by an edge, we have arranged that the two corresponding S^{2n}'s are indeed transversal and intersect in a single point; taking the orientation behaviour into account, we find that the intersection number in this case is -1. It remains to compute the self-intersection of each S^{2n}. In this case, we have to make the embedding transversal to itself. The way to do this is to consider the normal bundle of the embedding – in our case this is, by construction, given by TS^{2n} – and then we may push the embedding of the zero section in the normal direction; concretely by using a flow associated to a suitable section of the normal bundle. It follows that the self-intersection number can be described be the number (counted with signs according to the orientation) of zeros of such a section; this in turn can be shown to be the Euler number. In our situation, we therefore find that the self-intersection number is equal to the Euler number of TS^{2n}, which is equal to the Euler characteristic of S^{2n} which is equal to 2 (recall e.g. Exercise 1 Sheet 4 from [Lan25]).

It follows that the cup product pairing has representing matrix given by the E_8-matrix, which is famously known to be invertible. ☐

6.30. **Corollary** *Let $n \geq 2$. Then the smooth manifold $\partial E_8(n)$ is homotopy equivalent to S^n.*

6.31. **Remark** One can show that $\pi_1(\partial E_8(1))$ is non-trivial, and so that $\partial E_8(1)$ is indeed only a homology sphere[16] P, the Poincaré homology 3-sphere, and not a homotopy sphere. In fact, $\pi_1(\partial E_8(1))$ is the binary icosahedral group, a finite group of order 120. Milnor has shown that this is the only finite group which appears as the fundamental group of a homology 3-sphere.

6.32. **Theorem** *We have that $\partial E_8(2)$ is not diffeomorphic to S^7.*

Proof. Suppose that there is a diffeomorphism $\partial E_8(2) \cong S^7$. Then the topological manifold $\bar{E}_8(2) = E_8(2) \cup_{\partial E_8(2)} D^8$ admits a smooth structure. Since the inclusion $E_8(2) \subseteq \bar{E}_8(2)$ induces an isomorphism on H^4 we deduce from the exercise above that $p_1(\bar{E}_8(2)) = 0$ and that $\mathrm{sign}(E_8(2)) = \mathrm{sign}(\bar{E}_8(2))$. Hence, we find

$$8 = \mathrm{sign}(\bar{E}_8(2)) = L_2(p_1, p_2)(\bar{E}_8(2)) = \frac{7}{45} p_2(\bar{E}_8(2)).$$

Since $\bar{E}_8(2)$ is smooth, we deduce that $p_2(\bar{E}_8(2))$ is an integer. Therefore, we deduce that

$$8 \cdot 45 \equiv 0 \mod 7$$

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[16] That is, a smooth closed oriented 3-manifold whose homology is isomorphic to that of S^3.

which is evidently incorrect, yielding a contradiction as desired. □

6.33. **Remark** For all $n \geq 2$, we have that $\partial E_8(n)$ is not diffeomorphic to S^{4n-1}. Indeed, a similar argument will give

$$8 = \mathrm{sign}(\bar{E}_8(n)) = L_n(p_1, \dots, p_n)(\bar{E}_8(n)) = \alpha_n p_n(\bar{E}_8(n))$$

where α_n is the coefficient of p_n in the nth L-polynomial. A detailed analysis of this coefficient allows to deduce the same contradiction for all $n \geq 2$.

6.34. **Remark** It is a consequence of the h-cobordism theorem, proven by Smale, that any closed smooth manifold homotopy equivalent to S^n is in fact homeomorphic to S^n as long as $n \geq 6$ (in fact, the same is true if $n \geq 5$ and for closed topological manifolds rather than closed smooth ones). In particular, we deduce that $\partial E_8(2)$ is homeomorphic but not diffeomorphic to S^7. Such smooth manifolds are called exotic spheres (because they are topologically standard spheres with an exotic smooth structure).

6.35. **Remark** As a consequence of Remark 6.34, for $n \geq 2$ one can define closed oriented *topological* manifolds $\bar{E}_8(n)$ as in the proof of Theorem 6.32. Moreover, it is a deep theorem in point-set topology of Freedman (for which he was awarded the fields medal in 1982) that every homology 3-sphere Σ bounds a contractible topological 4-manifold $W(\Sigma)$. Hence, one may form the closed oriented topological manifold $E_8 = E_8(1) \cup_P W(P)$. This manifold admits a topological spin structure and has signature 8; this shows that an earlier theorem of Rokhlin's stating that a closed smooth spin 4-manifold has signature divisible by 16 is really special to smooth manifolds (also, a $K3$ surface has signature 16 so Rokhlin's theorem cannot be improved). I believe that Kervaire and Milnor's study of Rokhlin's theorem was when they realised they can in fact say much more about homotopy spheres than the mere existence in dimension 7. In fact, in a landmark result on the combination of surgery theory and homotopy theory, they essentially calculated[17] the group of homotopy spheres Θ_n in dimension n to participate in the exact sequence

$$0 \to \mathrm{bP}_{n+1} \to \Theta_n \to \mathrm{coker}(J)_n \to 0$$

where bP_{n+1} is the subgroup of those homotopy spheres which are boundaries of parallelisable smooth manifolds of dimension $n+1$; This turns out to be a cyclic group whose order is trivial if $n+1$ is odd, $\mathbb{Z}/2$ if $n+1 \equiv 2 \mod 4$ and for $n+1 = 4k$, it has order $2^{2k-2} \cdot (2^{2k-1}-1) \cdot \mathrm{num}(\frac{4B_k}{k})$ where B_k is a Bernoulli number.[18]

Finally, I cannot resist to mention that Rokhlin's theorem does not rule out that $E_8 \sharp E_8$ admits a smooth structure as its signature is 16. This, however, is ruled out by a theorem of Donaldson, for which he was awarded the fields medal in the same year as Freedman (1982). Using methods from analysis, what is often referred to as gauge theory, he showed that if the intersection form of a smooth closed 4-manifold is positive (or negative) definite, then the form is isomorphic to the standard form with only 1s (or -1s) on the diagonal – this rules out an enormous amount of definite forms as arising as the intersection form of a smooth manifold. Freedman's results in contrast showed that all unimodular forms appear as the intersection form of a closed topological 4-manifold.

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[17] With at the time some exceptions which by now we know to be finitely many exceptions.

[18] Topologists often write B_k rather than the convention in number theory which is B_{2k} instead – this is because the a priori defined numbers B_{2k+1} are in fact 0.

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