TOPOLOGY V

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Abstract. These are lecture notes for my lecture "Topology V" which I taught in the winter term 2025/26 at LMU Munich.

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1. Recollection/Prerequisites

There will be a biweekly exercise session where we discuss further examples and questions. There will be no formal exercise sheets. If you want to get credits for this course, you can do so under WP? for ?ECTS. The examination will be an oral exam at the end of the term.

This course will build on the lectures Topology I (WS 23/24), Topology II (SS 24), Topology III (WS 24/25), and Topology IV (SS 25) taught at LMU. We briefly recall the main topics that were covered, so a reader has an impression what will be the assumed background knowledge.

(1) Point-set topology
(2) Homotopy theory: homotopy groups, CW complexes, applications of cellular approximation, cofibrations, Seifert-van Kampen's theorem
(3) Covering theory; Fundamental theorem of covering theory
(4) Singular Homology; Definition, Properties, Applications.
(5) Singular Cohomology; Cup product, Universal coefficient theorems, Künneth theorem
(6) Topological Manifolds: Orientability and Poincaré duality, Applications
(7) Homotopy theory: Fibrations, long exact homotopy sequence, Whitehead's theorem, cellular approximation theorem, homotopy excision theorem, Freudenthal
(8) Hurewicz theorems
(9) Eilenberg–Mac Lane spaces and representability of cohomology
(10) Principal G-bundles
(11) Obstruction theory
(12) Steenrod operations
(13) The Leray–Hirsch theorem
(14) Thom isomorphism for spherical fibrations,
(15) Stiefel–Whitney and Wu classes, Chern classes, Pontryagin classes, the cohomology of BO, BU, remarks on BTop and BG,

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(16) Poincare duality complexes and Wu's formulas
(17) A survey on manifolds, tangent bundles, Pontryagin–Thom constructions.

Parts (1)–(4) were covered in Topology I [Lan23], parts (5)–(7) were covered in Topology II [Win24], parts (8)–(13) were covered in Topology III [Lan24], and parts (14)–(17) were covered in [Lan25]. The lecture notes for these courses are available on the course webpage.

Topic (16) is not relevant for this course (in particular all the higher categorical things we used to define Poincaré duality complexes) and Topic (17) will only be used in a minimalistic way. The rough plan for this term is to cover the following, (6) below only if time permits (which it almost surely will not);

(1) Spectral sequences and the Serre spectral sequence
(2) Rational homotopy theory
(3) Some stable homotopy groups of spheres, cohomology of EM spaces
(4) Computation of the rational oriented bordism ring, the signature theorem
(5) Construction of exotic spheres.
(6) Further applications to manifolds; geometric interpretation of cup product, existence of manifolds with certain cell structures, spin$^{\mathbb{C}}$-structures + intersection form on 4-manifolds, (obstructions to the) existence of submanifolds representing homology classes, Rokhlin's theorem

2. Spectral sequences

2.1. **Definition** A strongly convergent spectral sequence consists of the following data satisfying the following axioms:

(1) a complete and separated filtration F on a graded abelian group M called the *abutment* of the spectral sequence. That is, $M = \{M_n\}_{n\in\mathbb{Z}}$ is a graded abelian group and $F_\bullet M_n$ is a complete[1] and separated[2] filtration on M_n for every $n \in \mathbb{Z}$.
(2) for each $r \geq 1$ a bigraded abelian group $E^r_{p,q}$ equipped with a differential $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$, that is $(d^r)^2 = 0$.
(3) An isomorphism between the homology $H_*(E^r, d^r)$ of (E^r, d^r) and E^{r+1}.
(4) For every pair (p, q), there is an $N(p, q)$ such that for all $r \geq N(p, q)$, $d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $d^r \colon E^r_{p+r,q+r-1} \to E^r_{p,q}$ vanish. It follows that $E^{N(p,q)}_{p,q} \cong E^{N(p,q)+s}_{p,q}$ for all $s \geq 0$, so we call this common term $E^\infty_{p,q}$.
(5) An isomorphism between the associated graded $\mathrm{gr}(F_\bullet M)$ of the abutment (which is a bigraded abelian group) and E^∞ (which is also a bigraded abelian group). Explicitly, an isomorphism $F_k(M_n)/F_{k-1}(M_n) \cong E^\infty_{\dots}$.

A spectral sequence as above is called multiplicative if $F_\bullet M$ is a filtered graded commutative ring, all bigraded abelian groups $E^r_{*,*}$ are bigraded commutative rings and the differential satisfies the Leibniz rule

$$d^r(x \cdot y) = d^r(x) \cdot y + (-1)^{|x|} x \cdot d^r(y)$$

and the isomorphism $\mathrm{gr}(F_\bullet M) \cong E^\infty$ is one of bigraded commutative rings.

2.2. **Warning** Just like exact sequences, a spectral sequence is not itself capable of computing the graded abelian group M, only the associated graded with respect to some filtration on it

[1] That is $\mathrm{colim}_n\, M_n = M_n$
[2] That is $\lim_n M_n = 0$

which is part of the spectral sequence. Concretely, this means that in order to compute M itself, possible extension problems have to be solved. This is something one can then try to do by hand, but the spectral sequence is not a priori of any help in this task.

One way to obtain a spectral sequence is through filtered chain complexes. I recommend reading the relevant part of Weibel's book on the topic [Wei94] or Hatcher's account on spectral sequences [Hat04] or McLearys book [McC01]. In particular, the main example of a spectral sequence we will use in this course, the Serre spectral sequence, can be constructed from a filtered chain complex. An elegant construction using bisimplicial sets was found by Dress [Dre67].

However, not all spectral sequences that arise in practice arise naturally in this fashion, but they do arise naturally as the spectral sequence associated to a filtered *spectrum*. We briefly explain how a filtered spectrum gives rise to a spectral sequence now, see [Lur17] for details, but beware of the different indexing convention: We will have to make a choice whether a $\mathbb{Z}$-indexed filtration lowers or raises degree. To the best of my knowledge, either choice becomes annoying at some point, so we stick to the one that is closer to what we obtain from the examples that we shall consider, but which differs from the one appearing in [Lur17].

2.3. **Definition** A filtered spectrum is an object of $\mathrm{Fun}((\mathbb{Z},\geq),\mathrm{Sp}) =: \mathrm{Fil}(\mathrm{Sp})$, where we view $(\mathbb{Z},\geq)$ as a poset. This poset is in fact canonically a symmetric monoidal category under the sum of integers. Hence, $\mathrm{Fil}(\mathrm{Sp})$ is naturally a symmetric monoidal category under Day convolution so we may form $\mathrm{CAlg}(\mathrm{Fil}(\mathrm{Sp}))$. For $F \in \mathrm{Fil}(\mathrm{Sp})$ we write F_n for its evaluation at n. A filtered spectrum is called separated if $\lim_n F = 0$.

A graded spectrum is an object of $\mathrm{Fun}(\mathbb{Z}^\delta,\mathrm{Sp}) =: \mathrm{Gr}(\mathrm{Sp})$, where we view $\mathbb{Z}^\delta$ as a discrete category. This is also symmetric monoidal under the sum of integers, so $\mathrm{Gr}(\mathrm{Sp})$ also carries a Day convolution symmetric monoidal structure.

2.4. **Remark** Let us gather some facts about the above.

(1) There is a functor $\mathrm{gr}\colon \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$ called the associated graded of a filtration, sending F to $n \mapsto \mathrm{gr}^n(F) = \mathrm{cofib}(F_{n+1} \to F_n)$. This functor preserves colimits and limits and is equipped with a canonical symmetric monoidal structure.

(2) The colimit of a filtration gives rise to a functor $\mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$; we often call this the *underlying* spectrum of a filtered spectrum (and hence think of the filtered spectrum as a filtration on its colimit).

(3) Given a spectrum X, its Whitehead tower $\tau_{\geq\bullet}X$

$$\cdots \to \tau_{\geq n+1}X \to \tau_{\geq n}X \to \tau_{\geq n-1}X \to \dots$$

is a separated filtered spectrum with underlying spectrum X. The association $X \mapsto \tau_{\geq\bullet}X$ refines to a lax symmetric monoidal functor $\mathrm{Sp} \to \mathrm{Fil}(\mathrm{Sp})$. In particular, if X is a (commutative) algebra in Sp, its Whitehead tower $\tau_{\geq\bullet}X$ is a (commutative) algebra in $\mathrm{Fil}(\mathrm{Sp})$.

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