TOPOLOGY IV

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ABSTRACT. These are lecture notes for my lecture "Topology IV" which I taught in the summer term 2025 at LMU Munich.

CONTENTS

1. RECOLLECTION/PREREQUISITES

There will be **no lectures** on 18.11. and 20.11. and we will **reschedule** the lecture on 23.12. We will take some time to discuss exercises I pose during the lectures; either in the beginning of each lecture or regularly (roughly) every 3 weeks. If you want to get credits for this course, you can do so under WP37 for 6ECTS. The examination will be an oral exam at the end of the term.

This course will build on the lectures Topology I (WS 23/24), Topology II (SS 24), and Topology III (WS 24/25) taught at LMU. We briefly recall the main topics that were covered, so a reader has an impression what will be the assumed background knowledge.

(1) Point-set topology
(2) Homotopy theory: homotopy groups, CW complexes, applications of cellular approximation, cofibrations, Seifert-van Kampen's theorem
(3) Covering theory; Fundamental theorem of covering theory
(4) Singular Homology; Definition, Properties, Applications.
(5) Singular Cohomology; Cup product, Universal coefficient theorems, Künneth theorem
(6) Topological Manifolds: Orientability and Poincaré duality, Applications
(7) Homotopy theory: Fibrations, long exact homotopy sequence, Whitehead's theorem, cellular approximation theorem, homotopy excision theorem, Freudenthal
(8) Hurewicz theorems
(9) Eilenberg–Mac Lane spaces and representability of cohomology
(10) Principal G-bundles
(11) Obstruction theory

Date: July 22, 2025.

(12) Steenrod operations
(13) The Leray–Hirsch theorem

Parts (1)–(4) were covered in Topology I [Lan23], parts (5)–(7) were covered in Topology II [Win24], and parts (8)–(13) were covered in Topology III [Lan24]. The lecture notes for these courses are available on the course webpage.

The rough plan for this term is to cover the following (as much as fits into the timeframe).

(1) Thom isomorphism for spherical fibrations,
(2) Stiefel–Whitney and Wu classes, the cohomology of BO, BU, remarks on BTop,
(3) Poincare duality complexes and Wu's formulas
(4) A survey on manifolds, tangent bundles, Pontryagin–Thom constructions.
(5) Applications to manifolds; geometric interpretation of cup product, existence of manifolds with certain cell structures, spin$^{\mathbb{C}}$-structures + intersection form on 4-manifolds, (obstructions to the) existence of submanifolds representing homology classes, Rokhlin's theorem
(6) Construction of a non-standard homotopy 7-sphere.
(7) Serre spectral sequence
(8) The signature theorem (possibly in dimensions $4, 8$ only)
(9) Cohomology of Eilenberg–Mac Lane spaces
(10) Homotopy groups of spheres using Serre's method

Acknowledgements. Essentially everything in this script that concerns topological manifolds and the anima $\mathrm{Top}(d)$ was explained to me by Manuel Krannich. I thank him deeply for his patience in explaining the relevant results to me in language that I can properly parse. That being said, all errors about this material are of course due to me, and must have come from me misunderstanding what he surely explained properly to me.

2. Thom isomorphism for spherical fibrations

We now move towards the Thom isomorphism for spherical fibrations. We begin with all relevant definitions.

2.1. **Definition** Let $\pi\colon E \to B$ be a fibration with typical fibre S^{d-1} and B connected. Then π is called oriented if for all γ in $\pi_1(B)$, the induced homotopy self-equivalence of S^{d-1} is orientation preserving, that is, induces the identity on $H_{d-1}(S^{d-1};\mathbb{Z})$.

2.2. **Definition** Let $\pi\colon E \to B$ be a spherical fibration of rank $d-1$ and B connected. We call its mapping cone $\mathrm{C}(\pi)$ the Thom space of π and also write $\mathrm{Th}(\pi)$ for it.

2.3. **Lemma** *Given a pullback diagram*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r] & B \end{tikzcd}$$

of spherical fibrations of rank $d-1$, there is a canonically defined map $\mathrm{Th}(\pi') \to \mathrm{Th}(\pi)$. In particular, for $B' = \{b\}$ there is a map $S^d \to \mathrm{Th}(\pi)$ and the maps $\mathrm{Th}(\pi') \to \mathrm{Th}(\pi)$ are compatible with this map.

Proof. This follows by passing to vertical (homotopy) cofibres in the pullback diagram of the statement of the lemma, together with the observation that $\mathrm{Th}(S^{d-1} \to *) \simeq S^d$. $\square$

2.4. **Definition** For $d \geq 0$, let $\mathrm{Top}(d) = \mathrm{Homeo}(\mathbb{R}^d)$ and $\mathrm{Top}_0(d)$ be the subgroup of homeomorphisms preserving the origin.

2.5. **Remark** The inclusion $\mathrm{Top}_0(d) \to \mathrm{Top}(d)$ is a homotopy equivalence with homotopy inverse given by $f \mapsto f - f(0)$. It therefore induces a homotopy equivalence $\mathrm{BTop}_0(d) \to \mathrm{BTop}(d)$.

2.6. **Remark** A fibration $\pi\colon E \to B$ with typical fibre S^{d-1} is called a spherical fibration of rank $d-1$ over B. By [Lan24, Theorem 5.9] it is classified by a map $B \to \mathrm{BhAut}(S^{d-1})$. The group $\mathrm{hAut}(S^{d-1})$ is classically denoted by $\mathrm{G}(d)$, so a rank $d-1$-spherical fibration over B is classified by a map $B \to \mathrm{BG}(d)$. Being oriented means that the classifying map lifts to $\mathrm{BhAut}^+(S^{d-1}) =: \mathrm{BSG}(d)$.

There is related concept, that of a pointed spherical fibration of rank d which is a fibration $\pi\colon E \to B$ with typical fibre S^d which is equipped with a section $s\colon B \to E$, that is, such that $\pi s = \mathrm{id}_B$. These are classified by maps $B \to \mathrm{BhAut}_*(S^d) =: \mathrm{BF}(d)$. Likewise, there is an oriented version classified by maps to $\mathrm{BhAut}_*^+(S^d) =: \mathrm{BSF}(d)$. Suspending induces group homomorphisms $\mathrm{G}(d) \to \mathrm{F}(d)$ and $\mathrm{SG}(d) \to \mathrm{SF}(d)$ and therefore maps $\mathrm{BG}(d) \to \mathrm{BF}(d)$ and $\mathrm{BSG}(d) \to \mathrm{BSF}(d)$; we will discuss in the proof of the next lemma what these maps do concretely to a spherical fibration $E \to B$.

2.7. **Remark** We note that there are evident group homomorphisms

$$\mathrm{O}(d) \to \mathrm{GL}_d(\mathbb{R}) \to \mathrm{Top}_0(d) \to \mathrm{G}(d) \to \mathrm{F}(d)$$

which all induce the evident forgetful maps on classifying spaces. We note that the composite $\mathrm{O}(d) \to \mathrm{G}(d)$ factors as

$$\mathrm{O}(d) \to \mathrm{Homeo}(S^{d-1}) \to \mathrm{G}(d)$$

since an orthogonal isomorphism of $\mathbb{R}^d$ restricts to a homeomorphism on the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$. Moreover, the composite $\mathrm{Top}_0(d) \to \mathrm{F}(d)$ factors as

$$\mathrm{Top}_0(d) \to \mathrm{Homeo}_\infty(S^d) \to \mathrm{F}(d)$$

where the first map is given by sending a homeomorphism to its induced homeomorphism on one-point compactifications; the resulting homeomorphism then preserves the point at infinity as needed.

It turns out that there is no map $\mathrm{Top}_0(\mathbb{R}^d) \to \mathrm{Homeo}(S^{d-1})$ whose restriction along $\mathrm{O}(d) \to \mathrm{Top}_0(\mathbb{R}^d)$ and whose composite with $\mathrm{Homeo}(S^{d-1}) \to \mathrm{G}(d)$ are the maps described above.[1]

2.8. **Definition** For a pointed spherical fibration $\pi\colon E \to B$ with section $s\colon B \to E$, we can define its pointed Thom space $\mathrm{Th}_*(\pi) = \mathrm{C}(s)$ as the mapping cone of the section s.

2.9. **Lemma** *Let $\pi\colon E \to B$ be a spherical fibration of rank $d-1$ and $\Sigma(\pi)$ its associated pointed spherical fibration of rank d. Then there is a canonical equivalence* $\mathrm{Th}(\pi) \simeq \mathrm{Th}_*(\Sigma(\pi))$.

[1] This is not obvious, however.

Proof. Recall that $\mathrm{BG}(d) \subseteq \mathrm{An}$ is the full subgroupoid of An on the object S^{d-1}, likewise $\mathrm{BF}(d) \subseteq \mathrm{An}_*$ is the full subgroupoid on S^d. We then consider the functor $\Sigma\colon \mathrm{An} \to \mathrm{An}_*$ and observe that this induces the map $\mathrm{BG}(d) \to \mathrm{BF}(d)$ described above. For a base space B, recall the equivalence $\mathrm{An}_{/B} \simeq \mathrm{Fun}(B, \mathrm{An})$, which sends a spherical fibration $\pi\colon E \to B$ of rank $d-1$ to its classifying map $\theta\colon B \to \mathrm{BG}(d) \subseteq \mathrm{An}$. The fibrewise suspension is then obtained by forming the suspension of θ in the category $\mathrm{Fun}(B, \mathrm{An}) \simeq \mathrm{An}_{/B}$ and observing that the resulting object is canonically pointed. It follows that the total space $\Sigma^{\mathrm{fw}}(E)$ of $\Sigma(\pi)$ is given by the following homotopy pushout.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & \Sigma^{\mathrm{fw}}(E) \end{tikzcd}$$

It follows that the horizontal mapping cones in the above square are equivalent. The top horizontal cone is $\mathrm{Th}(\pi)$ and the lower one is $\mathrm{Th}_*(\Sigma(\pi))$, giving the claim. ☐

2.10. **Remark** In case π is a fibre bundle, we note that $\Sigma^{\mathrm{fw}}(E) = \mathrm{C}^{\mathrm{fw}}(E) \cup_E \mathrm{C}^{\mathrm{fw}}(E)$ where $\mathrm{C}^{\mathrm{fw}}(E)$ is the disk bundle classified by the composite $B \to \mathrm{BHomeo}(S^{d-1}) \to \mathrm{BHomeo}(D^d)$; its boundary is then given by π. Then the projection map $\mathrm{C}^{\mathrm{fw}}(E) \to B$ is a homotopy equivalence and $E \to \mathrm{C}^{\mathrm{fw}}(E)$ is a cofibration, showing the above lemma in the special case of fibre bundles.

2.11. **Remark** Taking the join and the suspension of homotopy equivalence and pointed homotopy equivalences yields maps

$$(\#) \qquad \mathrm{BG}(d) \times \mathrm{BG}(d') \xrightarrow{\star} \mathrm{BG}(d+d') \quad \text{and} \quad \mathrm{BF}(d) \times \mathrm{BF}(d') \xrightarrow{\wedge} \mathrm{BF}(d+d')$$

which are compatible with the previously mentioned maps $\mathrm{BF}(k) \to \mathrm{BG}(k)$. In addition, each of the above two maps restricts to the oriented versions. Given spherical fibrations of rank $d-1$ and $d'-1$ over B and B', respectively, with associated fibrations $\pi\colon E \to B$ and $\pi'\colon E' \to B'$, we write $E \star^{\mathrm{fw}} E' \to B \times B'$ for the resulting rank $d+d'-1$ spherical fibration over $B \times B'$; Likewise we write $E \wedge^{\mathrm{fw}} E'$ in the pointed case. The superscript fw in each case reflects the fact that the join and wedge are formed *fibrewise*. Similarly to earlier, we note for instance that there is a homotopy pushout

$$\begin{tikzcd} E \times B' \cup B \times E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \times B' \arrow[r] & E \wedge^{\mathrm{fw}} E' \end{tikzcd}$$

Indeed, we again consider the square as a homotopy pushout in spaces over $B \times B'$. Then passing to fibres over (b, b') we obtain the pushout

$$\begin{tikzcd} E_b \times \{b'\} \cup \{b\} \times E'_{b'} \arrow[r] \arrow[d] & E_b \times E'_b \arrow[d] \\ \{(b,b')\} \arrow[r] & (E \wedge^{\mathrm{fw}} E')_{(b,b')} \end{tikzcd}$$

but then this pushout is also $E_b \wedge E'_{b'}$ as needed.

Likewise, there is a homotopy pushout

$$\begin{tikzcd} E \times E' \arrow[r] \arrow[d] & E \times B' \arrow[d] \\ B \times E' \arrow[r] & E \star^{\mathrm{fw}} E' \end{tikzcd}$$

In both cases, these formulas are obtained by noting that when $E \to B$ and $E' \to B'$ are (pointed) spherical fibrations, viewed as functors $B \to \mathrm{An}_{(*)}$, we form the composite

$$B \times B' \to \mathrm{An}_{(*)} \times \mathrm{An}_{(*)} \to \mathrm{An}_{(*)}$$

where the latter functor is given by the join in the unpointed case and the smash product in the pointed case. Recall then that the join of two spaces and the smash product of pointed spaces can be described as the pushouts

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \star Y \end{tikzcd} \qquad \begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y \end{tikzcd}$$

It follows that the fibrewise join or smash product are similar pushouts in the category $\mathrm{Fun}(B \times B', \mathrm{An}_{(*)})$. Using then the straightening-unstraightening equivalence gives the above descriptions.

In case $B = B'$ one can further pull back these constructions along the diagonal $B \to B \times B$; By abuse of notation the resulting operation will be denoted by $E \oplus E'$.

The basepoints of $\mathrm{BF}(d')$ and $\mathrm{BG}(d')$ then induce *stabilization maps*

$$\sigma^{d'} \colon \mathrm{BF}(d) \to \mathrm{BF}(d + d') \quad \text{and} \quad \sigma^{d'} \colon \mathrm{BG}(d) \to \mathrm{BG}(d + d')$$

which concretely send a (pointed) spherical fibration π to $\sigma^{d'}(\pi)$, the fibrewise join (or wedge) with the trivial (pointed) fibration $S^{d'-1} \to *$ (or $S^d \to *$).

2.12. **Remark** Recall that there are group homormophisms

$$\mathrm{O}(d) \to \mathrm{GL}_d(\mathbb{R}) \to \mathrm{Top}_0(d) \to \mathrm{G}(d) \to \mathrm{F}(d).$$

The evident maps

- $\mathrm{O}(d) \times \mathrm{O}(d') \to \mathrm{O}(d + d')$,
- $\mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_{d'}(\mathbb{R}) \to \mathrm{GL}_{d+d'}(\mathbb{R})$
- $\mathrm{Top}_0(d) \times \mathrm{Top}_0(d') \to \mathrm{Top}_0(d + d')$

induce maps on classifying spaces which are compatible with each other as well as with the maps (#).

2.13. **Lemma** *Let $p \colon E \to B$ and $p' \colon E' \to B'$ be spherical fibrations. Then there is a canonical equivalence*

$$\Sigma^{\mathrm{fw}}(E \star^{\mathrm{fw}} E') \simeq \Sigma^{\mathrm{fw}}(E) \wedge^{\mathrm{fw}} \Sigma^{\mathrm{fw}}(E').$$

Proof. In terms of the classifying functors $B \to \mathrm{An}$ and $B' \to \mathrm{An}$, the left hand side is given by the composite

$$B \times B' \to \mathrm{An} \times \mathrm{An} \xrightarrow{- \star -} \mathrm{An} \xrightarrow{\Sigma} \mathrm{An}_*$$

whereas the right hand sind is given by

$$B \times B' \to \mathrm{An} \times \mathrm{An} \xrightarrow{\Sigma\times\Sigma} \mathrm{An}_* \times \mathrm{An}_* \xrightarrow{-\wedge-} \mathrm{An}_*$$

so it suffices to prove an equivalence of functors $\mathrm{An} \times \mathrm{An} \to \mathrm{An}_*$

$$\Sigma(- \star -) \simeq \Sigma(-) \wedge \Sigma(-)$$

Thanks to Gijs Heuts for telling me about the following argument: First, consider the following two pushout squares

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma Y \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X \times \Sigma Y \end{tikzcd}$$

where the right hand square is obtained from the left by applying the left adjoint $X \times -$. We deduce that the cofibre of $X \times Y \to X$ is equivalent to the cofibre of $X \to X \times \Sigma Y$, which is $X_+ \wedge \Sigma Y$. Then consider the following large diagram of functors in X and Y consisting of pushout squares:

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] & \\ X \arrow[r] \arrow[d] & X \star Y \arrow[r] \arrow[d] & X_+ \wedge \Sigma Y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \wedge Y_+ \arrow[r] \arrow[d] & \Sigma(X \times Y) \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ & * \arrow[r] & \Sigma Y \arrow[r] & \Sigma(X \star Y) \end{tikzcd}$$

We have used here that $\Sigma(-)$ preserves pushouts. Moreover, in this diagram, the composite $\Sigma X \wedge Y_+ \to \Sigma(X \times Y) \to \Sigma X$ is induced by applying the functor $\Sigma X \wedge -$ to the map $Y_+ \to S^0$. Since this functor preserves cofibres, and since the cofibre of $Y_+ \to S^0$ is ΣY, we obtain the desired natural equivalence $\Sigma(X \star Y) \simeq \Sigma X \wedge \Sigma Y$. $\square$

2.14. **Remark** For pointed spaces X and Y, one can also describe $X \star Y$ in terms of $X \wedge Y$ as follows: Consider the diagram

$$\begin{tikzcd} * & * \arrow[l] \arrow[r] & * \\ X \arrow[u] \arrow[d] & X \vee Y \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d] \\ X & X \times Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

This is a diagram natural in pointed spaces X and Y. The colimit of this diagram may be computed in two ways: By taking the pushout of the pushout of rows, or by taking the pushout of the pushout of the columns. The former gives $X \star Y$ since the pushout of the middle row is a point, and the latter gives $\Sigma(X \wedge Y)$. In particular, we obtain a canonical equivalence

$$X \star Y \simeq \Sigma(X \wedge Y).$$

Expanding on this argument, we then also have a diagram of pushout squares as follows:

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] \arrow[d] & \Sigma(X \wedge Y) \arrow[r] \arrow[d] & \Sigma(X \wedge Y) \vee \Sigma Y \arrow[d] \\
* \arrow[r] & \Sigma(X \wedge Y) \vee \Sigma X \arrow[r] & \Sigma(X \times Y)
\end{tikzcd}$$

where we have used that the maps $X \to \Sigma(X \wedge Y) \leftarrow Y$ are null homotopic. We deduce a canonical equivalence

$$\Sigma(X \times Y) \simeq \Sigma(X \vee Y \vee X \wedge Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

2.15. **Lemma** *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be pointed spherical fibrations of rank d and d', respectively. Then there is a canonical equivalence* $\mathrm{Th}_*(p \wedge p') \simeq \mathrm{Th}_*(p) \wedge \mathrm{Th}_*(p')$. *In particular, If $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ are spherical fibrations of rank $d-1$ and $d'-1$ respectively, there is a canonical equivalence* $\mathrm{Th}(\pi \star \pi') \simeq \mathrm{Th}(\pi) \wedge \mathrm{Th}(\pi')$.

Proof. The "in particular" follows from Lemma 2.13 and Lemma 2.9. To see the first one, recall that there is a homotopy pushout square

$$\begin{tikzcd}
E \times B' \cup B \times E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\
B \times B' \arrow[r] & E \wedge^{\mathrm{fw}} E'
\end{tikzcd}$$

and that the cofibre of the lower horizontal map is $\mathrm{Th}_*(p \wedge p')$. Since the above square is a pushout, we may equivalently compute the cofibre of the upper horizontal map. To that end, note there is a further commutative diagram induced by the sections of the fibrations p and p', respectively:

$$\begin{tikzcd}
B \times B' \arrow[r] \arrow[d] & B \times E' \arrow[r, dashed] \arrow[d] & B_+ \wedge \mathrm{Th}_*(p') \arrow[d, dashed] \\
E \times B' \arrow[r] & E \times E' \arrow[r, dashed] & E_+ \wedge \mathrm{Th}_*(p')
\end{tikzcd}$$

We aim to compute the cofibre of the map from the pushout of this square to $E \times E'$. Exercise: This agrees with the cofibre of the induced map on cofibres; these are indicated in the above diagram. To see that they are as claimed, (similar to above) recall that the functor $B \times -$ preserves colimits. We therefore have a pushout square

$$\begin{tikzcd}
B \times B' \arrow[r] \arrow[d] & B \times E' \arrow[d] \\
B \times * \arrow[r] & B \times \mathrm{Th}(p')
\end{tikzcd}$$

showing that the horizontal cofibres are as claimed. Now, since $- \wedge \mathrm{Th}_*(p')$ also preserves pushouts of pointed spaces, the induced vertical cofibre on horizontal cofibres then sits in a pushout diagram

$$\begin{tikzcd} B_+ \wedge \mathrm{Th}_*(p') \arrow[r] \arrow[d] & * \wedge \mathrm{Th}_*(p') \simeq * \arrow[d] \\ E_+ \wedge \mathrm{Th}_*(p') \arrow[r] & \mathrm{Th}_*(p) \wedge \mathrm{Th}_*(p') \end{tikzcd}$$

showing the lemma. $\square$

2.16. **Corollary** *For a (pointed) spherical fibration $\pi\colon E \to B$ we have a canonical equivalence $\mathrm{Th}_{(*)}(\sigma^k\pi) = \mathrm{Th}(\pi \oplus \epsilon^{\oplus k}) \simeq S^k \wedge \mathrm{Th}_{(*)}(\pi)$. Here, $\epsilon^{\oplus k}$ denotes the trivial fibration $S^{k-1} \to B$. In particular, $\mathrm{Th}(\epsilon^{\oplus k}) \simeq S^k \wedge B_+$.*

Proof. The first assertion is a special case of Lemma 2.15 using that $\mathrm{Th}(S^{k-1} \to *) \simeq S^k$. The second is a special case of the first, since π is the k-fold suspension of the empty fibration $\varnothing = S^{-1} \to B$ whose Thom space is given by $B/\varnothing = B_+$. $\square$

We will be interested in the (co)homology of Thom spaces of spherical fibrations. The following result is the basis of our analysis of these (co)homologies.

2.17. **Proposition** *Let $d \geq 1$ and let $\pi\colon E \to B$ be a rank $(d-1)$ spherical fibration with B connected. Then the map $S^d \to \mathrm{Th}(\pi)$ induces an isomorphism on $H^d(-;\mathbb{F}_2)$. If π is oriented, the same is true for $H^d(-;\mathbb{Z})$ instead.*

2.18. **Terminology** The (unique) classes corresponding under these isomorphisms to the (cohomological) fundamental classes of S^d are denoted by $u(\pi)$ and are called (mod 2) *Thom classes.*

Proof of Proposition 2.17. Let us begin with the case $d = 1$. In the orientable case, π is the trivial fibration $S^0 \times B \to B$ whose Thom space is equivalent to $S^1 \wedge B_+$. The claim then follows from the assumption that B is connected. In the possibly non-oriented case, with $\mathbb{F}_2$-coefficients, use Exercise 2 (e) from Exercise Sheet 14 from Topology I: This implies a quasi-isomorphism between $C^*_{\mathrm{sing}}(\mathrm{Th}(\pi), *; \mathbb{F}_2)$ and $C^*_{\mathrm{sing}}(\Sigma B, *; \mathbb{F}_2)$.

We focus on $d \geq 2$ now. Recall that $\mathrm{Th}(\pi) = \mathrm{C}(\pi)$ is the mapping cone of the projection π. Since $d \geq 2$ and π is $(d-1)$-connected, the relative Hurewicz theorem, see [**?**, Remark 2.33] for the exact version we use here, says that the map

$$\pi_d(B,E) \otimes_{\mathbb{Z}\pi_1(E)} \mathbb{Z} \to H_d(B,E;\mathbb{Z}) \cong H_d(\mathrm{Th}(\pi);\mathbb{Z})$$

is an isomorphism, since π is $(d-1)$-connected, and likewise, the universal coefficient theorem gives an isomorphism

$$\pi_d(B,E) \otimes_{\mathbb{Z}\pi_1(E)} \mathbb{F}_2 \to H_d(B,E;\mathbb{Z}) \cong H_d(\mathrm{Th}(\pi);\mathbb{F}_2).$$

Moreover, these maps are natural for pullback squares, and so for every pullback diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r] & B \end{tikzcd}$$

there is an induced commutative diagram

$$\begin{array}{ccc} \pi_d(B', E') \otimes_{\mathbb{Z}\pi_1(E')} \mathbb{Z} & \xrightarrow{\cong} & H_d(\mathrm{Th}(\pi'); \mathbb{Z}) \\ \downarrow & & \downarrow \\ \pi_d(B, E) \otimes_{\mathbb{Z}\pi_1(E)} \mathbb{Z} & \xrightarrow{\cong} & H_d(\mathrm{Th}(\pi); \mathbb{Z}) \end{array}$$

In particular, this is so for $B' = \{b\} \to B$ in which case the right vertical map is the map $S^d \to \mathrm{Th}(\pi)$ from the statement. Now, the map $\pi_d(B', E') \to \pi_d(B, E)$ is an isomorphism – both are canonically equivalent to $\pi_{d-1}(S^{d-1})$. By the assumption that $d \geq 2$ we find that the map $\pi_1(E) \to \pi_1(B)$ is surjective, so under the orientability assumption, the action on $\mathbb{Z}$ is trivial and with $\mathbb{F}_2$-coefficients, the action is trivial in any case. The cohomological universal coefficient theorem then gives the proposition. □

2.19. **Remark** The proof above shows that if $\pi\colon E \to B$ is a rank $(d-1)$ spherical fibration which is not orientable, then $H_d(\mathrm{Th}(\pi); \mathbb{Z}) \cong \mathbb{Z}/2$.

2.20. **Corollary** *Given a spherical fibration* $\pi\colon E \to B$ *with Thom class* $u(\pi)$ *and a map* $f\colon B' \to B$*. Denote by* $\pi'\colon E' \to B'$ *the pullback fibration. Then* $\mathrm{Th}(f)^*(u(\pi)) = u(\pi')$ *is again a Thom class; here we denote by* $\mathrm{Th}(f)\colon \mathrm{Th}(\pi') \to \mathrm{Th}(\pi)$ *the induced map on Thom spaces.*

Proof. In the composite

$$H^d(S^d) \to H^d(\mathrm{Th}(\pi')) \to H^d(\mathrm{Th}(\pi))$$

the first map and the composite are isomorphisms by Proposition 2.17 and the images of the fundamental class are the Thom classes of π' and π, respectively. □

2.21. **Definition** (Euler classes) Let $\pi\colon E \to B$ be a spherical fibration of rank $d-1$. Let $z\colon B \to \mathrm{Th}(\pi)$ be the canonical map. We define the its mod 2 Euler class $\bar{e}(\pi) = z^*(u(\pi)) \in H^d(B; \mathbb{F}_2)$. If π is oriented, then we have an Euler class $e(\pi) = z^*(u(\pi)) \in H^d(B; \mathbb{Z})$ which lifts the mod 2 Euler class (hence the name).

2.22. **Lemma** *The (mod 2) Euler class is a characteristic class for)oriented(spherical fibrations.*

Proof. We need to show that for a pullback diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

of (oriented) spherical fibration, we have $e(\pi') = f^*(e(\pi))$. To that end, note that we have a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{f} & B \\ \downarrow{\scriptstyle z'} & & \downarrow{\scriptstyle z} \\ \mathrm{Th}(\pi') & \xrightarrow{\mathrm{Th}(f)} & \mathrm{Th}(\pi) \end{array}$$

so we may compute

$$e(\pi') = (z')^*(u(\pi')) = (z')^*\mathrm{Th}(f)^*(u(\pi)) = f^*z^*(u(\pi)) = f^*(e(\pi))$$

as needed. □

2.23. **Remark** As a consequence, we obtain the universal (mod 2) Euler classes: $\bar{e} \in H^d(\mathrm{BG}(d); \mathbb{F}_2)$ and $e \in H^d(\mathrm{BG}(d); \mathbb{Z})$. Via the maps

$$\begin{tikzcd} \mathrm{SO}(d) \arrow[r] \arrow[d] & \mathrm{STop}_0(d) \arrow[r] \arrow[d] & \mathrm{SG}(d) \arrow[d] \\ \mathrm{O}(d) \arrow[r] & \mathrm{Top}_0(d) \arrow[r] & \mathrm{G}(d) \end{tikzcd}$$

we then also obtain (mod 2) Euler classes for $\mathbb{R}^d$-bundles and vector bundles, simply by pulling back.

2.24. **Terminology** To have unifying names, let us denote by $\mathrm{MSG}(d)$ the Thom space of the universal oriented spherical fibration over $\mathrm{BSG}(d)$, and likewise $\mathrm{MG}(d)$ the pointed Thom space of the universal spherical fibration over $\mathrm{BG}(d)$. Then we find that $e = z^*(u)$ and $\bar{e} = z^*(\bar{u})$ where $u \in H^d(\mathrm{MSG}(d); \mathbb{Z})$ is the Thom class and $\bar{u} \in H^d(\mathrm{MG}(d); \mathbb{F}_2)$ is the mod 2 Thom class.

2.25. **Lemma** *The composites* $\mathrm{BF}(d-1) \to \mathrm{BG}(d) \xrightarrow{\bar{e}} K(\mathbb{F}_2, d)$ *and* $\mathrm{BSF}(d-1) \to \mathrm{BSG}(d) \xrightarrow{e} K(\mathbb{Z}, d)$ *vanish. In particular, the (mod 2) Euler class is an obstruction to "destabilizing" a spherical fibration.*

Proof. To see the "in particular" simply recall that the stabilization map $\mathrm{BG}(d-1) \to \mathrm{BG}(d)$ factors as the composite

$$\mathrm{BG}(d-1) \xrightarrow{\Sigma} \mathrm{BF}(d-1) \to \mathrm{BG}(d)$$

of the suspension map followed by the forgetful map. In particular, the Euler class also vanishes upon restriction along $\mathrm{BG}(d-1) \to \mathrm{BG}(d)$. Now, in general, consider $\pi\colon E \to B$ a spherical fibration with section $s\colon B \to E$ (e.g. the universal one over $\mathrm{BF}(d-1)$). By definition, there is a cofibre sequence

$$E \xrightarrow{\pi} B \xrightarrow{z} \mathrm{Th}(\pi)$$

and

$$\bar{e}(\pi) = z^*(\bar{u}(\pi)) = (\pi s)^*(z^*(\bar{u}(\pi))) = s^*(z\pi)^*(\bar{u}(\pi)) = 0$$

since the composite $z\pi$ is null. The same argument applies in the oriented case. □

Let us again consider the map forgetful map $\mathrm{BSF}(d-1) \to \mathrm{BSG}(d)$ and note that it is a simple map in the sense of [Lan24, Def. 5.13] since $\mathrm{BSF}(d-1)$ is simply connected. Its fibre is equivalent to S^{d-1}. Hence, there is a primary obstruction to finding a section of the forgetful map, i.e. a lift in the diagram

$$\begin{tikzcd} & \mathrm{BSF}(d-1) \arrow[d] \\ \mathrm{BSG}(d) \arrow[ur, dashed] \arrow[r, equal] & \mathrm{BSG}(d) \end{tikzcd}$$

and this primary obstruction is an element $\theta \in H^d(\mathrm{BSG}(d);\pi_{d-1}(S^{d-1})) \cong H^d(\mathrm{BSG}(d);\mathbb{Z})$; see e.g. [Lan24, §5].

2.26. **Lemma** *We have the equality $\theta = e$ in $H^d(\mathrm{BSG}(d);\mathbb{Z})$, that is, the Euler class is the primary obstruction for a spherical fibration to admit a section.*

Proof. As above, we identify $\pi_{d-1}(S^{d-1})$ with $\mathbb{Z}$. By construction, see [Lan24, Cor. 5.14], θ takes part in a commutative diagram

$$\begin{tikzcd}
\mathrm{BSF}(d) \arrow[r] \arrow[d, "\pi"] & * \arrow[r] \arrow[d] & \mathrm{BhAut}_*(K(\mathbb{Z},d-1)) \arrow[d] \\
\mathrm{BSG}(d) \arrow[r, "\theta"] & K(\mathbb{Z},d) \arrow[r] & \mathrm{BhAut}(K(\mathbb{Z},d-1))
\end{tikzcd}$$

which on vertical fibres induces the map $S^{d-1} \to K(\mathbb{Z},d-1)$ classifying the cohomological fundamental class of S^{d-1}. Passing to vertical cofibres, we obtain a map $\tilde{u}\colon \mathrm{Th}(\pi) \to K(\mathbb{Z},d)$. Restricted to along the map $S^d \to \mathrm{Th}(\pi)$, we obtain a map $S^d \to K(\mathbb{Z},d)$ which is obtained from the map $S^{d-1} \to K(\mathbb{Z},d-1)$ (the one induced on vertical fibres of the left square) by suspending and composing with the tautological map $\Sigma K(\mathbb{Z},d-1) \to K(\mathbb{Z},d)$. It follows that the element $\tilde{u} \in H^d(\mathrm{Th}(\pi);\mathbb{Z})$ restricts to the cohomological fundamental class of S^d; hence $\tilde{u} = u(\pi)$ is the Thom class of π. Hence, upon restricting along the map $\mathrm{BSG}(d) \to \mathrm{Th}(\pi)$, we obtain $\theta = e(\pi)$ as claimed. □

2.27. **Remark** Denote by F the fibre of the map $\mathrm{BSG}(d-1) \to \mathrm{BSG}(d)$. We discuss in the exercises that for $d \geq 4$, F is $(d-2)$-connected and that the canonical map $F \to S^{d-1}$ induced from the suspension map $\mathrm{BSG}(d-1) \to \mathrm{BSF}(d-1)$, induces an isomorphism on π_{d-1}. It follows that the primary obstruction for admitting a section of $\mathrm{BSG}(d-1) \to \mathrm{BSG}(d)$ equals the primary obstruction for admitting a section of $\mathrm{BSF}(d-1) \to \mathrm{BSG}(d)$ which is the Euler class.[2]

Furthermore, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{BSO}(d-1) \arrow[r] \arrow[d] & \mathrm{BSF}(d-1) \arrow[d] \\
\mathrm{BSO}(d) \arrow[r] & \mathrm{BSG}(d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{BO}(d-1) \arrow[r] \arrow[d] & \mathrm{BF}(d-1) \arrow[d] \\
\mathrm{BO}(d) \arrow[r] & \mathrm{BG}(d)
\end{tikzcd}$$

On homotopy fibres (in both cases) we obtain a map $S^{d-1} \to S^{d-1}$ which turns out to be an equivalence: it is equivalent to the map $\mathrm{SO}(d)/\mathrm{SO}(d-1) \to S^{d-1}$ induced by $\mathrm{SO}(d) \to \mathrm{SG}(d) \to S^{d-1}$ where the latter map is the action on a basepoint of S^{d-1}. In other words, the above squares are pullback diagrams, so that the left vertical fibrations are the spherical fibrations underlying the universal (oriented) bundle γ_d over $\mathrm{BSO}(d)$ and $\mathrm{BO}(d)$, respectively. Consequently, for an oriented rank d vector bundle E, the Euler class is the primary obstruction to obtaining an isomorphism $E \cong E' \oplus \epsilon$.

[2] Exercise: Determine the cases $d = 2, 3$ as well. E.g. $d = 2$ the suspension map is a map $S^1 \to \Omega^2_{\{1\}} S^2$. Which map is this?

In particular, if X is a d-dimensional CW complex, or more generally, has trivial cohomology in degrees $> d$, then the Euler class is the precise obstruction for the lifting problems

$$\begin{tikzcd} & \mathrm{BSO}(d-1) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{BSO}(d) \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{BSF}(d-1) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{BSG}(d) \end{tikzcd}$$

2.28. **Remark** It turns out that the map $\frac{\mathrm{Top}(d-1)}{\mathrm{O}(d-1)} \to \frac{\mathrm{Top}(d)}{\mathrm{O}(d)}$ is $d+1$-connected, see [KS77, Essay V §5] if $d \geq 6$. Equivalently, the square

$$\begin{tikzcd} \mathrm{BSO}(d-1) \arrow[r] \arrow[d] & \mathrm{BSTop}(d-1) \arrow[d] \\ \mathrm{BSO}(d) \arrow[r] & \mathrm{BSTop}(d) \end{tikzcd}$$

is $d+1$-cartesian (i.e. its total fibre is d-connected) so that for $d \geq 6$, the Euler class is also the primary obstruction for the lifting problem

$$\begin{tikzcd} & \mathrm{BSTop}(d-1) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{BSTop}(d) \end{tikzcd}$$

and the precise obstruction if X has trivial cohomology in degrees $> d$.

We continue with our aim to compute the cohomology of Thom spaces.

2.29. **Theorem** (Thom isomorphism) *Let $\pi\colon E \to B$ be a spherical fibration of rank $d-1$. Then the map*

$$H^*(B;\mathbb{F}_2) \to \widetilde{H}^{*+d}(\mathrm{Th}(\pi);\mathbb{F}_2), \quad x \mapsto x \cdot \bar{u}$$

given by the $H^*(B;\mathbb{F}_2)$-module multiplication on the mod 2 Thom class $\bar{u} \in \widetilde{H}^d(\mathrm{Th}(\pi);\mathbb{F}_2)$ is an isomorphism. Similarly, if π is oriented, then the map

$$H^*(B;\mathbb{Z}) \to \widetilde{H}^{*+d}(\mathrm{Th}(\pi);\mathbb{Z}), \quad x \mapsto x \cdot u$$

is an isomorphism, where $u \in \widetilde{H}^d(\mathrm{Th}(\pi);\mathbb{Z})$ is the Thom class.

That is, the reduced cohomology of a Thom space a spherical fibration of rank $d-1$ is free of rank one over $H^*(B)$ on a degree d generator (namely, the Thom class).

Proof. Let R be $\mathbb{F}_2$ or, in the oriented case, $\mathbb{Z}$. Consider the following situation for the Leray–Hirsch theorem:

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, equal] \\ * \arrow[r] & B \arrow[r, "\mathrm{id}"] & B \end{tikzcd}$$

Proposition 2.17 implies that we may apply the Leray–Hirsch theorem with the class $u \in H^d(B,E;R) \cong \widetilde{H}^d(\mathrm{Th}(\pi);R)$. Note that $H^*(*,S^{d-1};R) = \widetilde{H}^*(S^d;R) = R[d]$[3] so we obtain

[3]This denotes the graded R-module having R in degree d and 0 in all other degrees.

that the map
$$H^*(B;R)[d] \to \widetilde{H}^*(\mathrm{Th}(\pi);R)$$
is an isomorphism. The concrete formula is then obtained simply by spelling out the map in the Leray–Hirsch theorem. □

2.30. **Corollary** *Let $\pi\colon E \to B$ be an (oriented) rank $d-1$ spherical fibration. Then there exists long exact sequence* Gysin *sequence*
$$\cdots \to H^{n-1}(E) \to H^{n-d}(B) \xrightarrow{\cdot e(\pi)} H^n(B) \xrightarrow{\pi^*} H^n(E) \to H^{n+1-d}(B) \to \dots$$
where the coefficients can be arbitrary if π is oriented, and an $\mathbb{F}_2$-algebra if π is non-orientable.

Proof. We consider the diagram
$$\begin{tikzcd} \dots \arrow[r] & H^{n-1}(E) \arrow[r] & \widetilde{H}^n(\mathrm{Th}(\pi)) \arrow[r, "z^*"] & H^n(B) \arrow[r, "\pi^*"] & H^n(E) \arrow[r] & \dots \\ & & H^{n-d}(B) \arrow[u, "\cdot u(\pi)", "\cong"'] \arrow[ur, "\cdot e(\pi)"'] & & & \end{tikzcd}$$
the top horizontal sequence is long exact, since $E \to B \to \mathrm{Th}(\pi)$ is a cofibre sequence. The vertical maps are isomorphisms by the Thom isomorphism Theorem 2.29 and the diagonal maps are indeed given by multiplication with the Euler class, by definition of the Euler class. □

2.31. **Corollary** *Let $\pi\colon E \to B$ be an oriented rank $d-1$ spherical fibration with trivial Euler class, e.g. assume that π has a section. Then for all $n \geq 0$ there are short exact sequences*
$$0 \to H^n(B) \xrightarrow{\pi^*} H^n(E) \to H^{n+1-d}(B) \to 0$$

3. Characteristic classes

3.1. **Definition** (Stiefel–Whitney and Wu classes) Let $\pi\colon E \to B$ be a spherical fibration. We define its Stiefel–Whitney classes $w_n(\pi) \in H^n(B;\mathbb{F}_2)$ and Wu classes $v_n(\pi) \in H^n(B;\mathbb{F}_2)$ as the unique classes satisfying
$$w_n(\pi)\cdot u = \mathrm{Sq}^n(u) \quad \text{and} \quad v_n(\pi)\cdot u = \chi(\mathrm{Sq}^n)(u)$$
in $H^{d+n}(\mathrm{Th}(\pi);\mathbb{F}_2)$. Similarly as before, let us denote by $w(\pi)$ and $v(\pi)$ the *total* Stiefel–Whitney and Wu classes; they lie in the completed cohomology $H^*(\mathrm{Th}(\pi);\mathbb{F}_2)^\wedge$, where one completes at the ideal of positively graded elements.[4]

3.2. **Theorem** *The Stiefel–Whitney classes satisfy the following relations: Let $f\colon B' \to B$, $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ spherical fibrations. Then*
1. *Naturality: $f^*(w(\pi)) = w(f^*(\pi))$ and $f^*(v(\pi)) = v(f^*(\pi))$.*
2. *Triviality: $w_n(\pi) = 0$ if $n > \mathrm{rank}(\pi)$.*
3. *Non-triviality: $w_1(\gamma) = t$ where γ is the spherical fibration underlying the universal line bundle on $\mathbb{RP}^\infty$ and $t \in H^1(\mathbb{RP}^\infty;\mathbb{F}_2)$ is the non-trivial element.*

[4] This completed cohomology agrees with the usual cohomology ring for instance if positively graded element is nilpotent, as is the case if $\mathrm{Th}(\pi)$ has trivial cohomology above a fixed degree. The underlying graded abelian group of the completion is simply $\prod_n H^*(-;\mathbb{F}_2)$ and so the completion map in general is at least injective. In particular, proving equations between elements of the uncompleted cohomology within the completed cohomology really amounts to proving these equations in the uncompleted cohomology.

(4) *Stability:* $w(\pi) = w(\pi \oplus \epsilon^{\oplus k})$ *where* ϵ *denotes the trivial rank* $(k-1)$*-spherical fibration.*
(5) *Cartan formula:* $w(\pi \star \pi') = w(\pi) \times w(\pi') \in H^*(B \times B'; \mathbb{F}_2)^\wedge$.
(6) *Euler formula: For* $n = \mathrm{rank}(\pi)$, *we have* $w_n(\pi) = \bar{e}(\pi)$.

3.3. **Remark** The Cartan formula is equivalent to the statement that for each $n \geq 0$, we have $w_n(\pi \times \pi') = \sum_{i+j=n} w_i(\pi) \times w_j(\pi') \in H^*(B \times B'; \mathbb{F}_2)$. Moreover, it is also equivalent to the statement that if $B = B'$ then $w(\pi \oplus \pi') = w(\pi) \cdot w(\pi') \in H^*(B; \mathbb{F}_2)^\wedge$.

Proof of Theorem 3.2. (1): f induces a map $\mathrm{Th}(f^*(\pi)) \to \mathrm{Th}(\pi)$ which by Corollary 2.20 sends $u(\pi)$ to $u(f^*(\pi))$. The statement then follows from the naturality of the Steenrod squares. (2): $w_n(\pi) \cdot u(\pi) = \mathrm{Sq}^n(u(\pi)) = 0$ if $n > \mathrm{rank}(\pi) = |u(\pi)$. (3) By naturality and the fact that $H^1(\mathbb{RP}^\infty; \mathbb{F}_2) \to H^1(\mathbb{RP}^n; \mathbb{F}_2)$ is an isomorphism for $n \geq 1$ it suffices to show the same result for the $\mathbb{RP}^n$ in place of $\mathbb{RP}^\infty$. Note that γ is the C_2-covering map $S^n \to \mathbb{RP}^n$. Its mapping cone is equivalent to $\mathbb{RP}^{n+1}$, by the concrete analysis of the cell structure on $\mathbb{RP}^k$'s. Hence $\mathrm{Th}(\gamma_{|\mathbb{RP}^n}) \simeq \mathbb{RP}^{n+1}$ and the Thom class $u(\gamma)$ is the unique non-trivial element in degree 1. Moreover, $w_1(\gamma) = t$ is equivalent to $w_1(\gamma) \neq 0$, which by the Thom isomorphism is equivalent to the statement that $u^2 = \mathrm{Sq}^1(u) \neq 0$ which is true. (4): By Corollary 2.16, we have $\mathrm{Th}(\pi \oplus \epsilon^{\oplus k}) \simeq S^k \wedge \mathrm{Th}(\pi)$ and the Thom class $u(\pi \oplus \epsilon^{\oplus k})$ corresponds under this isomorphism to the k-fold suspension of the Thom class $u(\pi)$. The result then follows from the stability of the Steenrod squares. (5): First, under the equivalence $\mathrm{Th}(\pi \star \pi') \simeq \mathrm{Th}(\pi) \wedge \mathrm{Th}(\pi')$ the Thom class $u(\pi \star \pi')$ corresponds to $u(\pi) \wedge u(\pi')$. The result then follows from the Cartan formula for Steenrod squares. (6): By definition, we have $w_n(\pi) \cdot u(\pi) = \mathrm{Sq}^n(u(\pi)) = u(\pi)^2$ and $\bar{e}(\pi) \cdot u(\pi) = z^*(u(\pi)) \cdot u(\pi) = u(\pi)^2$ by inspection of the module multiplication used in the product $z^*(u(\pi)) \cdot u(\pi)$. The result then follows from the Thom isomorphism, as the map $- \cdot u(\pi)$ is in particular injective. ☐

3.4. **Remark** By the splitting principle for real vector bundles, the Stiefel–Whitney classes for real vector bundles are uniquely determined by the above properties. Moreover, for an oriented spherical fibration, the proof of part (6) gives the equality $e(\pi) \cdot u(\pi) = u(\pi)^2$. In particular, for an oriented spherical fibration π of rank $d - 1$ with d odd, we have $2e(\pi) = 0$.

3.5. **Remark** By the above, we have constructed a compatible system of maps $\mathbb{F}_2[w_1, w_2, \dots] \to H^*(\mathrm{BG}(d); \mathbb{F}_2)$ for all $d \geq 1$. Concretely, we claim that the diagram

$$\begin{tikzcd} & H^*(\mathrm{BG}(d); \mathbb{F}_2) \arrow[d] \\ \mathbb{F}_2[w_1, w_2, \dots] \arrow[ur] \arrow[r] & H^*(\mathrm{BG}(d-1); \mathbb{F}_2) \end{tikzcd}$$

commutes. This is a consequence of the stability of the Stiefel–Whitney classes, since the map $\mathrm{BG}(d-1) \to \mathrm{BG}(d)$ classifies the once stabilized universal fibration over $\mathrm{BG}(d-1)$. Note that the map constructed above factors as

$$\mathbb{F}_2[w_1, w_2, \dots] \to \mathbb{F}_2[w_1, \dots, w_d] \to H^*(\mathrm{BG}(d); \mathbb{F}_2).$$

It is a consequence of the exercises that the system

$$\cdots \to H^k(\mathrm{BG}(d+2); \mathbb{F}_2) \to H^k(\mathrm{BG}(d+1); \mathbb{F}_2) \to H^k(\mathrm{BG}(d); \mathbb{F}_2)$$

eventually consists of isomorphisms (i.e. for fixed k, once d is large enough, the maps are isomorphisms): Indeed, the forgetful map $\mathrm{BF}(d) \to \mathrm{BG}(d+1)$ and the stabilization map

$\mathrm{BF}(d) \to \mathrm{BF}(d+1)$ are d-connected and participate in a commutative diagram

$$\begin{tikzcd} \mathrm{BF}(d) \arrow[r] \arrow[d] & \mathrm{BF}(d+1) \arrow[d] \\ \mathrm{BG}(d+1) \arrow[r] & \mathrm{BG}(d+2) \end{tikzcd}$$

showing the claim. It follows that we have an isomorphism

$$H^*(\mathrm{BG};\mathbb{F}_2) \cong \lim_d H^*(\mathrm{BG}(d);\mathbb{F}_2)$$

so we have in fact constructed a map

$$\mathbb{F}_2[w_1, w_2, \dots] \to H^*(\mathrm{BG};\mathbb{F}_2).$$

Again, via pullback, this determines classes with the same name in $H^*(\mathrm{BTop};\mathbb{F}_2)$ and $H^*(\mathrm{BO};\mathbb{F}_2)$.

3.6. **Theorem** *The maps*

(1) $\mathbb{F}_2[w_1, \dots, w_d] \to H^*(\mathrm{BO}(d);\mathbb{F}_2)$, *and*
(2) $\mathbb{F}_2[w_1, w_2, \dots] \to H^*(\mathrm{BO};\mathbb{F}_2)$

are isomorphisms.

Proof. Similarly to the arguments in Remark 3.5, (1) implies (2) by letting d go to infinity (the map $\mathrm{BO}(d) \to \mathrm{BO}(d+1)$ is also d-connected). We now prove (1) by induction over d. In case $d = 1$, we have $\mathrm{BO}(1) = \mathbb{RP}^\infty$. By part (3) of Theorem 3.2, we get that the map $\mathbb{F}_2[w_1] \to H^*(\mathbb{RP}^\infty;\mathbb{F}_2)$ is indeed an isomorphism. Now let us consider the fibre sequence

$$S^{d-1} \to \mathrm{BO}(d-1) \xrightarrow{i} \mathrm{BO}(d)$$

which is the underlying spherical fibration of the universal bundle γ_d over $\mathrm{BO}(d)$ as noted in Remark 2.27. It follows from the Euler formual Theorem 3.2 (6) that the Euler class of this spherical fibration is given by w_d. We now consider the Gysin sequence associated to this fibration, which gives

$$\dots \to H^{n-d}(\mathrm{BO}(d);\mathbb{F}_2) \xrightarrow{\cdot w_d} H^n(\mathrm{BO}(d);\mathbb{F}_2) \to H^n(\mathrm{BO}(d-1) \to \dots$$

By induction, $H^*(\mathrm{BO}(d-1);\mathbb{F}_2)$ is polynomial on $w_1, \dots, w_{d-1}$, and in particular, the map $H^*(\mathrm{BO}(d);\mathbb{F}_2) \to H^*(\mathrm{BO}(d-1);\mathbb{F}_2)$ is surjective. It follows that the above long exact Gysin sequence splits up into a short exact sequences which participate in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^*(\mathrm{BO}(d);\mathbb{F}_2)[d] \arrow[r, "\cdot w_d"] & H^*(\mathrm{BO}(d);\mathbb{F}_2) \arrow[r, "i^*"] & H^*(\mathrm{BO}(d-1);\mathbb{F}_2) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{F}_2[w_1, \dots, w_d][d] \arrow[r, "\cdot w_d"] \arrow[u] & \mathbb{F}_2[w_1, \dots, w_d] \arrow[r, "p"] \arrow[u] & \mathbb{F}_2[w_1, \dots, w_{d-1}] \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

of $\mathbb{Z}$-graded abelian groups; Here $(-)[d]$ denotes the shift operator on $\mathbb{Z}$-graded abelian groups, i.e. such that $M[d]_n = M_{n-d}$. The right vertical map is an isomorphism by inductive assumption, and the map p admits a tautological section, the inclusion. It follows that the map i^* also admits a section s. We now show that the middle vertical map is surjective by induction over the degree. In degrees $< d$, the left most terms are trivial, so i^*, as well as the p is an isomorphism, so the map is in fact an isomorphism as follows from the (separate) inductive hypothesis that the right most vertical map is an isomorphism (in all degrees). Now let

$x \in H^n(\mathrm{BO}(d); \mathbb{F}_2)$. Then, by exactness of the upper sequence, we have $x = si^*(x) + w_d \cdot y$ for some y with $|y| < |x|$. Then $si^*(x)$ is in the image of the middle vertical map, and y is by the inductive assumption, so it follows that also x is in the image. Finally, we can use induction over the the degree yet again to see that in each degree, source and target of the middle vertical map are finite dimensional $\mathbb{F}_2$-vector spaces of the same dimension. It follows that the middle vertical map is an isomorphism as needed. □

Exercise. Compute $H^*(\mathrm{BSO}(d); \mathbb{F}_2)$ and $H^*(\mathrm{BSO}; \mathbb{F}_2)$.

3.7. **Remark** We have just proven, in particular, the following: The composite

$$\mathbb{F}_2[w_1, w_2, \dots] \to H^*(\mathrm{BG}; \mathbb{F}_2) \to H^*(\mathrm{BTop}; \mathbb{F}_2) \to H^*(\mathrm{BO}; \mathbb{F}_2)$$

is an isomorphism. It follows that the middle two terms contain $\mathbb{F}_2[w_1, \dots,]$ as a retract (in graded commutative rings). It turns out that neither $H^*(\mathrm{BG}; \mathbb{F}_2)$ nor $H^*(\mathrm{BTop}; \mathbb{F}_2)$ are polynomial on the Stiefel–Whitney classes. However, some things can be said from what we have achieved so far, relying on the following fact: There exists a space $\mathrm{B(G/O)}$ and a fibre sequence

$$\mathrm{BO} \to \mathrm{BG} \to \mathrm{B(G/O)}.$$

We have just argued that the map $\mathrm{BO} \to \mathrm{BG}$ induces a surjection on $\mathbb{F}_2$-cohomology. Hence, the Leray–Hirsch theorem gives an (additive) isomorphism

$$H^*(\mathrm{BG}; \mathbb{F}_2) \cong H^*(\mathrm{BO}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(\mathrm{B(G/O)}; \mathbb{F}_2).$$

This, however, is perhaps not the most clever way to compute $H^*(\mathrm{BG}; \mathbb{F}_2)$ fully, as it is not clear how to access the cohomology of $\mathrm{B(G/O)}$ directly. Similarly, Leray–Hirsch gives an (additive) isomorphism

$$H^*(\mathrm{BTop}; \mathbb{F}_2) \cong H^*(\mathrm{BO}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(\mathrm{B(Top/O)}; \mathbb{F}_2)$$

but again, it is not immediate how to access the cohomology of $\mathrm{B(Top/O)}$.

In fact, in both cases, the answers are known explicitly, see e.g. [BMM73, Theorem 9.9] for the computation of the Hopf algebra $H_*(\mathrm{BSTop}; \mathbb{F}_2)$ and [MM79, Theorem 3.45] for $H^*(\mathrm{BSG}; \mathbb{F}_2)$; the argument is, however, not along the lines indicated above; for instance, for BSTop, one rather uses the fibre sequence $\mathrm{SG} \to \mathrm{G/Top} \to \mathrm{BSTop} \to \mathrm{BSG}$ and what is called the (homological) Eilenberg–Moore spectral sequence.

We now compute more characteristic classes of various kinds of vector bundles. First, we treat the case of characteristic classes for complex vector bundles.

3.8. **Theorem** *There is a unique class $c_d \in H^{2d}(\mathrm{BU}; \mathbb{Z})$ whose restriction to $H^{2d}(\mathrm{BU}(d); \mathbb{Z})$ equals the Euler class of the universal bundle. The resulting maps $\mathbb{Z}[c_1, c_2, \dots] \to H^*(\mathrm{BU}; \mathbb{Z})$ and $\mathbb{Z}[c_1, \dots, c_d] \to H^*(\mathrm{BU}(d); \mathbb{Z})$ are isomorphisms.*

3.9. **Terminology** The classes $c_i \in H^{2i}(\mathrm{BU}; \mathbb{Z})$ are called the Chern classes. The *total Chern class* c is defined as the sum $c = \sum_{d \geq 0} c_d$.[5] The element c_0 is understood to be the unit of $H^*(\mathrm{BU}; \mathbb{Z})$.

[5]Similarly as before, this is an element in the completed cohomology of BU.

Proof. For the first part, we note that the map $\mathrm{BU}(d) \to \mathrm{BU}$ induces an isomorphism on $H^n(-;\mathbb{Z})$ for $n \leq 2d+1$. Indeed, consider the fibrations

$$S^{2d+1} \to \mathrm{BU}(d) \to \mathrm{BU}(d+1)$$

and its associated Gysin sequence

$$H^{n-2(d+1)}(\mathrm{BU}(d+1)) \xrightarrow{\cdot c_{d+1}} H^n(\mathrm{BU}(d+1)) \to H^n(\mathrm{BU}(d)) \to H^{n+1-2(d+1)}(\mathrm{BU}(d+1))$$

where we have denoted the Euler class by c_{d+1}. If $n \leq 2d+1$, then the two outer terms vanish and the middle map is an isomorphism as claimed. From here, we prove inductively, just as in Theorem 3.6 that the map $\mathbb{Z}[c_1, \dots, c_d] \to H^*(\mathrm{BU}(d);\mathbb{Z})$ is an isomorphism (for the induction start, we use that $\mathrm{BU}(1) = \mathbb{CP}^\infty$ indeed has polynomial cohomology ring on the Euler class of the tautological line bundle on $\mathbb{CP}^\infty$). The computation for $H^*(\mathrm{BU};\mathbb{Z})$ is then again a formal consequence of this, and the connectivity of the maps $\mathrm{BU}(d) \to \mathrm{BU}$. □

3.10. **Theorem** *The Chern classes satisfy the Cartan formula: If E, E' are complex vector bundles we have $c(E \oplus E') = c(E) \cdot c(E')$.*

Proof. Exercise. Consider the canonical map $\mathrm{BU}(d) \times \mathrm{BU}(d') \to \mathrm{BU}(d+d')$ classiying the exterior product $\gamma^d_{\mathbb{C}} \times \gamma^{d'}_{\mathbb{C}}$ of the universal bundles over $\mathrm{BU}(d)$ and $\mathrm{BU}(d')$. On cohomology, using Theorem 3.8 and Künneth, this map induces a map

$$\phi\colon \mathbb{Z}[c_1, \dots, c_{d+d'}] \to \mathbb{Z}[c^l_1, \dots, c^l_d, c^r_1, \dots, c^r_{d'}]$$

where the superscripts l and r indicate that the Chern classes are pulled back from the left or right product factor of $\mathrm{BU}(d) \times \mathrm{BU}(d')$. The aim is to show that this map sends the total Chern class c to the product of the total Chern classes $c^l \cdot c^r$. We prove this by induction over $d + d'$. The inductive start is the case $d = d' = 0$ which is clear. For the inductive step, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{BU}(d-1) \times \mathrm{BU}(d') \arrow[r] \arrow[d] & \mathrm{BU}(d) \times \mathrm{BU}(d') \arrow[d] \\
\mathrm{BU}(d-1+d') \arrow[r] & \mathrm{BU}(d+d').
\end{tikzcd}$$

On cohomology, for the top right composite we obtain the map

$$\mathbb{Z}[c_1, \dots, c_{d+d'}] \xrightarrow{\phi} \mathbb{Z}[c^l_1, \dots, c^l_d, c^r_1, \dots, c^r_{d'}] \to \mathbb{Z}[c^l_1, \dots, c^l_{d-1}, c^r_1, \dots, c^r_{d'}]$$

where the latter map is the quotient by the element c^l_d. For the bottom left composite, we obtain the map

$$\mathbb{Z}[c_1, \dots, c_{d+d'}] \to \mathbb{Z}[c_1, \dots, \hat{c}_d, c_{d+1}, \dots, c_{d+d'}] \to \mathbb{Z}[c^l_1, \dots, c^l_{d-1}, c^r_1, \dots, c^r_{d'}]$$

where $\hat{c}_d$ indicates that this polynomial generator is not present, the first map is the projection and the second map, by the inductive hypothesis, sends the total Chern class

$$c = 1 + c_1 + \dots c_{d-1} + c_{d+1} + \dots c_{d+d'}$$

to the product of the total Chern classes

$$(1 + c^l_1 + \dots c^l_{d-1}) \cdot (1 + c^r_1 + \dots c^r_{d'}).$$

We deduce that $\phi(c) \equiv c^l \cdot c^r \mod c^l_d$ and by the same argument with the roles of d and d' reversed, that $\phi(c) \equiv c^l \cdot d^r \mod c^r_{d'}$. Since c^l_d and $c^r_{d'}$ are coprime and the polynomial ring has unique factorizations, we find that $\phi(c) \equiv c^l \cdot c^r \mod c^l_d c^r_{d'}$, i.e. that

$$\phi(c) = c^l \cdot c^r + p c^l_d c^r_{d'}$$

for some $p \in \mathbb{Z}[c^l_1, \dots, c^l_d, c^r_1, \dots, c^r_{d'}]$. Note that ϕ is a map of graded rings, where the grading of each polynomial generator is 2. The highest degree term appearing in c is $c_{d+d'}$ which has degree $2(d+d')$, so the highest degree of a non-trivial homogenous summand of $\phi(c)$ is also $2(d+d')$. Since $|c^l_d c^r_{d'}| = 2(d+d')$, this implies that the degree of p is 0. In addition, $c_{d+d'} = e \in H^{2(d+d')}(\mathrm{BU}(d+d'))$. Since ϕ is the map induced by classifying the bundle $\gamma^d_{\mathbb{C}} \times \gamma^{d'}_{\mathbb{C}}$, we find that

$$\phi(e) = e(\gamma^d_{\mathbb{C}} \times \gamma^{d'}_{\mathbb{C}}) = e(\gamma^d_{\mathbb{C}}) \cdot e(\gamma^{d'}_{\mathbb{C}}) = c^l_d c^r_{d'}.$$

This shows that $p = 0$ and hence the theorem. □

3.11. **Remark** By the splitting principle for complex vector bundles, the Chern classes are uniquely determined by the these properties (naturality, stability, compatibility with the Euler class, and the Cartan formula).

3.12. **Lemma** *Under the map $H^*(\mathrm{BU}; \mathbb{Z}) \to H^*(\mathrm{BU}; \mathbb{Z}/2)$, the Chern class c_d is sent to w_{2d}.*

Proof. It suffices to show the same claim for $\mathrm{BU}(d)$ in place of BU. Then c_d is the Euler class e, and we have shown that the mod 2 reduction of the Euler class is the Stiefel–Whitney class w_{2d} in Theorem 3.2 (6). □

3.13. **Lemma** *For a complex vector bundle E denote by $\overline{E}$ its complex conjugate bundle. Then $c_d(\overline{E}) = (-1)^d c_d(E)$.*

Proof. It suffices to prove the wanted formula in $H^{2d}(\mathrm{BU}) \cong H^{2d}(\mathrm{BU}(d))$. Then the statement becomes one about the Euler class of the underlying orientable real vector bundle. Complex conjugation then corresponds to reversing the orientation of the underlying real vector bundle if the complex dimension is odd, and keeping the same orientation if d is even. Hence it suffices to note that $e(\bar{E}) = -e(E)$ where $\bar{E}$ denotes the same real vector bundle but with reversed orientation. □

3.14. **Definition** Consider the complexification map $\mathfrak{c}\colon \mathrm{BO} \to \mathrm{BU}$. For $d \geq 0$, we define the Pontryagin class $p_d \in H^{4d}(\mathrm{BO}; \mathbb{Z})$ to be $(-1)^d \mathfrak{c}^*(c_{2d})$. Similarly as before, let p denote the total Pontryagin class.

Concretely, this means that for $E \to B$ a real vector bundle, we have

$$p_d(E) = (-1)^d c_{2d}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4d}(B; \mathbb{Z}).$$

Exercise. Show that $e(TS^n) = \chi(S^n)$, i.e. that it is 0 when n is odd and 2 when n is even.

3.15. **Theorem** *The elements $\mathfrak{c}^*(c_{2d+1}) \in H^{4n+2}(\mathrm{BO}; \mathbb{Z})$ are (in general non-trivial) 2-torsion elements and we have isomorphisms*

(1) $H^*(\mathrm{BSO}(2n); \mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_n, e]/(e^2 = p_n) = \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_{n-1}, e]$, *and*
(2) $H^*(\mathrm{BSO}(2n+1); \mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_n, e]/e = \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_n]$.

Proof. To see the first claim, note that for E a real vector bundle, $E \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to its complex conjugate bundle. Therefore,

$$c_{2d+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = c_{2d+1}(\overline{E \otimes_{\mathbb{R}} \mathbb{C}}) = -c_{2d+1}(E \otimes_{\mathbb{R}} \mathbb{C})$$

showing that this element is 2-torsion. To see that these elements are non-trivial, let us compute $\mathrm{red}_2(\mathfrak{c}^*(c_d)) \in H^{2d}(\mathrm{BO}; \mathbb{F}_2)$ to be non-trivial. By Lemma 3.12, we have

$$\mathrm{red}_2(\mathfrak{c}^*(c_d)) = \mathfrak{c}^*(\mathrm{red}_2(c_d)) = \mathfrak{c}^*(w_{2d}).$$

Now, the elements $w_{2d} \in H^*(\mathrm{BU}; \mathbb{F}_2)$ are themselves pulled back from the cohomology of BO via the forgetful map $\mathrm{BU} \to \mathrm{BO}$. Denoting by γ the universal bundle over BO, we therefore obtain

$$\mathrm{red}_2(\mathfrak{c}^*(c_d)) = w_{2d}(\gamma \oplus \gamma) = \sum_{i=0}^{2d} w_i w_{2d-i} = w_d^2.$$

which is non-trivial as claimed.

We now prove the isomorphisms by induction. Recall that $\mathrm{SO}(2) = \mathrm{U}(1)$ so even $H^*(\mathrm{BSO}(2); \mathbb{Z}) \cong \mathbb{Z}[c_1] = \mathbb{Z}[e]$. Let γ be the universal oriented bundle over $\mathrm{SO}(2)$. Since γ is a complex bundle, we find

$$p_1(\gamma) = -c_2(\gamma \oplus \overline{\gamma}) = -(2c_2(\gamma) - c_1(\gamma)^2) = e^2$$

as needed. For fixed n, we now show the implication (1) $\Rightarrow$ (2). After that, we will show that statement (2) for fixed n implies statement (1) for $n+1$. Consider the fibre sequence

$$S^{2n} \to \mathrm{BSO}(2n) \to \mathrm{BSO}(2n+1)$$

and recall from the exercise sheets that the first map in this sequence classifies the tangent bundle of S^{2n}. We claim that the second left most map in the Gysin sequence

$$H^{2n}(\mathrm{BSO}(2n+1)) \to H^{2n}(\mathrm{BSO}(2n)) \to H^0(\mathrm{BSO}(2n+1)) \xrightarrow{\cdot e} H^{2n+1}(\mathrm{BSO}(2n+1))$$

identifies with the map

$$H^{2n}(\mathrm{BSO}(2n); \mathbb{Z}) \to H^{2n}(S^{2n}; \mathbb{Z}) \cong \mathbb{Z}$$

given by restriction along the map classifying the tangent bundle, and hence sends the Euler class to ± 2, see Exercise 1 Sheet 4, and the Pontryagin classes to 0. As a result, the Euler class $e \in H^{2n+1}(\mathrm{BSO}(2n+1); \mathbb{Z})$ is a non-trivial 2-torsion class. Working with $\mathbb{Z}[\frac{1}{2}]$ as coefficients, we deduce that the Euler class $e \in H^{2n+1}(\mathrm{BSO}(2n+1); \mathbb{Z}[\frac{1}{2}])$ vanishes. By Corollary 2.31, the Gysin sequence therefore splits into short exact sequences

$$0 \to H^*(\mathrm{BSO}(2n+1); \mathbb{Z}[\tfrac{1}{2}]) \to H^*(\mathrm{BSO}(2n); \mathbb{Z}[\tfrac{1}{2}]) \to H^{*-2n}(\mathrm{BSO}(2n+1); \mathbb{Z}[\tfrac{1}{2}]) \to 0.$$

From this, by induction over k, one deduces that the first map in this sequence identifies with the inclusion of the subring $\mathbb{Z}[\frac{1}{2}][p_1, \dots, p_n] \subseteq \mathbb{Z}[\frac{1}{2}][p_1, \dots, p_{n-1}, e]$. To see the claim, we consider the cofibre sequence

$$\mathrm{MSO}(2n+1) \to \Sigma\mathrm{BSO}(2n) \to \Sigma\mathrm{BSO}(2n+1)$$

part of the definition of the Thom space $\mathrm{MSO}(2n+1)$ of the universal bundle over $\mathrm{BSO}(2n+1)$. By construction, the map under investigation is given by the applying $H^{2n+1}(-; \mathbb{Z})$ to the first map in this cofibre sequence and then by applying the suspension isomorphism and the Thom isomorphism and target and source, respectively. Let F denote the fibre of $\Sigma\mathrm{BSO}(2n) \to \Sigma\mathrm{BSO}(2n+1)$. Then there are canonical maps

$$\mathrm{MSO}(2n+1) \to F \leftarrow S^{2n+1}$$

and the composite $S^{2n+1} \to F \to \Sigma\mathrm{BSO}(2n)$ is the suspension of the map $S^{2n} \to \mathrm{BSO}(2n)$ classifying the tangent bundle of S^{2n}. The homotopy excision theorem implies that both of the maps appearing in the above display induce isomorphisms of $H^{2n+1}(-;\mathbb{Z})$, showing the claim.

To finish the proof of the theorem, we then consider the fibre sequence

$$S^{2n-1} \to \mathrm{BSO}(2n-1) \to \mathrm{BSO}(2n)$$

and its associated Gysin sequence. By induction, we can deduce that the map $H^*(\mathrm{BSO}(2n);\mathbb{Z}[\frac{1}{2}]) \to H^*(\mathrm{BSO}(2n-1);\mathbb{Z}[\frac{1}{2}])$ is surjective (since the target of this map is generated solely by Pontryagin classes), so the Gysin sequence splits into short exact sequences of the kind

$$0 \to H^{*-2n}(\mathrm{BSO}(2n);\mathbb{Z}[\tfrac{1}{2}]) \xrightarrow{\cdot e} H^*(\mathrm{BSO}(2n);\mathbb{Z}[\tfrac{1}{2}]) \to H^*(\mathrm{BSO}(2n-1);\mathbb{Z}[\tfrac{1}{2}]) \to 0.$$

As before, one deduces inductively over the cohomological degree that the map

$$\mathbb{Z}[\tfrac{1}{2}][p_1,\dots,p_{n-1},e] \to H^*(\mathrm{BSO}(2n);\mathbb{Z}[\tfrac{1}{2}])$$

is an isomorphism. It then finally remains to show that $e^2 = p_n$. To that end, let γ be the universal oriented bundle over $\mathrm{BSO}(2n)$. Then we have

$$p_n(\gamma) = (-1)^n c_{2n}(\gamma \otimes \mathbb{C}) = (-1)^n e(\gamma \otimes \mathbb{C}).$$

Now, the underlying oriented vector bundle of $\gamma \otimes \mathbb{C}$ is isomorphic to $\gamma \oplus \gamma$ when n is even and $\gamma \oplus \bar{\gamma}$ when n is odd. Therefore, we find

$$p_n(\gamma) = (-1)^n e(\gamma) \cdot (-1)^n e(\gamma) = e(\gamma)^2$$

as needed. □

Let us now consider the map $\mathrm{BU} \xrightarrow{\mathfrak{u}} \mathrm{BO}$ classifying the underlying real bundle of the universal complex bundle $\gamma_{\mathbb{C}}$.

3.16. **Lemma** *We have* $\mathfrak{u}^*(p_d) = \sum\limits_{a=0}^{2d} (-1)^{a+d} c_a c_{2d-a} \in H^{4d}(\mathrm{BU};\mathbb{Z})$.

Proof. The composite

$$\mathrm{BU} \xrightarrow{\mathfrak{u}} \mathrm{BO} \xrightarrow{\mathfrak{c}} \mathrm{BU}$$

classifies the bundle $\gamma_{\mathbb{C}} \oplus \overline{\gamma}_{\mathbb{C}}$. It then follows from Lemma 3.13 and Theorem 3.10 that the composite

$$H^*(\mathrm{BU}) \xrightarrow{\mathfrak{c}^*} H^*(\mathrm{BO}) \xrightarrow{\mathfrak{u}^*} H^*(\mathrm{BU})$$

sends c_d to $\sum_{a=0}^{d} (-1)^a c_a c_{d-a}$. Since the first map sends c_{2d} to $(-1)^d p_d$, the lemma follows. □

3.17. **Remark** In the proof of Theorem 3.15, we have shown that the image of p_d under the reduction mod 2 map $H^{4d}(\mathrm{BO};\mathbb{Z}) \to H^{4d}(\mathrm{BO};\mathbb{F}_2)$ is given by w_{2d}^2.

3.18. **Remark** Recall that $\mathrm{BSO}(d) \to \mathrm{BO}(d)$ is a double cover. From Exercise Sheet 14 of the course [Lan23], we can deduce an isomorphism $H^*(\mathrm{BO}(d);\mathbb{Z}[\frac{1}{2}]) \cong H^*(\mathrm{BSO}(d);\mathbb{Z}[\frac{1}{2}])^{C_2}$. The C_2-action on $\mathrm{BSO}(d)$ comes from the orientation reversal, which acts by the identity on the Pontryagin classes and by a sign on the Euler class. Hence, we find

$$H^*(\mathrm{BO}(d);\mathbb{Z}[\tfrac{1}{2}]) \cong \mathbb{Z}[\tfrac{1}{2}][p_1,\dots,p_{\lfloor \frac{d}{2} \rfloor}] \quad \text{and} \quad H^*(\mathrm{BO};\mathbb{Z}[\tfrac{1}{2}]) \cong \mathbb{Z}[\tfrac{1}{2}][p_1,p_2,\dots].$$

3.19. **Remark** It turns out that all the homotopy groups of the simple space Top/O are finite abelian groups.[6] As a result, one can show (Exercise) that $H^*(\mathrm{Top/O};\mathbb{Q}) = \mathbb{Q}$. In particular, the Leray–Hirsch theorem applied to the fibration

$$\mathrm{Top/O} \to \mathrm{BTop} \to \mathrm{BO}$$

gives that the map $H^*(\mathrm{BTop};\mathbb{Q}) \to H^*(\mathrm{BO};\mathbb{Q})$ is an isomorphism. As a consequence, the rational Pontryagin classes have unique lifts to the cohomology of BTop, and the resulting map

$$\mathbb{Q}[p_1, p_2, \dots] \to H^*(\mathrm{BTop};\mathbb{Q})$$

is an isomorphism.

On the other hand, it also turns out that the homotopy groups of BG are all finite abelian groups (they agree up to a shift with the stable homotopy groups of spheres), so that the same exercise as above yields $H^*(\mathrm{BG};\mathbb{Q}) = \mathbb{Q}$. This suggests the correct fact that among vector bundles, euclidean bundles, and spherical fibrations, the characteristic classes of euclidean bundles is the richest and most complicated one.

3.20. **Remark** From the above, for every $d \geq 0$, we have a map $\mathbb{Q}[p_1, p_2, \dots] \to H^*(\mathrm{BTop}(d);\mathbb{Q})$. By Remark 3.18 this map fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}[p_1, p_2, \dots] & \longrightarrow & H^*(\mathrm{BTop}(d);\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathbb{Q}[p_1, \dots, p_{\lfloor \frac{d}{2} \rfloor}] & \longrightarrow & H^*(\mathrm{BO}(d);\mathbb{Q}) \end{array}$$

and it was conjectured for quite some time that the top horizontal map also factors through the lower left corner, i.e. that the rational Pontryagin classes $p_n(\gamma_d^{\mathrm{top}})$ of the universal euclidean bundle γ_d^{top} over $\mathrm{BTop}(d)$ vanish when $n > 2d$. In spectacular work, Michael Weiss disproved this conjecture (although he tried for many years to prove it!) [Wei21]. Recently, based on very different and in fact very classical techniques, Galatius–Randal-Williams even showed that the upper horizontal map in the above diagram is injective for $d \geq 6$ [GRW23].

4. Wu formulas

4.1. **Fact** Let π be a spherical fibration over a finite CW complex B. Then there exists a spherical fibration π^{-1} over B such that $\pi \oplus \pi^{-1}$ is the trivial spherical fibration. Up to stabilizing π^{-1}, this spherical fibration is unique, we refer to it as the inverse of π.

[6] In fact, these homotopy groups can be identified with the groups of exotic spheres, which have been calculated by Kervaire and Milnor (and this computation in particular reveals that they are finite). In fact, to see only the finiteness of these groups, one need not go through all of their work. Indeed, equivalently to the finiteness of the homotopy of Top/O on can show the map $\mathrm{G/O} \to \mathrm{G/Top}$ is a rational equivalence. To see this, one can use topological surgery theory to compute the homotopy groups of G/Top to be finitely generated, torsion in degrees not divisible by 4 and $\mathbb{Z}$ in degrees divisible by 4 – the space G/O has the same rational homotopy groups by Bott's computation of the homotopy of O and Serre's computation of the homotopy groups of G. Then it suffices to show that the map is non-trivial on homotopy in degrees divisible by 4. Concretely, this amounts to constructing suitable almost stably framed smooth manifolds with non-trivial signature. Such manifolds were constructed by Kervaire and Milnor in their work on exotic spheres.

I will refer to the following proposition as Wu's first formula. I am not sure that everyone calls it like that, but am sure that Wu was well aware of it. We will see two further Wu formula's later. As the name suggests, they are all due to Wen-Tsun Wu.

4.2. **Theorem** (Wu's first formula) *Let $\pi\colon E \to B$ be a spherical fibration over a finite CW complex B and let π^{-1} be its inverse. Then we have the following relation between then Steenrod operation, the Wu class, and the Stiefel–Whitney class:*

$$\mathrm{Sq}(v(\pi)) = w(\pi^{-1}).$$

4.3. **Remark** One can define the notion of a *stable spherical fibration* as a map $B \to \mathrm{BG} = \mathrm{colim}_n \mathrm{BG}(n)$.[7] Defining $B_n = B \times_{\mathrm{BG}} \mathrm{BG}(n)$, this amounts to giving a filtration on B, namely the family of subspaces B_n, together with rank $(n-1)$-spherical fibrations $\pi_n\colon E_n \to B_n$ and equivalences $(\pi_{n+1})_{|B_n} \simeq \pi_n \oplus \epsilon$. One can add stable spherical fibrations and any spherical fibration ξ of finite rank can then be viewed as a stable spherical fibration with constant filtration on the base space, by simply considering the family of iterations of fibrewise suspensions of ξ. Moreover, for a stable spherical fibration π the inverse π^{-1} always exists regardless of whether or not B is finite (in fact, this formally follows from the case for finite B) – this is a consequence of the fact that, under the operation of direct sum of stable spherical fibrations, BG is a *grouplike* H-space, in fact a group-like $\mathbb{E}_\infty$-space. Group-likeness here is itself a trivial consequence of the fact that BG is connected. Since the Stiefel–Whitney classes and the Wu classes are stable, they depend only on the underlying stable spherical fibration of a finite rank spherical fibration. Finally, for a stable spherical fibration π, the formula

$$\mathrm{Sq}(v(\pi)) = w(\pi^{-1})$$

always holds; indeed the very same argument as the one we are about to give applies – the only fact we need to believe is that π^{-1} exists. Note also that

$$1 = w(\epsilon) = w(\pi \oplus \pi^{-1}) = w(\pi) \cdot w(\pi^{-1})$$

so that we also have $w(\pi^{-1}) = w(\pi)^{-1}$.

Proof of Theorem 4.2. Let us compute $\mathrm{Sq}(v(\pi) \cdot u(\pi))$, where we recall that this is an element in the (completed) cohomology of $\mathrm{Th}(\pi)$. By definition of $v(\pi)$ we find

$$\mathrm{Sq}(v(\pi) \cdot u(\pi)) = \mathrm{Sq}[\chi(\mathrm{Sq})(u(\pi))] = u(\pi)$$

where the second equality was established last term, see [Lan24, Lemma 6.41]. On the other hand, using the compatibility of the Steenrod squares with the module multiplication $v(\pi) \cdot u(\pi)$ also established last term, see [Lan24, Lemma 6.24], we also have

$$\mathrm{Sq}(v(\pi) \cdot u(\pi)) = \mathrm{Sq}(v(\pi)) \cdot \mathrm{Sq}(u(\pi)) = [\mathrm{Sq}(v(\pi)) \cdot w(\pi)] \cdot u(\pi).$$

The Thom isomorphism for π then implies that

$$1 = \mathrm{Sq}(v(\pi)) \cdot w(\pi)$$

so the fact that $w(\pi^{-1}) = w(\pi)^{-1}$ implies the theorem. □

We now come to the second Wu formula. It is about the interaction of the Steenrod squares with the Stiefel–Whitney classes. Originally, this is due to Wen Tsün Wu [Wu50] who proved it for spherical fibrations underlying a real vector bundle. The following proof, which applies to general spherical fibrations, is due to Wu Chung Hsiang [Hsi63].

[7] This turns out to be a component of the group completion of $\coprod_{n\geq 0} \mathrm{BG}(n)$.

4.4. **Theorem** (Wu's second formula) *Let $\pi\colon E \to B$ be a spherical fibration. Then we have the following formula in $H^*(B;\mathbb{F}_2)$:*

$$\mathrm{Sq}^i(w_j) = \sum_{k=0}^{i} \binom{j+k-i-1}{k} w_{j+k}w_{i-k}.$$

Here, we use the convention that the binomial coefficient $\binom{a}{b}$ is the usual binomial coefficient if $a \geq b$, is 1 if $a = -1$ and $b = 0$, and is 0 otherwise. In particular, the right hand side vanishes by convention if $i > j$.

4.5. **Remark** For vector bundles $E \to B$ rather than spherical fibrations, one can argue as follows: By the splitting principle as discussed in [Lan24, Cor. 7.9], it suffices to treat the case where E is a sum of line bundles. One can then prove the result by induction over the rank of E.[8]

The proof we present here relies on the following arithmetic lemmata about binomial coefficients:

4.6. **Proposition** *Let $m \geq n \geq 0$ and $k > 0$. Then*

$$\binom{m}{n} + \binom{m+k}{n} \equiv \sum_{k \leq 2l \leq \min(2k, n+k)} \binom{m+k-l}{n+k-2l} \cdot \binom{l-1}{k-l} \mod 2$$

In particular, for $0 < a < b$ with $a + b \leq i \leq j$, we have

$$\binom{j-b-1}{i-a-b} + \binom{j-a-1}{i-a-b} \equiv \sum_{n=0}^{\lfloor i/2 \rfloor} \binom{j-n-1}{i-2n}\binom{n-a-1}{b-n} \mod 2$$

Similarly,

$$\binom{j-1}{i-a} + \sum_{n=0}^{\lfloor i/2 \rfloor} \binom{j-n-1}{i-2n}\binom{n-1}{a-n} \equiv \binom{j-a-1}{i-a} \mod 2$$

Proof sketch. The main statement is proven as follows: The statement to prove is a statement $S(m,n,k)$ depending on three numbers. Let $S(t)$ be the union of all statements $S(m,n,k)$ with $m \geq n \geq 0$ and $k \leq t$. One prove $S(t)$ by induction over t. The case $t = 1$ is the well-known identity

$$\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}.$$

Similarly, one shows $S(2)$ directly, and then in general $S(t)$ by induction. The "in particular" is the special case $m = j - a - 1$, $n = i - a - b$, $k = b - a$ and the "similarly" is the special case $m = j - a - 1$, $n = i - a$, $k = a$. I'll add the details here later. □

Proof sketch of Theorem 4.4. It will suffice to prove Wu's formula in $H^*(\mathrm{BG}(d);\mathbb{F}_2)$ for arbitrary d. So fix such a d once and for all and denote by u the Thom class of $\mathrm{MG}(d)$.

Let us define an ordering on pairs by setting $(i,j) \leq (i',j')$ if $j < j'$ or if $j = j'$ and $i \leq i'$. We aim to prove Wu's formula for $\mathrm{Sq}^i(w_j)$ by induction over the pair (i,j). The case $i = 1$ was shown on the exercise sheets. Then suppose that $\mathrm{Sq}^a(w_b)$ satisfies Wu's formula for all $(a,b) \leq (i-1,j)$ for $i-1 \geq 1$, the aim is to show that $\mathrm{Sq}^i(w_j)$ then also satisfies Wu's formula,

[8] The inductive step also requires some relations on sums of binomial coefficients..

so that the theorem follows by induction. We may assume that $i \leq j$, else both sides vanish as observed earlier. Now let's compute $\mathrm{Sq}^i\mathrm{Sq}^j(u)$. By definition, on the one hand we obtain

$$\mathrm{Sq}^i(w_j \cdot u) = \mathrm{Sq}^i(w_j) \cdot u + \sum_{n=1}^{i} \mathrm{Sq}^{i-n}(w_j) \cdot w_n \cdot u$$

Then Adem relations then give

$$\mathrm{Sq}^i\mathrm{Sq}^j(u) = \sum_{m=0}^{\lfloor i/2 \rfloor} \binom{j-m-1}{i-2m} \mathrm{Sq}^{i+j-m}\mathrm{Sq}^m(u).$$

We get

$$\begin{aligned}
\mathrm{Sq}^i(w_j) \cdot u &= \sum_{n=1}^{i} \mathrm{Sq}^{i-n}(w_j) \cdot w_n \cdot u + \sum_{m=0}^{\lfloor i/2 \rfloor} \binom{j-m-1}{i-2m} \mathrm{Sq}^{i+j-m}(w_m \cdot u) \\
&= \sum_{n=1}^{i} \mathrm{Sq}^{i-n}(w_j) \cdot w_n \cdot u + \sum_{m=0}^{\lfloor i/2 \rfloor} \sum_{t=0}^{i+j-m} \binom{j-m-1}{i-2m} \mathrm{Sq}^t(w_m) \cdot w_{i+j-m-t} \cdot u
\end{aligned}$$

and consequently

$$\mathrm{Sq}^i(w_j) = \sum_{n=1}^{i} \mathrm{Sq}^{i-n}(w_j) \cdot w_n + \sum_{m=0}^{\lfloor i/2 \rfloor} \sum_{t=0}^{i+j-m} \binom{j-m-1}{i-2m} \mathrm{Sq}^t(w_m) \cdot w_{i+j-m-t}$$

We may apply the inductive hypothesis to $\mathrm{Sq}^{i-n}(w_j)$ since $n > 0$; Similarly, $\mathrm{Sq}^t(w_m)$ is non-zero at most if $t \leq m \leq i/2$ so again, in these cases, the inductive hypothesis can be used. The first summand above then gives

$$\sum_{n=1}^{i} \sum_{k=0}^{i-n} \binom{j+k-i+n-1}{k} w_{j+k} w_{i-k} w_n$$

and the second summand above gives

$$\sum_{m=0}^{\lfloor i/2 \rfloor} \sum_{t=0}^{i+j-m} \sum_{l=0}^{t} \binom{j-m-1}{i-2m} \binom{j+l-t-1}{l} w_{j+l} w_{t-l} w_{i+j-m-t}$$

We now rewrite this as a sum of four types of products of SW-classes:

(A) $w_a w_b w_{i+j-a-b}$, with $0 < a < b < i$ and $a + b \leq i$,
(B) $w_a^2 w_{i+j-2a}$, with $0 < a \leq \lfloor i/2 \rfloor$,
(C) $w_a w_{i+j-a}$, with $0 < a \leq i$, and
(D) w_{i+j}

These types come with the following coefficients:

(A) $\binom{j-b-1}{i-a-b} + \binom{j-a-1}{i-a-b} + \sum_{n=0}^{\lfloor i/2 \rfloor} \binom{j-n-1}{i-2n}\binom{n-a-1}{b-n}$
(B) 0
(C) $\binom{j-1}{i-a} + \sum_{n=0}^{\lfloor i/2 \rfloor} \binom{j-n-1}{i-2n}\binom{n-1}{a-n}$,
(D) $\binom{j-i-1}{i}$

By Proposition 4.6 the coefficient in (A) vanishes and the coefficient of (C) is $\binom{j-a-1}{i-a}$. Unravelling, we find that $\mathrm{Sq}^i(w_j)$ satisfies the desired formula. $\square$

5. Poincaré duality complexes and Wu's formula

In this section, we will use some ∞-categorical language: Let $X \in \mathrm{An}^\omega$ be a compact anima (e.g. represented by a finite CW complex or more generally a finitely dominated CW complex). Denote by $r\colon X \to *$ the terminal map. Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ and denote by $\mathrm{Mod}(R)$ its ∞-category of modules in Sp. $\mathrm{Mod}(R)$ is presentable[9], stable[10], symmetric monoidal under $\otimes_R$ and the tensor functor $- \otimes_R -$ preserves colimits in each variable; $M \otimes_R -$ then has a right adjoint $\mathrm{map}_R(M, -)$. Given a map $R \to S$ in $\mathrm{CAlg}(\mathrm{Sp})$, the extension of scalars functor is a symmetric monoidal left adjoint $- \otimes_R S\colon \mathrm{Mod}(R) \to \mathrm{Mod}(S)$, its right adjoint is the restriction of scalars functor $\mathrm{Mod}(S) \to \mathrm{Mod}(R)$. This functor preserves colimits and has itself a right adjoint. The composite of $\mathrm{map}_R(M, -)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ with the forgetful functors $\mathrm{Mod}(R) \to \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{An}$ is equivalent to the mapping anima functor $\mathrm{Map}_{\mathrm{Mod}(R)}(M, -)\colon \mathrm{Mod}(R) \to \mathrm{An}$. In particular, the mapping anima in $\mathrm{Mod}(R)$ canonically refine to mapping spectra. This turns out to be a general fact: For a stable ∞-category $\mathcal{C}$, there is a unique lift of the functor $\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ to a functor $\mathrm{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ along the forgetful functor $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$.

Now given $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $X \in \mathrm{An}$ we have left and right Kan extension adjunctions

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left=2] & \mathrm{Fun}(X, \mathrm{Mod}(R)) \arrow[l, shift right=4, "r_!^R"'] \arrow[l, shift left=4, "r_*^R"] \end{tikzcd}$$

in which the unlabelled arrow is given by r_R^*, the restriction along $r\colon X \to *$. This functor is symmetric monoidal and in particular makes $\mathrm{Fun}(X, \mathrm{Mod}(R))$ into a $\mathrm{Mod}(R)$-module in Pr^{L}. Since $\mathrm{Mod}(R)$ is generated under colimits from its dualizable objects $\mathrm{Perf}(R)$, it follows that both of its adjoints, that is $r_!^R$ and r_*^R are canonically $\mathrm{Perf}(R)$-linear that is, are morphisms in $\mathrm{Mod}_{\mathrm{Perf}(R)}(\mathrm{Cat}_\infty^{\mathrm{st}})$, the ∞-category of $\mathrm{Perf}(R)$-modules in the ∞-category of stable ∞-categories.[11] Since $r_!$ is a left adjoint and hence also preserves colimits, in fact $r_!$ is $\mathrm{Mod}(R)$-linear, that is, a morphism in $\mathrm{Mod}_{\mathrm{Mod}(R)}(\mathrm{Pr}^{\mathrm{L}})$. The same holds for r_* in case it preserves colimits. In case $R = \mathbb{S}$, we have $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}}) = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is the ∞-category of presentable and stable ∞-categories and Sp-linearity is equivalent to preserving colimits, while $\mathrm{Perf}(\mathbb{S})$-linearity is equivalent to preserving finite colimits, i.e. simply being exact. Here, concretely $r_*^R = \lim_X$ and $r_!^R = \mathrm{colim}_X$. In case X is compact, r_* indeed preserves colimits, and for a

[9]That is, it is generated by filtered colimits from a small category, and is cocomplete. It follows that $\mathrm{Mod}(R)$ is also complete.

[10]That is, it is pointed (i.e. there exists an object which is both initial and terminal, we call such an object a zero object), it is semi-additive (i.e. the canonical map from a finite coproduct to a finite product, which uses that there is a zero object, is an equivalence, we call such finite (co)products finite sums), and a square is a pushout if and only if it is a pullback.

[11]In particular, $r_!$ and r_* satisfy a projection formula.

map $R \to S$ in CAlg(Sp), we have commutative diagrams

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left, "r_!^R"] \arrow[d] & \mathrm{Fun}(X, \mathrm{Mod}(R)) \arrow[l, shift left, "r_*^R"] \arrow[d] \\ \mathrm{Mod}(S) \arrow[r, shift left, "r_!^S"] & \mathrm{Fun}(X, \mathrm{Mod}(S)) \arrow[l, shift left, "r_*^S"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mod}(S) \arrow[r, shift left, "r_!^S"] \arrow[d] & \mathrm{Fun}(X, \mathrm{Mod}(S)) \arrow[l, shift left, "r_*^S"] \arrow[d] \\ \mathrm{Mod}(R) \arrow[r, shift left, "r_!^R"] & \mathrm{Fun}(X, \mathrm{Mod}(R)) \arrow[l, shift left, "r_*^R"] \end{tikzcd}$$

where the left vertical maps are the extension of scalar and the right vertical maps are the restriction of scalars. Here, we mean the vertical maps make the each diagrams involving $r_!$, r_*, and r^* commute (only the case of r_* requires the compactness of X). Indeed, for the left square and the case r_*^X, this uses that $r_* = \lim_X$ is a retract of a finite limit (since X is a retract of a finite anima), and that the extension of scalars functor preserves finite limits (since it is a left adjoint, it preserves in particular finite colimits, so the result follows from the stability of the categories $\mathrm{Mod}(R)$). For that reason, we will drop the super/subscript and merely right $r_*, r_!$ and r^*; in each case such a functor appears a fixed $R \in \mathrm{CAlg(Sp)}$ is therefore meant implicitly.

Using again that X is compact, a similar argument gives that $r_*\colon \mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Mod}(R)$ preserves colimits and that $r_!\colon \mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Mod}(R)$ preserves limits.

Moreover, for any $R \in \mathrm{CAlg(Sp)}$, the functor

$$\mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Fun}^{\mathrm{Mod}(R)}(\mathrm{Fun}(X, \mathrm{Mod}(R)), \mathrm{Mod}(R)), \quad \mathcal{F} \mapsto r_!(- \otimes \mathcal{F})$$

is an equivalence, where the superscript $\mathrm{Mod}(R)$ refers to $\mathrm{Mod}(R)$-linear functors; this is often referred to as Morita theory.[12] Using this equivalence, we make the following definition.

5.1. **Definition** The unique object $D_X^R \in \mathrm{Fun}(X, \mathrm{Mod}(R))$ such that $r_*(-) \simeq r_!(- \otimes_R D_X^R)$ is called the R-dualizing spectrum of X. For $R = \mathbb{S}$ we write D_X in place of $D_X^{\mathbb{S}}$ and call it the dualizing spectrum of X.

5.2. **Proposition** *Let $X \in \mathrm{An}^\omega$ be a compact anima and $R \in \mathrm{CAlg(Sp)}$. Then its suspension R-module $R[X]$ is dualizable in $\mathrm{Mod}(R)$ with dual given by $r_!(D_X^R)$.*

Proof. The functor $R[-]\colon \mathrm{An} \to \mathrm{Mod}(R)$ is left adjoint to the composite $\mathrm{Mod}(R) \to \mathrm{Sp} \to \mathrm{An}$ which preserves filtered colimits. Hence $R[-]$ preserves compact objects, so $R[X]$ is compact if X is. Exercise: The compact objects in $\mathrm{Mod}(R)$ are retracts of finite R-modules (that is, of modules that can be built by finite colimits from R itself). Hint: mimic the CW approximation and use the compactness of $R[X]$. Since dualizable objects are closed under finite (co)limits and retracts, we deduce that $R[X]$ is dualizable. Since $\mathrm{Mod}(R)$ is closed symmetric monoidal, the dual of a dualizable D object is given by the internal hom object $\mathrm{map}_R(D, R)$. Then we compute

$$\mathrm{map}_X(r^*(R), r^*(R)) = \mathrm{map}_R(R, r_*r^*(\mathbb{S})) = r_*r^*(R) = r_!(D_X^R)$$

as well as

$$\mathrm{map}_X(r^*(R), r^*(R)) = \mathrm{map}_R(r_!r^*(R), R) = \mathrm{map}(R[X], R). \qquad \square$$

[12] You have seen in the exercises that there is a canonical equivalence $\mathrm{Fun}(X, \mathrm{Mod}(R)) \simeq \mathrm{Mod}(R[\Omega X])$ if X is connected. Rewritten in this fashion, the above is then a special case of to the statement that $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Mod}(R), \mathrm{Mod}(S)) = \mathrm{BiMod}(R, S)$.

5.3. **Remark** Under the equivalences

$$r_!(D_X^R) \to \mathrm{map}_X(r^*(R), r^*(R)) \simeq \mathrm{map}(R[X], R)$$

the identity of $r^*(R)$ on the left hand side determines a map $c\colon R \to r_!(D_X^R)$; this map is by construction the unit map $R \to r_* r^*(R)$ followed by the equivalence $r_* r^*(R) \simeq r_!(D_X^R)$. On the right hand side, the identity map determines the map $R[X] \to R$ induced by $X \to *$.

5.4. **Remark** Since $r_!\colon \mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Mod}(R)$ preserves limits (and colimits) and the ∞-categories $\mathrm{Fun}(X, \mathrm{Mod}(R))$ are presentable for all X, we find that $r_!$ admits a left adjoint, or equivalently, is corepresentable. The next lemma identifies the corepresting object as D_X.

5.5. **Lemma** *The composite*

$$\mathrm{map}_X(D_X^R, -) \to \mathrm{map}_R(r_!(D_X^R), r_!(-)) \xrightarrow{c} r_!(-)$$

is an equivalence.

Proof. Consider the functor

$$[\mathrm{Fun}(X, \mathrm{Mod}(R))^\omega]^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{Mod}(R)}(\mathrm{Fun}(X, \mathrm{Mod}(R)), \mathrm{Mod}(R)), \quad \mathcal{F} \mapsto \mathrm{map}_X(\mathcal{F}, -)$$

which is well-defined since we restrict to compact objects of $\mathrm{Fun}(X, \mathrm{Mod}(R))$ in the source; here we use again the stability of $\mathrm{Mod}(R)$ and the resulting fact that $\mathrm{map}_X(\mathcal{F}, -)$ preserves *finite* colimits for arbitrary $\mathcal{F} \in \mathrm{Fun}(X, \mathrm{Mod}(R))$. This functor preserves limits (i.e. thought of as a functor in the variable $\mathcal{F}$, it sends colimits in $\mathrm{Fun}(X, \mathrm{Mod}(R))^\omega$ to limits). Under the Morita theory equivalence described above, this gives a (limit preserving) functor

$$T\colon [\mathrm{Fun}(X, \mathrm{Mod}(R))^\omega]^{\mathrm{op}} \to \mathrm{Fun}(X, \mathrm{Mod}(R))$$

described by $r_!(T(\mathcal{F}) \otimes_R -) \simeq \mathrm{map}(\mathcal{F}, -)$; For $\mathcal{F} = r^*(R)$ this simply recovers the existence of the dualizing spectrum $D_X = T(r^*(R))$. As a consequence of the (enriched) Yoneda lemma, T is fully faithful. Moreover, we claim that T in fact lands inside compact objects, giving rise to a functor

$$T\colon [\mathrm{Fun}(X, \mathrm{Mod}(R))^\omega]^{\mathrm{op}} \to \mathrm{Fun}(X, \mathrm{Mod}(R))^\omega$$

This for instance follows from the fact that the left hand side is generated (before taking op under finite colimits and retracts) by objects of the form $i_!(R)$ where $i\colon \{x\} \to X$ is the inclusion of a point and the computation that $T(i_!(R)) \simeq i_!(R)$. It then also follows that T is essentially surjective. In addition, we have

$$\mathrm{map}(\mathcal{E}, T(\mathcal{F})) \simeq r_!(T(E) \otimes_R T(F)) \simeq \mathrm{map}(\mathcal{F}, T(\mathcal{E}))$$

from which it follows that $T^2 \simeq \mathrm{id}$. Hence, $T(D_X) = T(T(r^*(R))) = R$, showing that there is a canonical equivalence $\mathrm{map}_X(D_X, -) \simeq r_!(-)$. To see that the map we claim is the equivalence, since it is a natural map and source and target commute with colimits, it suffices to show that the map is an equivalence when evaluated on $i_!(R)$ where $i\colon \{x\} \to X$ for arbitrary $x \in X$. From what we have discussed so far, the first map in this composite is then identified with a map from

$$r_!(T(D_X^R) \otimes i_!(R)) \simeq r_! i_!(i^*(T(D_X^R))) = i^*(T(D_X^R)) = R$$

to

$$\mathrm{map}_R(r_!(D_X^R, r_! i_!(R)) = \mathrm{map}_R(r_!(D_X^R), R) \simeq R[X]$$

by Proposition 5.2; under the above identifications, this map turns out to be the canonical map $R \simeq R[\{x\}] \to R[X]$. By Remark 5.3, under these equivalences, the second map in the composite under investigation is then induced by $X \to *$, showing that the composite is indeed an equivalence as claimed. $\square$

5.6. **Remark** The equivalence T discussed above is called *Costenoble–Waner duality.*[13] Moreover, in the proof above, we have used the following general projection formula for $f\colon X \to Y$ a map of anima (this is part of the statement that $f_!$ is in fact f^*-linear; where $f^*\colon \mathrm{Fun}(Y, \mathrm{Mod}(R)) \to \mathrm{Fun}(X, \mathrm{Mod}(R))$ is symmetric monoidal, and in particular makes the codomain a module over the domain): There is a canonical equivalence of bifunctors

$$f_!(-) \otimes_R - \simeq f_!(- \otimes f^*(-))\colon \mathrm{Fun}(X, \mathrm{Mod}(R)) \times \mathrm{Fun}(Y, \mathrm{Mod}(R)) \to \mathrm{Fun}(Y, \mathrm{Mod}(R)).$$

To see this, one notes that there is a canonical equivalence from right to left (using that f^* is symmetric monoidal and the unit map $\mathrm{id} \to f^*f_!$). Note that $\mathrm{Fun}(T, \mathrm{Mod}(R))$ is closed symmetric monoidal with internal hom object given by the pointwise mapping spectrum, i.e. $\mathrm{Hom}(\mathcal{F}, \mathcal{G})_t = \mathrm{map}_R(\mathcal{F}_t, \mathcal{G}_t)$. It follows that $f^*\colon \mathrm{Fun}(Y, \mathrm{Mod}(R)) \to \mathrm{Fun}(X, \mathrm{Mod}(R))$ is also closed symmetric monoidal, i.e. $f^*\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(f^*(\mathcal{F}), f^*(\mathcal{G}))$. From this, the projection formula follows formally by Yoneda: Simply map the putative equivalence into a test object.

Let us move on by recording the following:

5.7. **Lemma** *Let $X \in \mathrm{An}$ be an anima (not necessarily compact), $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $\mathcal{L} \in \mathrm{Fun}(X, \mathrm{Mod}(R))$. Then then composite*

$$\mathrm{map}(r_*(-), r_!(- \otimes \mathcal{L})) \to \mathrm{map}(r_*r^*(R), r_!(\mathcal{L})) \xrightarrow{\eta^*} r_!(\mathcal{L})$$

where $\eta\colon R \to r_*r^*(R)$ is the unit of the adjunction, is an equivalence.

Proof. Indeed, note that r_* is corepresented by $r^*(R)$, so the claim is essentially the (enriched) Yoneda lemma. $\square$

5.8. **Corollary** *Let $\theta\colon r_* \to r_!(- \otimes_R \mathcal{L})$ be a natural transformation and $c\colon R \to r_*r^*(R) \to r_!(\mathcal{L})$ the induced map. Then θ identifies with the composite*

$$r_*(-) = \mathrm{map}(r^*R, -) \to \mathrm{map}(\mathcal{L}, - \otimes \mathcal{L}) \to \mathrm{map}(r_!(\mathcal{L}), r_!(- \otimes \mathcal{L})) \xrightarrow{c^*} r_!(- \otimes \mathcal{L}).$$

Proof. By the above, it suffices to show that both θ and the displayed composite canonically agree upon evaluating on $r^*(R)$ and furthermore restricting along the unit map $R \to r_*r^*(R)$. For the former we obtain the map c (by definition). For the latter, by Remark 5.3, under the equivalence $r_*r^*(R) \simeq \mathrm{map}(r^*(R), r^*(R))$, the unit map is sent to the identity of $r^*(R)$, which is sent to the identity of $r_!(\mathcal{L})$, precomposed with c, giving the desired result. $\square$

5.9. **Remark** For any $\mathcal{M}, \mathcal{L} \in \mathrm{Fun}(X, \mathrm{Mod}(R))$, we have in particular constructed above a map

$$r_*(\mathcal{M}) \to \mathrm{map}(r_!(\mathcal{L}), r_!(\mathcal{M} \otimes \mathcal{L}))$$

or equivalently, a map

$$r_*(\mathcal{M}) \otimes r_!(\mathcal{L}) \to r_!(\mathcal{M} \otimes \mathcal{L}).$$

This map is a general form of the cap product.

[13]Perhaps this is only classically done so in case $R = \mathbb{S}$, but we shall use the same terminology for all R.

5.10. **Remark** For an anima X, the cap product was constructed in Topology II [Win24] as the map induced by the composite

$$C^*(X) \otimes_{\mathbb{Z}} C_*(X) \to C^*(X) \otimes_{\mathbb{Z}} C_*(X) \otimes_{\mathbb{Z}} C_*(X) \to C_*(X)$$

on homology, where the first map is induced the diagonal, and the second by the evaluation map $C^*(X) \otimes C_*(X) \to \mathbb{Z}$. Exercise: Show that this map agrees with the one we've defined above in case $\mathcal{M} = \mathcal{L} = r^*(\mathbb{Z})$.

5.11. **Remark** To make more explicit the above map being a cap product, let us also recall explicitly the notion of (co)homology with coefficients in local coefficient systems for an anima X. A local coefficient system is defined to be a functor $X \to \mathrm{Ab}$, where we think of X as an ∞-groupoid. Since Ab is an ordinary category, the universal property of Postnikov truncations shows that a local coefficient system is the same thing as a functor $\tau_{\leq 1}(X) \to \mathrm{Ab}$ and $\tau_{\leq 1}(X)$ is just the usual fundamental groupoid of X. In particular, a local coefficient system in our definition is really the same thing as a local coefficient system in classical terms. Now, recall that there are functors $\mathrm{Ab} \subseteq \mathrm{Mod}_{\mathrm{Sp}}(\mathbb{Z}) \to \mathrm{Sp}$, the first and the composite of which are in fact fully faithful. In particular, a local coefficient system may be viewed as a functor $\mathcal{M}\colon X \to \mathrm{Mod}(\mathbb{Z})$[14] i.e. as an object of $\mathrm{Fun}(X, \mathrm{Mod}(\mathbb{Z}))$. We then define

$$C^*(X;\mathcal{M}) = r_*(\mathcal{M}) = \lim_X \mathcal{M} \quad \text{and} \quad C_*(X;\mathcal{M}) = r_!(\mathcal{M}) = \operatorname*{colim}_X \mathcal{M}.$$

Moreover, we set

$$H^k(X;\mathcal{M}) = \pi_{-k}(r_*(\mathcal{M})) \quad \text{and} \quad H_k(X;\mathcal{M}) = \pi_k(r_!(\mathcal{M})).$$

These definitions coincide with the more classical notion of singular homology with coefficients in a local coefficient system $\mathcal{M}\colon X \to \mathrm{Ab}$, obtained by adjusting the definition of the singular chain complex appropriately. For instance, let us argue here that for connected X, we have

$$C_*(X;\mathcal{M}) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{M}$$

a formula which is also satisfied by the classical definition. Indeed, since $\mathcal{M}\colon X \to \mathrm{Ab}$, we find that $\mathcal{M}$ factors as $X \to \tau_{\leq 1}(X) \simeq B\pi_1(X) \to \mathrm{Ab} \subseteq \mathrm{Mod}(\mathbb{Z})$. We may then compute the colimit over X by first left Kan extension along $t\colon X \to B\pi_1(X)$, and the further left Kan extending the result along $B\pi_1(X) \to *$. Since the fibre of $X \to B\pi_1(X)$ is given by $\widetilde{X}$, we have

$$t_!(\mathcal{M}) = \operatorname*{colim}_{\widetilde{X}} \mathcal{M} = C_*(\widetilde{X}) \otimes_{\mathbb{Z}} \mathcal{M}$$

where we think of $\mathcal{M}$ as a constant coefficient system on $\widetilde{X}$. This object is naturally acted upon by $\pi_1(X)$ using the geometric action of $\pi_1(X)$ on $\widetilde{X}$ and the given action on $\mathcal{M}$. Then we have

$$r_!(\mathcal{M}) \simeq \operatorname*{colim}_{B\pi_1(X)} \operatorname*{colim}_{\widetilde{X}} \mathcal{M} \simeq \operatorname*{colim}_{B\pi_1(X)} C_*(\widetilde{X}) \otimes_{\mathbb{Z}} \mathcal{M} \simeq C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{M}$$

as claimed. Similarly, one finds

$$r_*(\mathcal{M}) = \mathrm{map}_{\mathbb{Z}[\pi_1(X)]}(C_*(\widetilde{X}), \mathcal{M}).$$

We also record the following compatibility of the dualizing spectrum with the coefficient commutative algebra:

[14]From here on, when we write $\mathrm{Mod}(\mathbb{Z})$ we always mean $\mathrm{Mod}_{\mathrm{Sp}}(\mathbb{Z})$, i.e. $\mathbb{Z}$-modules in spectra, not $\mathbb{Z}$-modules in abelian groups.

5.12. **Lemma** *For every $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $X \in \mathrm{An}^\omega$, there is a canonical equivalence $R \otimes_{\mathbb{S}} D_X \simeq D_X^R$.*

Proof. For $\mathcal{F} \in \mathrm{Fun}(X, \mathrm{Mod}(R))$, we have

$$r_!^R(\mathcal{F} \otimes_R R \otimes_{\mathbb{S}} D_X) = r_!^R(\mathcal{F} \otimes_{\mathbb{S}} D_X) = \operatorname*{colim}_X(\mathcal{F} \otimes_{\mathbb{S}} D_X) \simeq \lim_X(\mathcal{F}) \simeq r_*^R(\mathcal{F})$$

since the restriction of scalars functor is compatible with $\lim_X$ and colim_X. □

Recall now that $\mathrm{Pic}(\mathbb{S})$ denotes the groupoid of $\otimes$-invertible spectra. On the exercise sheet, we show that $\pi_0(\mathrm{Pic}(\mathbb{S})) = \mathbb{Z}$. Moreover, there is an equivalence of anima $\mathrm{Pic}(\mathbb{S}) = \mathbb{Z} \times \mathrm{BG}$. We write $\mathrm{Pic}^+(\mathbb{S})$ for the anima $\mathbb{Z} \times \mathrm{BSG}$, so that there is a map $\mathrm{Pic}^+(\mathbb{S}) \to \mathrm{Pic}(\mathbb{S})$. The latter classifies stable spherical fibrations (of arbitrary virtual rank) and the former classifies oriented spherical fibrations (of arbitrary virtual rank).

5.13. **Definition** A compact anima X is called a Poincaré duality complex if

$$D_X \in \mathrm{Fun}(X, \mathrm{Pic}(\mathbb{S})) \subseteq \mathrm{Fun}(X, \mathrm{Sp}),$$

i.e. if D_X takes values in invertible spectra. In this case, D_X is called the Spivak normal fibration of X.[15] Its inverse $T_X = D_X^{-1}$ is called the Spivak tangent fibration of X. If X is a PD complex, its *dimension* is $-\mathrm{rk}(D_X)$ where $\mathrm{rk}(D_X)$ is the function $\pi_0(X) \to \pi_0(\mathrm{Pic}(\mathbb{S})) \cong \mathbb{Z}$ induced by D_X. We say that X is *oriented* if D_X comes equipped with a lift through $\mathrm{Pic}^+(\mathbb{S}) \to \mathrm{Pic}(\mathbb{S})$.

For a d-dimensional PD complex X, write $\omega_X = \Sigma^d D_X^{\mathbb{Z}}$ so that $\pi_*(r_!(\omega_X)) = H_*(X; \omega_X)$ is the homology with local coefficients. Recall that ω_X is pointwise infinite cyclic, and $w_1 \colon \pi_1(X) \to \mathbb{Z}/2\mathbb{Z} = \mathrm{Aut}(\mathbb{Z})$ determines ω_X as a $\mathbb{Z}\pi_1(X)$-module. The following lemma explains why Poincaré duality complexes are called Poincaré duality complexes:

5.14. **Lemma** *Let X be a connected Poincaré duality complex of dimension d. Then the unit map $\mathbb{S} \to r_* r^*(\mathbb{S}) \simeq r_!(D_X)$ determines, via the Hurewicz homomorphism, a class $[X] \in H_d(X; \omega_X)$ capping with which induces an isomorphism*

$$H^*(X; \mathcal{M}) \xrightarrow{\cong} H_{d-*}(X; \mathcal{M} \otimes_{\mathbb{Z}} \omega_X)$$

for any local coefficient system $\mathcal{M} \in \mathrm{Fun}(X, \mathrm{Ab})$. An orientation on X gives an equivalence $\omega_X = r^*\mathbb{Z}$ so that in this case, there is Poincaré duality without twisted coefficients. Moreover, $D_X^{\mathbb{F}_2} = r^*(\mathbb{F}_2)$ is always true, so there is then Poincaré duality without twisted coefficients.

Proof. The Hurewicz image of the unit map $\mathbb{S} \to r_* r^*(\mathbb{S}) \simeq r_!(D_X)$ is a map $\mathbb{Z} \to r_!(D_X) \otimes \mathbb{Z} \simeq r_!(D_X^{\mathbb{Z}}) = \Omega^d \omega_X$ by Lemma 5.12 and the definition of ω_X. This map gives an element $[X] \in H_d(X; \omega_X)$. Corollary 5.8 and Remark 5.9 then imply that cap product with $[X]$ is precisely the equivalence

$$C^*(X; \mathcal{M}) = r_*(\mathcal{M}) \simeq r_!(\mathcal{M} \otimes_{\mathbb{Z}} \Omega^d(\omega_X)) = \Omega^d C_*(X; \mathcal{M} \otimes_{\mathbb{Z}} \omega_X)$$

of the definition of $D_X^{\mathbb{Z}} = \Omega^d \omega_X$. □

[15] This is because it is a theorem of Spivak that a space X whose (co)homology satisfies Poincaré duality for all local coefficient systems admits a spherical fibration which turns out to be the dualizing spectrum of X.

By Lemma 5.14, for a connected PD complex of dimension d, we obtain that the map

$$H^k(X;\mathbb{F}_2) \to \mathrm{Hom}_{\mathbb{F}_2}(H^{n-k}(X;\mathbb{F}_2),\mathbb{F}_2), x \mapsto (y \mapsto \langle xy,[X]\rangle)$$

is an isomorphism. Then we notice that Sq^k canonically gives rise to an element in the right hand side of this isomorphism. The following definition then follows standard terminology for manifolds:

5.15. **Definition** Let X be a connected Poincaré duality complex. We define its Wu classes $v_i(X)$ as the unique class in $H^i(X;\mathbb{F}_2)$ such that for all $x \in H^{n-i}(X;\mathbb{F}_2)$, one has

$$\langle v_i(X)\cdot x,[X]\rangle = \langle \mathrm{Sq}^i(x),[X]\rangle.$$

Moreover, we define its Stiefel–Whitney classes $w_i(X)$ as the Stiefel–Whitney classes $w_i(T_X)$ of its Spivak tangent fibration.

5.16. **Remark** We note that if $f\colon X \to Y$ is an equivalence between compact anima, then $f^*(D_Y) \simeq D_X$, so that $f^*(w(Y)) = w(X)$ and $f^*(v(Y)) = v(X)$. In particular, X is a PD complex if and only if Y is a PD complex, and the Stiefel–Whitney classes and Wu classes are invariants under homotopy equivalences.

5.17. **Theorem** (Wu's third formula) *Let X be connected Poincaré duality complex of dimension d. Then $v(D_X) = v(X)$. In particular, we have*

$$\mathrm{Sq}(v(X)) = w(X).$$

Proof. Denote by $v(X)$ the total Wu class of X, so that we need to show $v(X) = v(D_X)$. By definition, we find that for all $x \in H^*(X;\mathbb{F}_2)$, we have

$$\langle \mathrm{Sq}(x),[X]\rangle = \langle v(X)\cdot x,[X]\rangle$$

and this equation characterizes $v(X)$ uniquely. So let us show that $v(D_X)$ also satisfies the equation. On the exercise sheets, we show that there is a preferred equivalence $\phi^{-1}\colon M(D_X)\otimes \mathbb{F}_2 \simeq X\otimes\mathbb{F}_2$ implementing on cohomology the Thom isomorphism (which we recall is $H^*(X;\mathbb{F}_2)$-linear). This equivalence is in addition compatible with the evaluation pairing $\langle -,-\rangle$ between homology and cohomology. By Lemma 5.14 with $\mathbb{F}_2$-coefficients, we find that $\phi[X] \in H_0(M(D_X);\mathbb{F}_2)$ is the image of $[1] \in H_0(\mathbb{S};\mathbb{F}_2) \cong \mathbb{F}_2$ under the map $c\colon \mathbb{S} \to M(D_X)$ part of the definition of the dualizing spectrum. Then we can simply compute:

$$\begin{aligned}
\langle v(D_X)\cdot x,[X]\rangle &= \langle \phi(v(D_X)\cdot x),\phi[X]\rangle\\
&= \langle \phi(v(D_X))\cdot x, c_*[1]\rangle\\
&= \langle \mathrm{Sq}^{-1}(u(D_X))\cdot x, c_*[1]\rangle\\
&= \langle c^*(\mathrm{Sq}^{-1}(u(D_X)\cdot \mathrm{Sq}(x))),[1]\rangle\\
&= \langle \mathrm{Sq}^{-1}c^*(\phi(\mathrm{Sq}(x))),[1]\rangle\\
&= \langle c^*(\phi(\mathrm{Sq}(x))),[1]\rangle\\
&= \langle \phi(\mathrm{Sq}(x)), c_*[1]\rangle\\
&= \langle \phi(\mathrm{Sq}(x)),\phi[X]\rangle\\
&= \langle \mathrm{Sq}(x),[X]\rangle
\end{aligned}$$

Here, the second equality uses that the Thom isomorphism is $H^*(X;\mathbb{F}_2)$-linear, the third equality is from the definition of the Wu class of D_X (here, $u(D_X)$ is the Thom class of

D_X. Now, since $H^*(\mathbb{S};\mathbb{F}_2)$ is trivial for $* \neq 0$, we find that $\mathrm{Sq}^{-1}\colon H^*(\mathbb{S};\mathbb{F}_2) \to H^*(\mathbb{S};\mathbb{F}_2)$ is simply the identity, which gives the 6th equality. The other equalities are immediate, showing the first assertion of the theorem. The "in particular" is now a direct consequence of Theorem 4.2. □

5.18. **Corollary** *Let X be a d-dimensional PD complex and $2i > d$. Then $v_i(D_X) = 0$.*

Proof. Indeed, $v_i(D_X) = v_i(X)$ represents Sq^i on $H^{d-i}(-;\mathbb{F}_2)$ which vanishes if $i > d-i$. □

Let us spell out some direct consequences of the above Wu formula.

5.19. **Example** Let X be a PD complex. Then

(1) $w_1(X) = v_1(X)$; In particular, X is orientable of dimension d if and only if the map $\mathrm{Sq}^1\colon H^{d-1}(X;\mathbb{F}_2) \to H^d(X;\mathbb{F}_2)$ is trivial.
(2) $v_2(X) = w_1^2(X) + w_2(X)$; In particular, if X is orientable of dimension d, then $w_2(X)$ vanishes if and only if $\mathrm{Sq}^2\colon H^{d-2}(X;\mathbb{F}_2) \to H^d(X;\mathbb{F}_2)$ is trivial.
(3) $w_{2d}(X) = v_d^2(X)$ if X is of dimension $2d$; Indeed, the Wu formula gives
$$w_{2d}(X) = v_{2d}(X) + \cdots + \mathrm{Sq}^d(v_d(X)) + \cdots + \mathrm{Sq}^{2d}v_0(X)$$
where the term in the middle is $v_d^2(X)$, all terms to the left of it vanish by Corollary 5.18 and all terms to the right of it vanish as the degree of the Steenrod operation we apply is larger that the degree of the cohomology class we apply it to.
(4) $v_{2i+1}(X) = 0$ if X is orientable of dimension d; Indeed, we need to show that $\mathrm{Sq}^{2i+1}\colon H^{d-2i-1}(X;\mathbb{F}_2) \to H^d(X;\mathbb{F}_2)$ vanishes, which follows from (1) and the Adem relation $\mathrm{Sq}^{2i+1} = \mathrm{Sq}^1\mathrm{Sq}^{2i}$.

5.20. **Corollary** *Let X be an orientable PD complex of dimension 3. Then all Stiefel–Whitney classes of X vanish.*

Proof. By assumption, $w_1(X) = 0$ and $v_2(X) = 0$ by Corollary 5.18. Hence $w_2(X) = 0$ by Example 5.19 (2) so also $0 = \mathrm{Sq}^1(w_2) = w_1^2(X) + w_3(X)$ which implies that $w_3(X) = 0$. All other Stiefel-Whitney classes vanish for degree reasons. □

5.21. **Remark** One can in fact show the Spivak normal fibration of a 3-dimensional PD complex is the underlying spherical fibration of stable vector bundle, see e.g. [Lan22, Theorem 2.8 or Remark 3.11]. We will see later that this implies that if X is orientable, then D_X is trivial, so that in fact *all* characteristic classes of spherical fibrations vanish on D_X.[16]

5.22. **Corollary** *Let X be an orientable PD complex of dimension 4. Then $w_2(X) = 0$ implies that the intersection form on $H^2(X;\mathbb{Z})$ is* even. *The converse holds if $H_1(X;\mathbb{Z})$ does not contain 2-torsion.*

Proof. The intersection form is the symmetric bilinear unimodular form μ on $H^2(X;\mathbb{Z})/\mathrm{tors}$ given by $\mu(x,y) = \langle x \cdot y, [X]\rangle$. Such a form is called even, if for all $x \in H^2(X;\mathbb{Z})/\mathrm{tors}$, or equivalently for all $x \in H^2(X;\mathbb{Z})$, we have $\mu(x,x) \in 2\mathbb{Z}$, or equivalently, $\mu(x,x) \equiv 0 \mod (2)$, or yet equivalently, that $0 = x^2 = \mathrm{Sq}^2(x) = v_2(X) \cdot x \in H^4(X;\mathbb{F}_2)$ for all x in the image of the map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{F}_2)$. By Poincaré duality and the long exact coefficient-change exact sequence, the cokernel of this map is the 2-torsion in $H_1(X;\mathbb{Z})$. The Wu formula then implies the result. □

[16] There is a characteristic class $e_1 \in H^3(\mathrm{BG};\mathbb{F}_2)$ which in principle could be non-trivial.

5.23. **Example** We give an example that the condition on $H_1(X;\mathbb{Z})$ being 2-torsion free cannot be dropped: Let E be an Enriques surface. This is a smooth closed orientable (in fact complex) 4-manifold with $\pi_1(E) = \mathbb{Z}/2$ and universal cover a $K3$ surface. The intersection form $\mu(E)$ turns out to be $H \oplus E_8$, the sum of a hyperbolic form and the positive definite E_8 form, both of which are even forms. The long exact sequence

$$0 = H^1(E;\mathbb{Z}) \to \mathbb{F}_2 \cong H^1(E;\mathbb{F}_2) \to H^2(E;\mathbb{Z}) \xrightarrow{\cdot 2} H^2(E;\mathbb{Z})$$

shows that $H^2(E;\mathbb{Z})$ contains a unique non-trivial 2-torsion element which turns out to be sent to $w_2(E)$ under the reduction mod 2 map, see Remark 5.24 for some more perspectives on this example.

5.24. **Remark** One can show that the signature of an even unimodular symmetric bilinear form over $\mathbb{Z}$ is divisible by 8. In particular, it follows from the above result that an oriented PD complex of dimension 4 with trivial w_2 has signature divisible by 8. Later, we will show that closed topological manifolds are PD complexes. Moreover, for a closed, orientable *smooth*[17] 4-manifolds, w_2 vanishes if and only if X admits what is called a spin-structure, and in the presence of such a structure, the signature of X is even divisible by 16; this is a theorem due to Rokhlin. In particular, since the signature of the Enriques surface E is 8 we recover the fact that E is not spin. Moreover, since E is in fact a complex manifold its tangent bundle is canonically a complex vector bundle. Its first Chern class $c_1(E) \in H^2(E;\mathbb{Z})$ then turns out to be the unique non-trivial 2-torsion class discussed above, compatible with the general result that the mod 2 reduction of c_1 is w_2, see Lemma 3.12.

More restrictions on the intersection forms of smooth oriented closed 4-manifolds have been established by Donaldson (worth a fields medal!) using methods nowadays called gauge theory (largely intricate analytic arguments). You can learn about these methods in the course of Kotschick on the topic.

For closed, oriented topologically spin manifolds (i.e. topological manifolds with $w_1 = w_2 = 0$), it turns out that the signature is in fact only divisible by 8 in general, that is, there is an orientable closed topological 4-manifold with $w_2(X) = 0$ and signature 8; this a famous and very deep result due to Freedman (it was worth another fields medal!).

5.25. **Remark** One can show that the Spivak normal fibration any oriented PD complex of dimension 4 is the underlying spherical fibration of a stable vector bundle, see [Lan22, Theorem 2.7]. For not necessarily orientable PD complexes of dimension 4, the same is not true, a counter example was given by Hambleton–Milgram, see [Lan22, §4.1]

We now aim to give examples of Poincaré duality complexes, in particular, to give a recognition principle for Poincaré duality complexes. First, we need:

5.26. **Lemma** *For $X \in \mathrm{An}^\omega$ and $x \in X$, we have $(D_X)_x \simeq \lim_X \mathbb{S}[\Omega X]$. In particular, D_X takes values in bounded below spectra.*

Proof. We need to compute $i^*(D_X)$ where $i\colon \{x\} \to X$ is the inclusion. To that end, we have $i_!(\mathbb{S}) \otimes D_X = i_!(\mathbb{S} \otimes i^* D_X)$ by the projection formula (Exercise). Hence

$$r_*(i_!(\mathbb{S})) = r_!(i_!(\mathbb{S}) \otimes D_X) = r_! i_!(i^*(D_X)) = i^*(D_X).$$

[17] In fact, formally smooth is sufficient here, that is, it suffices that the topological tangent bundle admits a reduction to a vector bundle.

The right hand side is what we wish to compute, and the left hand side is $\lim_X \mathbb{S}[\Omega X]$. The in particular is a consequence of the fact that bounded below spectra are closed under finite (and hence compact) limits and $\mathbb{S}[\Omega X]$ is connective (and hence bounded below). $\square$

5.27. **Proposition** *Let $X \in \mathrm{An}^\omega$ be a compact anima. Suppose given a local coefficient system $\mathcal{L} \in \mathrm{Fun}(X, \mathrm{Ab})$ of pointwise infinite cyclic abelian groups on X and a class $[X] \in H_d(X; \mathcal{L})$ such that the map $- \cap [X] \colon H^*(X; \mathcal{M}) \to H_{*-d}(X; \mathcal{M} \otimes_{\mathbb{Z}} \mathcal{L})$ is an isomorphism for all local coefficient systems $\mathcal{M} \in \mathrm{Fun}(X, \mathrm{Ab})$ on X. Then X is a Poincaré duality complex.*

Proof. We need to show that D_X takes values in invertible spectra. Since D_X takes values in bounded below spectra by Lemma 5.26, the stable Hurewicz theorem implies that it suffices to show that $D_X \otimes \mathbb{Z} = D_X^{\mathbb{Z}}$ takes values in invertible $\mathbb{Z}$-modules, that is, is pointwise infinite cyclic (possibly shifted in homological degree). We will show that $D_X^{\mathbb{Z}} \simeq \Omega^d \mathcal{L}$ so the claim follows. To see this, by Lemma 5.12 and the uniqueness of $D_X^{\mathbb{Z}}$, it suffices to show that

$$r_*^{\mathbb{Z}}(\mathcal{G}) \simeq r_!(\mathcal{G} \otimes_{\mathbb{Z}} \Omega^d \mathcal{L})$$

for all $\mathcal{G} \in \mathrm{Fun}(X, \mathrm{Mod}(\mathbb{Z}))$. First, we claim that $[X]$ determines a canonical map from left to right. To see this, we note that $[X]$ determines a map $c \colon \Sigma^d \mathbb{Z} \to r_!^{\mathbb{Z}}(\mathcal{L})$ by definition of homology with local coefficients. Therefore, we may consider again the composite

$$r_*^{\mathbb{Z}}(-) = \mathrm{Map}_X(r^*(\mathbb{Z}), -) \to \mathrm{Map}_X(\mathcal{L}, - \otimes_{\mathbb{Z}} \mathcal{L}) \to \mathrm{Map}(r_!(\mathcal{L}), r_!(- \otimes_{\mathbb{Z}} \mathcal{L})) \xrightarrow{c^*} r_!(- \otimes_{\mathbb{Z}} \Omega^d \mathcal{L})$$

This map is just an abstract way to write to the cap product with $[X]$ as discussed in the proof of Lemma 5.14. Hence, by assumption, the just constructed map induces an isomorphism for any local coefficient system $\mathcal{M} \in \mathrm{Fun}(X, \mathrm{Ab})$, shifted in arbitrary degrees. Since both $r_*^{\mathbb{Z}}$ and $r_!^{\mathbb{Z}}$ commute with limits and colimits, we may use the pointwise Postnikov and Whitehead towers for $\mathcal{G}$ to deduce that the map in fact induces an equivalence for any $\mathcal{G} \in \mathrm{Fun}(X, \mathrm{Mod}(\mathbb{Z}))$. $\square$

5.28. **Theorem** *A closed topological manifold is a Poincaré duality complex.*

Proof sketch. The claim is that a closed topological manifold M satisfies the assumptions of Proposition 5.27. First, we need to argue that its underlying anima is compact (in fact, it is even finite by a theorem of West), this is true much more generally for compact ANR's of which topological manifolds are examples; see also [Win24, Prop. 4.1.12]. Alternatively, one can use the (also rather deep result) that $M \times D^6$ admits a finite handle decomposition. Then, one notes that M has an orientation local system ω_M and a fundamental class $[M] \in H_d(M; \omega_M)$, capping with which indeed is an isomorphism for *all* local coefficient systems $\mathcal{M}$ on M.

A shadow of this was shown in Topology II [Win24, Theorem 4.3.11]; namely the case where M is orientable, i.e. $\omega_M = r^*(\mathbb{Z})$ and where in addition $\mathcal{M} = r^*(\mathbb{Z})$. The more general claim is true as well, however, and can in fact be proven by a similar argument. $\square$

5.29. **Remark** We will show later that a topological manifold M has a (stable) tangent bundle $TM \colon M \to \mathrm{ktop}$. Its inverse ν_M, the stable normal bundle, has an underlying stable spherical fibration which turns out to be D_M, the Spivak normal fibration of M. Similarly, a smooth manifold M has a (stable) tangent vector bundle $TM \colon M \to \mathrm{ko}$ lifting the topological (stable) tangent bundle. In particular, if a PD complex X is homotopy equivalent to a closed topological manifold, its Spivak normal fibration must come with a lift $X \to \mathrm{ktop} \to \mathrm{Pic}(\mathbb{S})$; similarly if X is homotopy equivalent to a closed smooth manifold, its Spivak normal fibration must come with a lift $X \to \mathrm{ko} \to \mathrm{Pic}(\mathbb{S})$.

We now aim to show that this allows us to construct PD complexes which are not homotopy equivalent to closed manifolds.

To that end, we begin with the following recognition principle for the Spivak normal fibration to be trivial.

5.30. **Proposition** *Consider a pushout diagram*

$$\begin{tikzcd} S^{d-1} \arrow[r, "f"] \arrow[d] & X_0 \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

where X_0 is a $(d-1)$-dimensional CW complex and assume that X is a connected d-dimensional PD complex.[18] *Then $D_X \simeq \Omega^d r^*(\mathbb{S})$ if and only if f is stably null-homotopic.*

Proof. Suppose we have $D_X \simeq \Omega^d r^*(\mathbb{S})$. We have argued earlier that the collapse map $\Sigma^d\mathbb{S} \to \Sigma^d r_!(D_X) \simeq r_! r^*(\mathbb{S}) \simeq \mathbb{S}[X]$ induces on homology the fundamental class of X. In particular, the composite $\Sigma^d\mathbb{S} \to \mathbb{S}[X] \to \mathbb{S}[S^d]$ is an equivalence, showing that $X \to S^d$ splits stably; from this it follows that f is stably null-homotopic. Conversely, we run a similar argument as in the proof of Proposition 5.27: If f is stably null-homotopic, we first observe that this implies that $\mathrm{Sq}^1\colon H^{d-1}(X;\mathbb{F}_2) \to H^d(X;\mathbb{F}_2)$ vanishes, so by Example 5.19, we find that X is orientable. Now, since f is stably null-homotopic, we have a map $c\colon \mathbb{S} \to \Omega^d r_! r^*(\mathbb{S})$ which splits of the top cell of X stably. As usual, this map induces a transformation

$$r_*(-) \to r_!(- \otimes \Omega^d r^*(\mathbb{S}))$$

which we aim to show is an equivalence; it then follows from the uniqueness of the dualizing spectrum that $\Omega^d r^*(\mathbb{S}) \simeq D_X$. After applying $- \otimes_{\mathbb{S}} \mathbb{Z}$, we find that c classifies a generator of $H_d(X;\mathbb{Z})$, so up to a sign, the fundamental class $[X]$ of X as we have argued that X is orientable. Hence, the map under investigation is an equivalence after applying $- \otimes_{\mathbb{S}} \mathbb{Z}$. Since both sides commute with colimits and $\mathrm{Fun}(X, \mathrm{Sp})$ is generated under colimits by $i_!(\mathbb{S})$, it suffices to check that the map is an equivalence when evaluated on $i_!(\mathbb{S})$ which is pointwise connective. Hence, evaluated on $i_!(\mathbb{S})$ both left and right hand side are bounded below, and the map is an equivalence on homology, so it is in fact an equivalence. $\square$

5.31. **Example** Consider the anima given by the pushout

$$\begin{tikzcd} S^4 \arrow[r, "{[i_2,i_3]+\alpha}"] \arrow[d] & S^2 \vee S^3 \arrow[d] \\ * \arrow[r] & X(\alpha) \end{tikzcd}$$

where α is either the composite $S^4 \xrightarrow{\eta^2} S^2 \to S^2 \vee S^3$, or $S^4 \xrightarrow{\eta} S^3 \to S^2 \vee S^3$, or trivial. We have $X(0) = S^2 \times S^3$. We have shown in the exercises that $X(\eta^2)$ and $X(\eta)$ are PD complexes of dimension 5, and it follows from Proposition 5.30 that their Spivak normal fibration is not trivial, since both η and η^2 are not null homotopic. Moreover, we have shown on the exercise sheets that $X(\eta)$ is homotopy equivalent to a closed (smooth) 5-dimensional manifold.

In the next section, we will show that if a PD complex is homotopy equivalent to a closed smooth manifold, then there is a canonical factorization of the Spivak normal fibration as a

[18] Wall showed that every finite d-dimensional connected PD complex is of this form if $d \neq 3$.

composite $X \to \mathrm{BO} \to \mathrm{BG}$, similarly, if X is homotopy equivalent to a closed topological manifold, there is a canonical factorization of the Spivak normal fibration as a composite $X \to \mathrm{BTop} \to \mathrm{BG}$. Let us now argue that this implies that $X(\eta^2)$ is not homotopy equivalent to a smooth/topological closed manifold. Indeed, let us first show that $D_{X(\eta^2)}$ does not admit a lift along $\mathrm{BO} \to \mathrm{BG}$, so that X is not homotopy equivalent to a closed smooth manifold. To that end, let us compute Stiefel–Whitney classes of $D_{X(\eta^2)}$: first, w_1 vanishes simply because $H^1(X;\mathbb{F}_2) = 0$. Consequently, $w_2(D_{X(\eta^2)} = w_2(T_{X(\eta^2)} = v_2(X)$ by the Wu formula, and $v_2(X) = 0$ since $\mathrm{Sq}^2\colon H^3(X(\eta^2);\mathbb{F}_2) \to H^5(X(\eta^2);\mathbb{F}_2)$ vanishes, as follows from the naturality of Steenrod squares together with the observation that there is a map $X(\eta^2) \to S^3 \vee S^5$ inducing an isomorphism on $H^3(-)$ and the fact that Sq^2 vanishes on $S^3 \vee S^5$. Hence $D_{X(\eta^2)}$ lifts to the fibre of the map $\mathrm{BO} \xrightarrow{(w_1,w_2)} K(\mathbb{F}_2,1) \times K(\mathbb{F}_2,2)$. This fibre is given by $\tau_{\geq 3}\mathrm{BO} = \mathrm{BSpin}$. Now we need to use some computations that we have not yet proven: Namely $\pi_2(\mathrm{BSpin}) = \pi_3(\mathrm{BSpin}) = \pi_5(\mathrm{BSpin}) = 0$. The first one is simply because Spin is the universal cover of SO; the second because $\pi_3(\mathrm{BSpin}) = \pi_2(\mathrm{Spin}(n))$ for $n \geq 4$ and $\mathrm{Spin}(n)$ is a Lie group, then we can recall that we have mentioned in [Lan24] (I think) that Lie groups have trivial π_2 (which can be deduced from Morse theory). Alternatively, we also have $\pi_2(\mathrm{Spin}(4)) = \pi_2(\mathrm{SO}(4))$ and $\mathrm{Spin}(4) \cong S^3 \times S^3$ as topological space. Finally, there is the following theorem of Bott's, namely $\Omega^4\mathrm{O} = \mathrm{Sp}$. Hence $\pi_5(\mathrm{BSpin}) = \pi_4(\mathrm{O}) = \pi_0(\mathrm{Sp})$ which vanishes as all symplectic groups $\mathrm{Sp}(n)$ are connected. As a consequence of the defining cofibre sequence for $X(\eta^2)$, we then find that $[X(\eta^2), \mathrm{BSpin}] = \{*\}$ so if $D_{X(\eta^2)}$ lifts to BO it also lifts to BSpin, and therefore must be trivial – a contradiction.

Finally, $D_{X(\eta^2)}$ also does not lift along $\mathrm{BTop} \to \mathrm{BG}$, from which it follows that X is also not homotopy equivalent to a closed topological manifold: There is a map $\mathrm{ks}\colon \mathrm{BTop} \to K(\mathbb{F}_2,4)$ called the Kirby–Siebenmann invariant. It has the property that the composite $\mathrm{BO} \to \mathrm{BTop} \to K(\mathbb{F}_2,4)$ is canonically null and that the induced map $\mathrm{BO} \to \mathrm{fib}(\mathrm{BTop} \to K(\mathbb{F}_2,2)$ is 7-connected: Its fibre is a space traditionally called PL/O whose homotopy groups are the groups of exotic spheres. It is known that PL/O is 6-connected resulting in the claimed connectivity of $\mathrm{BO} \to \mathrm{fib}(\mathrm{ks})$. In particular, since $H^4(X(\eta^2);\mathbb{F}_2) = 0$, we find that if $D_{X(\eta^2)}$ lifts to BTop, then it also lifts to BO. As we have just ruled out the latter, also the former is not the case.

6. A survey on (topological) manifolds

6.1. **Definition** A topological manifold is a second countable Hausdorff space X which can be covered by open sets homeomorphic to $\mathbb{R}^d$ for some d's. A choice of such a covering $\{U_i, f_i\colon U_i \to \mathbb{R}^n\}_{i\in I}$ is called an atlas. A smooth atlas on a topological manifold consists of an atlas for which the transition maps, i.e. the composites

$$f_i^{-1}\colon f_i(U_i \cap U_j) \to U_i \cap U_j \xrightarrow{f_j} f_j(U_i \cap U_j)$$

are smooth functions for all pairs $(i,j) \in I \times I$. Two smooth atlases are equivalent if their union is a smooth atlas. A smooth structure on a topological manifold consists of a maximal smooth atlas.

6.2. **Remark** For manifolds with boundary, one replaces in the above definition the role of $\mathbb{R}^d$ by open subsets of $\mathbb{R}^d_+$, the half space with dth coordinate non-negative.

6.3. **Definition** Let M, W be a smooth or topological manifolds of dimension m and w, respectively. An embedding of M in W is a smooth or continuous map $i\colon M \to W$ which is a homeomorphism onto its image and which is, locally and M and on W, given by the standard inclusion $\mathbb{R}^m \subseteq \mathbb{R}^w$.[19]

6.4. **Definition** Let M and W be a smooth or topological manifolds. The sets $\mathrm{Emb}^{(t)}(W, M)$ of smooth or topological embeddings of W into M are canonically topological spaces: The topological embedding space is topologized as a subspace of the mapping space with compact open topology, and the smooth one as the subspace of the mapping space with the (Whitney) C^∞-topology.

6.5. **Theorem** *Let M and W be (topological) manifolds and $K \subseteq W$ a compact (topological) submanifold. Then the restriction map*

$$\mathrm{Emb}^{(t)}(W, M) \to \mathrm{Emb}^{(t)}(K, M)$$

is a Serre fibration.

6.6. **Corollary** *The evaluation at 0 maps give fibrations $\mathrm{Emb}^{(t)}(\mathbb{R}^d, M) \to M$.*

6.7. **Proposition** *Consider the fibre $\mathrm{Emb}_m^{(t)}(\mathbb{R}^d, M)$ of the evaluation at 0. Then there is a map*

$$\mathrm{Emb}_0^{(t)}(\mathbb{R}^d, \mathbb{R}^d) \times \mathrm{Emb}_m^{(t)}(\mathbb{R}^d, M) \to \mathrm{Emb}_m^{(t)}(\mathbb{R}^d, M)$$

simply by composing embeddings. Then, if $d = \dim(M)$, for each $e \in \mathrm{Emb}_m^{(t)}(\mathbb{R}^d, M)$, the resulting map

$$\mathrm{Emb}_0^{(t)}(\mathbb{R}^d, \mathbb{R}^d) \to \mathrm{Emb}_m^{(t)}(\mathbb{R}^d, M)$$

is an equivalence.

Finally, note that $\mathrm{Top}(d)$ acts on $\mathrm{Emb}_0^t(\mathbb{R}^d, M)$ in the topological case, and $\mathrm{O}(d)$ acts on $\mathrm{Emb}_0(\mathbb{R}^d, M)$ in the smooth case, by precomposing an embedding with a homeomorphism or an orthogonal transformation, respectively.

6.8. **Theorem** *The maps obtained by acting on the identity $\mathrm{Top}(d) \to \mathrm{Emb}_0^t(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathrm{O}(d) \to \mathrm{Emb}_0(\mathbb{R}^d, \mathbb{R}^d)$ are homotopy equivalences.*

Proof. The first claim is a result of Kister–Mazur. The second one is easier and can in fact be done as an exercise (Hint: differentiate a smooth embedding at 0 to obtain a homotopy inverse). □

6.9. **Corollary** *On the fibre sequences*

$$\mathrm{Emb}_0^{(t)}(\mathbb{R}^d, \mathbb{R}^d) \to \mathrm{Emb}^{(t)}(\mathbb{R}^d, M) \to M$$

we have $\mathrm{Top}(d)$, resp. $\mathrm{O}(d)$-actions on the fibre and the total space, making all maps equivariant. In particular, we find that $\mathrm{Emb}(\mathbb{R}^d, M)_{h\mathrm{O}(d)} \to M$ and $\mathrm{Emb}^t(\mathbb{R}^d, M)_{h\mathrm{Top}(d)} \to M$ are equivalences.

[19] Some authors call such embeddings locally flat, but we shall not ever seriously consider non locally-flat embeddings.

Proof. It is a direct check to see that the first map is equivariant with respect to the $\mathrm{Top}(d)$ or $\mathrm{O}(d)$-action. Moreover, the evaluation map is equivariant with respect to the trivial action on M. We may then apply homotopy orbits to the sequence; we spell out the case of $\mathrm{O}(d)$, the other one is verbatim the same. First we note that $M_{h\mathrm{O}(d)} = M \times \mathrm{BO}(d)$ which comes with a canonical projection map to M. We then claim that there is a fibre sequence

$$\mathrm{Emb}_0(\mathbb{R}^d, \mathbb{R}^d)_{h\mathrm{O}(d)} \to \mathrm{Emb}(\mathbb{R}^d, M)_{h\mathrm{O}(d)} \to M.$$

To see this, we expand out the following diagram

$$\begin{tikzcd}
\mathrm{Emb}_0(\mathbb{R}^d, \mathbb{R}^d) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^d, M) \arrow[r] \arrow[d] & M \arrow[d] \\
\mathrm{Emb}_0(\mathbb{R}^d, \mathbb{R}^d)_{h\mathrm{O}(d)} \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^d, M)_{h\mathrm{O}(d)} \arrow[r] \arrow[d] & M \arrow[d] \\
\mathrm{BO}(d) \arrow[r] & \mathrm{BO}(d) \arrow[r] & *
\end{tikzcd}$$

in which all vertical sequences are fibre sequences, and the top and bottom sequence are also fibre sequences. By a diagram-chase, the middle sequence is also a fibre sequence.

Now we can use Theorem 6.8, i.e. that the map $\mathrm{O}(d) \to \mathrm{Emb}_0(\mathbb{R}^d, \mathbb{R}^d)$ is an $\mathrm{O}(d)$-equivariant homotopy equivalence. Since homotopy orbits preserve equivalences, we also find that

$$* = \mathrm{O}(d)_{h\mathrm{O}(d)} \to \mathrm{Emb}_0(\mathbb{R}^d, \mathbb{R}^d)_{h\mathrm{O}(d)}$$

is an equivalence, and consequently, that $\mathrm{Emb}(\mathbb{R}^d, M)_{h\mathrm{O}(d)} \to M$ is an equivalence. □

6.10. **Definition** We define the tangent bundles T^tM and TM of a topological and smooth d-manifold to be classified by the map

$$M \xleftarrow{\simeq} \mathrm{Emb}(\mathbb{R}^d, M)_{h\mathrm{O}(d)} \to *_{h\mathrm{O}(d)} = \mathrm{BO}(d)$$

and

$$M \xleftarrow{\simeq} \mathrm{Emb}^t(\mathbb{R}^d, M)_{h\mathrm{Top}(d)} \to *_{h\mathrm{Top}(d)} = \mathrm{BTop}(d).$$

6.11. **Remark** Let M be a smooth manifold. Then the diagram

$$\begin{tikzcd}
\mathrm{Emb}(\mathbb{R}^d, M)_{h\mathrm{O}(d)} \arrow[r, "\simeq"] \arrow[d] & \mathrm{BO}(d) \arrow[d] \\
\mathrm{Emb}^t(\mathbb{R}^d, M)_{h\mathrm{Top}(d)} \arrow[r] & \mathrm{BTop}(d)
\end{tikzcd}$$

commutes and the left vertical map identifies with the identity of M. That is, the underlying euclidean bundle of the tangent vector bundle of a smooth manifold identifies with the topological tangent bundle; in particular, it does not depend on the smooth structure on M.

6.12. **Remark** Let M be a closed topological d-manifold, that is compact and without boundary. Then we will see a bit later that the composite

$$M \to \mathrm{BTop}(d) \to \mathrm{BTop} \to \mathrm{BG}$$

is the Spivak tangent fibration of the anima underlying M.

Next we want to discuss the notion of normal bundles of embeddings. To begin, we consider the following definition.

6.13. **Definition** Let $M \subseteq W$ be a submanifold. Consider the group $\mathrm{Top}(w;m) \subseteq \mathrm{Top}(w)$ of homeomorphisms $f\colon \mathbb{R}^w \to \mathbb{R}^w$ which preserve the subset $\mathbb{R}^m \subseteq \mathbb{R}^w$. Similarly, let $\mathrm{GL}(w;m)$ be the subspace of linear isomorphism of $\mathbb{R}^w$ which preserve the subset $\mathbb{R}^m$.

6.14. **Remark** There are evident maps of groups $\mathrm{Top}(w-m) \times \mathrm{Top}(m) \to \mathrm{Top}(w;m)$ and $\mathrm{GL}(w-m) \times \mathrm{GL}(m) \to \mathrm{GL}(w;m)$ by taking products of homeomorphisms and linear isomorphisms, respectively. Exercise: The latter of the two is an equivalence.

6.15. **Remark** There are also evident maps of groups $\mathrm{Top}(w;m) \to \mathrm{Top}(w) \times \mathrm{Top}(m)$ and $\mathrm{GL}(w;m) \to \mathrm{GL}(w) \times \mathrm{GL}(m)$, which forget that a homeomorphism respects $\mathbb{R}^m$ or takes the induced homeomorphism of $\mathbb{R}^m$; same with linear isomorphisms. The composite

$$\mathrm{Top}(w-m) \times \mathrm{Top}(m) \to \mathrm{Top}(w;m) \to \mathrm{Top}(w) \times \mathrm{Top}(m)$$

sends $(f,g) \to (f \times g, g)$. Similarly for GL in place of Top.

6.16. **Notation** Let us write $\mathrm{Emb}^t(\mathbb{R}^w;\mathbb{R}^m, W; M)$ for the subspace of $\mathrm{Emb}^t(\mathbb{R}^w, W)$ consisting of those topological embeddings $\mathbb{R}^w \to W$ which restrict to an embedding $\mathbb{R}^m \to M$. Likewise, write $\mathrm{Emb}(\mathbb{R}^w;\mathbb{R}^m, W; M)$ for the subspace of $\mathrm{Emb}(\mathbb{R}^w, W)$ on those smooth embeddings $\mathbb{R}^w \to W$ which restrict to a smooth embedding $\mathbb{R}^m \to M$.

Similarly as in the case where $M = \varnothing$, we have:

6.17. **Proposition** *There is are actions of* $\mathrm{Top}(w;m)$ *on* $\mathrm{Emb}^t(\mathbb{R}^w;\mathbb{R}^m, W; M)$ *and* $\mathrm{GL}(w;m)$ *on* $\mathrm{Emb}(\mathbb{R}^w;\mathbb{R}^m, W; M)$ *which induce equivalences*

$$\mathrm{Emb}^t(\mathbb{R}^w;\mathbb{R}^m, W; M)_{h\mathrm{Top}(w;m)} \xrightarrow{\simeq} M \xleftarrow{\simeq} \mathrm{Emb}(\mathbb{R}^w;\mathbb{R}^m, W; M)_{h\mathrm{GL}(w;m)}.$$

Proof. The fibre of the evaluation at 0 map

$$\mathrm{Emb}^{(t)}(\mathbb{R}^w;\mathbb{R}^m, W; M) \to M$$

over $m \in M$ is given by $\mathrm{Emb}^{(t)}_m(\mathbb{R}^w;\mathbb{R}^m, W; M)$ which, similarly as in Proposition 6.7 is equivalent to $\mathrm{Emb}^{(t)}_0(\mathbb{R}^w;\mathbb{R}^m, \mathbb{R}^w; \mathbb{R}^m)$. Finally, a relative form of the Kister–Mazur theorem gives that the action map $\mathrm{Top}(w;m) \to \mathrm{Emb}^t(\mathbb{R}^w;\mathbb{R}^m, \mathbb{R}^w; \mathbb{R}^m)$ is an equivalence, and $\mathrm{GL}(w;m) \to \mathrm{Emb}(\mathbb{R}^w;\mathbb{R}^m, \mathbb{R}^w, \mathbb{R}^m)$ is also one. The argument is then same as in Corollary 6.9. $\square$

6.18. **Notation** For an embedding $M \to W$, we denote by $T(W;M)\colon M \to \mathrm{BTop}(w;m)$ the resulting map and call it the *relative tangent bundle.*

6.19. **Remark** The diagrams

$$\begin{tikzcd}
\mathrm{Emb}^t(\mathbb{R}^w;\mathbb{R}^m, W; M) \arrow[r] \arrow[d] & M \arrow[d] \\
\mathrm{Emb}^t(\mathbb{R}^w, W) \arrow[r] & W
\end{tikzcd}$$

evidently commutes and the left vertical map is equivariant for the map $\mathrm{Top}(w;m) \to \mathrm{Top}(w)$. Therefore, we find that also the square

$$\begin{tikzcd} M \arrow[r, "T(W;M)"] \arrow[d] & \mathrm{BTop}(w;m) \arrow[d] \\ W \arrow[r, "TW"] & \mathrm{BTop}(w) \end{tikzcd}$$

also commutes. That is, the restriction of the tangent bundle of W to M admits a canonical lift to $\mathrm{BTop}(w;m)$. Similarly, the map $\mathrm{Emb}^t(\mathbb{R}^w;\mathbb{R}^m,W;M) \to \mathrm{Emb}^t(\mathbb{R}^m,M)$ is equivariant for the map $\mathrm{Top}(w;m) \to \mathrm{Top}(m)$. Hence, we obtain that the topological tangent bundle TM of M is classified by the composite $M \to \mathrm{BTop}(w;m) \to \mathrm{BTop}(m)$.

Same results hold true in the smooth case.

With this at hand, we make the following definitions:

6.20. **Definition** Let $i\colon M \to W$ be a smooth embedding and let $M \to \mathrm{BGL}(w;m)$ be the associated map. Using the above exercise, we have that $\mathrm{BGL}(w-m)\times\mathrm{BGL}(m) \to \mathrm{BGL}(w;m)$ is an equivalence. Hence we obtain a canonical map

$$M \to \mathrm{BGL}(w-m) \times \mathrm{BGL}(m) \simeq \mathrm{BO}(w-m) \times \mathrm{BO}(m)$$

or equivalently, a map $M \to \mathrm{BO}(m)$ (this is just the tangent bundle as observed in Remark 6.19) and a map $\nu(i)\colon M \to \mathrm{BO}(w-m)$ which we call the *normal bundle* of the embedding.

6.21. **Definition** Let $i\colon M \to W$ be a topological embedding and let $T(M;W)\colon M \to \mathrm{BTop}(w;m)$ be the associated map. A *normal bundle* of the embedding consists of a map $\nu(i)\colon M \to \mathrm{BTop}(w-m)$ and a homotopy between the composite

$$M \to \mathrm{BTop}(w-m) \times \mathrm{BTop}(m) \to \mathrm{BTop}(w;m)$$

and the above associated map

In both cases, the normal bundles $\nu(i)$ satisfy $\nu(i) \oplus TM = i^*(TW)$.

Now, unlike in the smooth case, the map $\mathrm{Top}(w-m) \times \mathrm{Top}(m) \to \mathrm{Top}(w;m)$ is not an equivalence. However:

6.22. **Theorem** *The map* $\mathrm{Top}(w-m) \times \mathrm{Top}(m) \to \mathrm{Top}(w;m)$ *is roughy ...-connected.*

In particular, normal bundles for topological embeddings need not exist in general, but they do exist in large enough codimension.

6.23. **Theorem** *Let M be a manifold. There exists $N > 0$ such that M embeds into S^N. In fact, the space $\mathrm{Emb}^{(t)}(M,S^N)$ has a connectivity which grows with N. Concretely, this means that for N large enough, any two embeddings $M \to S^N$ are isotopic. In particular, any manifold admits a stable normal bundle by considering the normal bundle of an embedding $M \to S^N$ for large enough N,*

6.24. **Definition** Let $i\colon M \to W$ be an embedding. A mapping cylinder neighbourhood consists of a codimension 0 embedding $K \subseteq W$ with $M \subseteq K$ an embedding such that K is homeomorphic to the mapping cylinder of a map $r\colon \partial K \to M$ having homotopy fibres equivalent to S^{w-m-1}; It is hence classified by a map $n(i)\colon M \to \mathrm{BG}(w-m-1)$. In particular, there is a deformation retraction $\bar{r}\colon K \to M$.

6.25. **Remark** If i is a smooth embedding, then K can be chosen to be a tubular neighbourhood, that is, such that $r\colon K \to M$ is even homeomorphic to the projection of a disk-bundle. In this case, $r\colon \partial K \to M$ is in fact a S^{w-m-1}-fibre bundle.

6.26. **Theorem** *Every topological embedding $i\colon M \to W$ admits a mapping cylinder neighbourhood. Moreover, if i admits a normal bundle $\nu(i)\colon M \to \mathrm{BTop}(w-m)$, then its composite with $\mathrm{BTop}(w-m) \to \mathrm{BG}(w-m-1)$ identifies with $n(i)$. As in the case of normal bundles, we have that $n(i) \oplus TM \simeq i^*(TW)$; here we view TM and $i^*(TW)$ as spherical fibrations via the maps $\mathrm{Top}(k) \to \mathrm{G}(k)$.*

6.27. **Remark** Nevertheless, a topological embedding need not have a tubular neighbourhood. That is, the mapping cylinder neighbourhood $M \subseteq K \subseteq W$ and the map $r\colon K \to M$ are in general not homeomorphic to the projection map of D^{w-m}-bundle.

6.28. **Theorem** *A smooth embedding $i\colon M \to W$ admits a tubular neighborhood. That is, a mapping cylinder neighborhood $M \subseteq K \subseteq W$ where $r\colon K \to M$ is diffeomorphic to the projection $D(\nu(i)) \to M$, where $D(\nu(i))$ is the disk bundle of the normal bundle of i.*

6.29. **Remark** Let $i\colon N \to M$ and $i'\colon M \to W$ be embeddings. If i and i' admit normal bundles, then we have $\nu(i'i) \oplus TN \simeq i^*i'^*TW = i^*(\nu(i') \oplus TM) = i^*(\nu(i')) \oplus i^*(TM) = i^*(\nu(i')) \oplus \nu(i) \oplus TN$. In particular, we find an equivalence of stable bundles $\nu(i'i) \simeq i^*(\nu(i')) \oplus \nu(i)$ of bundles over N.

The same statement holds true for the spherical fibrations associated to mapping cylinder neighborhoods of i and i' in case i and/or i' do not have normal bundles: In this case we have an equivalence $n(i'i) \simeq i^*(n(i')) \oplus n(i)$ of spherical fibrations over N.

6.30. **Theorem** *The underlying spherical fibration of the stable normal bundle of a closed manifold M identifies with the Spivak normal fibration.*

Proof. There exists $N > 0$ so that there exists an embedding $M \to S^N$ which admits a normal bundle $\nu(i)$. Consider the mapping cylinder neighbourhood $M \subseteq K \subseteq S^N$. Note that there is a canonical map $S^N \to K/\partial K$: it is the identity on the interior of K and collapses everything else to the point represented by ∂K. Since $\partial K \to K$ is a cofibration and $K \to M$ is a homotopy equivalence, we find that $K/\partial K \simeq \mathrm{Th}(n(i)) \simeq \mathrm{Th}(\nu(i))$. Using the geometric description of the degree, we find that $S^N \to \mathrm{Th}(n(i))$ induces on $H^N(-;\mathbb{Z})$ a map $\mathbb{Z} \to H^m(M;\mathcal{L}) \cong \mathbb{Z}$ (where $\mathcal{L}$ is determined by $w_1(\nu(i))$) sending 1 to $\pm[M]$, the fundamental class of M. This map gives rise, as discussed earlier, to a comparison transformation $r_* \to r_!(- \otimes \nu(i))$ and the same argument as in the proof of Proposition 5.27 shows that this map is an equivalence, exhibiting $\nu(i)$ as the Spivak normal fibration of M. □

6.31. **Theorem** *Let $i\colon M \to W$ be an embedding with both M and W oriented closed. Then there is a canonical map $c\colon W \to \mathrm{Th}(n(i))$ and $c^*(u(n(i))) \cap [W] = i_*[M]$.*

Proof. Choose an embedding $j\colon W \to S^N$ for large N. Write ν_M for the normal bundle of the embedding ji and ν_W for the normal bundle of the embedding j. As discussed above, $\mathrm{Th}(n(i)) \simeq K/\partial K$, so we have the collapse map $c\colon W \to \mathrm{Th}(n(i))$. What we have discussed in Remark 6.29 implies that there is a pullback diagram

$$\begin{tikzcd}
\nu_M \arrow[r] \arrow[d] & \nu_W \times n(i) \arrow[d] \\
M \arrow[r] & W \times M
\end{tikzcd}$$

and hence a map $\mathrm{Th}(\nu_M) \to \mathrm{Th}(\nu_W) \wedge \mathrm{Th}(n(i))$. Similarly, recall that there is the Thom diagonal map $\mathrm{Th}(\nu_W) \to \mathrm{Th}(\nu_W) \wedge W_+$ induced from the pullback diagram

$$\begin{tikzcd}
\nu_W \arrow[r] \arrow[d] & \nu_W \times \epsilon^0 \arrow[d] \\
W \arrow[r] & W \times W
\end{tikzcd}$$

by passing to Thom spaces. Recall that there are also collapse maps $S^N \to \mathrm{Th}(\nu_W)$ and $S^N \to \mathrm{Th}(\nu_M)$; interpreting ν_W as a spherical fibration of virtual rank $-w$ instead, this determines a collapse map $\mathbb{S} \to \mathrm{M}(\nu_W)$, likewise we get a map $\mathbb{S} \to \mathrm{M}(\nu_M)$. The Thom classes are then maps $\mathrm{M}(\nu_W) \to \Omega^w\mathbb{Z}$ and $\mathrm{M}(\nu_M) \to \Omega^m\mathbb{Z}$; combined with the Thom diagonals, one then obtains the maps

$$\mathbb{S} \to \mathrm{M}(\nu_W) \to \mathrm{M}(\nu_W) \otimes W \to \Omega^w(\mathbb{Z} \otimes W)$$

and

$$\mathbb{S} \to \mathrm{M}(\nu_M) \to \mathrm{M}(\nu_M) \otimes M \to \Omega^m(\mathbb{Z} \otimes M).$$

which represent the fundamental classes $[W]$ and $[M]$, respectively. The claim is then that there is a commutative diagram

$$\begin{tikzcd}
S^N \arrow[rr, "c_M"] \arrow[d, "c_W"] & & \mathrm{Th}(\nu_M) \arrow[d] \\
\mathrm{Th}(\nu_W) \arrow[r] & \mathrm{Th}(\nu_W) \wedge W_+ \arrow[r, "\mathrm{id}\wedge c"] & \mathrm{Th}(\nu_W) \wedge \mathrm{Th}(n(i))
\end{tikzcd}$$

relating the various collapse maps. Using further Thom diagonals and Thom classes, and rewriting this diagram in terms of spectra, we find that also the diagram

$$\begin{tikzcd}
\mathbb{S} \arrow[rr, "c_M"] \arrow[d, "c_W"] & & \mathrm{M}(\nu_M) \otimes M_+ \arrow[r] \arrow[d] & \Omega^m(\mathbb{Z} \otimes M) \arrow[dr] & \\
\mathrm{M}(\nu_W) \arrow[r] & \mathrm{M}(\nu_W) \otimes W \arrow[r] & \mathrm{M}(\nu_W) \otimes \mathrm{M}(n(i)) \otimes W \arrow[r] & \Omega^w\mathbb{Z} \otimes \Omega^{w-m}\mathbb{Z} \otimes W \arrow[r] & \Omega^m(\mathbb{Z} \otimes W)
\end{tikzcd}$$

commutes. The composite going over the top horizontal arrow classifies $i_*[M]$. The lower composite unravels to classify $c^*(u(n(i)) \cap [W]$ as needed. $\square$

6.32. **Theorem** *Let W be a topological or smooth manifold, B a topological space and $\pi\colon E \to B$ an $\mathbb{R}^n$-bundle or a vector bundle, respectively, with Thom space* $\mathrm{Th}(\pi)$ *and zero section $z\colon B \to \mathrm{Th}(\pi)$. Then any map $f\colon W \to \mathrm{Th}(\pi)$ can be made* transversal *to z by a small* *homotopy*[20]*; in the case of a transversal map f, in the pullback diagram*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "i"] & B \arrow[d, "z"] \\ W \arrow[r, "f"] & \mathrm{Th}(\pi) \end{tikzcd}$$

the map $i\colon M \to W$ is a embedding (topological or smooth, respectively). Moreover, $g^*(\pi)$ is a normal bundle for i. This construction participates in a bijection

$$[W, \mathrm{Th}(\pi)] \cong \{(M, g) \mid M \subseteq W \mid \text{ submanifolds with normal bundle } \nu(i) \overset{\theta}{\cong} g^*(\pi)\}/\sim$$

where $\sim$ denotes cobordism of such objects, that is submanifolds $\bar{M} \subseteq W \times [0,1]$ with normal bundle identified with $\bar{g}^*(\pi)$ and whose intersection with $W \times \{0,1\}$ are the given pairs (M_0, g_0) and (M_1, g_1). The inverse of the above construction is obtained by sending (M, g) to the collapse map

$$W \to \mathrm{Th}(\nu(i)) \xrightarrow{\theta} \mathrm{Th}(\pi).$$

6.33. **Theorem** *Let W be a closed oriented manifold and $x \in H^{w-m}(W; \mathbb{Z})$ a cohomology class. Then x is Poincaré dual to $i_*[M]$ for some embedding $i\colon M \to W$ of a closed oriented manifold M if the classifying map $W \to K(\mathbb{Z}, w-m)$ factors as*

$$W \to \mathrm{MSTop}(w-m) \xrightarrow{u} K(\mathbb{Z}, w-m)$$

where $\mathrm{MSTop}(w-m)$ *is the Thom space of the universal euclidean bundle over* $\mathrm{BSTop}(w-m)$ *and u is its Thom class. In that case i admits a normal bundle.*

Proof. If the classifying map factors as a map $W \to \mathrm{MSTop}(w-m)$, then we may apply Theorem 6.32 it and obtain a pullback diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "i"] & \mathrm{BSTop}(w-m) \arrow[d] \\ W \arrow[r] & \mathrm{MSTop}(w-m) \end{tikzcd}$$

so that $g^*(\gamma) = \nu(i)$. The result then follows from Theorem 6.31 □

We hence obtain a complete answer to the question: Which cohomology classes in $H^*(W; \mathbb{Z})$ are the Poincaré dual classes associated to embedded submanifolds $M \subseteq W$ equipped with a normal bundle. In particular, in the smooth case where every embedding admits a normal bundle, we do not have to explicitly assume the existence of a normal bundle.

Moreover, from Theorem 6.31, we also find an obstruction for classes to be Poincaré dual to embedded submanifolds (possibly without normal bundle): In that case, the classifying map of the cohomology class factors as

$$W \to \mathrm{MSG}(d) \to K(\mathbb{Z}, d)$$

where $\mathrm{MSG}(d)$ is the Thom space of the universal spherical fibration over $\mathrm{BSG}(d)$.

[20]That is, f is homotopic to f' with $d(f(w), f'(w)) < \epsilon$ for any fixed ϵ.

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