TOPOLOGY IV

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Abstract. These are lecture notes for my lecture "Topology IV" which I taught in the summer term 2025 at LMU Munich.

Contents

1. Recollection/Prerequisites

There will be **no lectures** on 18.11. and 20.11. and we will **reschedule** the lecture on 23.12. We will take some time to discuss exercises I pose during the lectures; either in the beginning of each lecture or regularly (roughly) every 3 weeks. If you want to get credits for this course, you can do so under WP37 for 6ECTS. The examination will be an oral exam at the end of the term.

This course will build on the lectures Topology I (WS 23/24), Topology II (SS 24), and Topology III (WS 24/25) taught at LMU. We briefly recall the main topics that were covered, so a reader has an impression what will be the assumed background knowledge.

(1) Point-set topology
(2) Homotopy theory: homotopy groups, CW complexes, applications of cellular approximation, cofibrations, Seifert-van Kampen's theorem
(3) Covering theory; Fundamental theorem of covering theory
(4) Singular Homology; Definition, Properties, Applications.
(5) Singular Cohomology; Cup product, Universal coefficient theorems, Künneth theorem
(6) Topological Manifolds: Orientability and Poincaré duality, Applications
(7) Homotopy theory: Fibrations, long exact homotopy sequence, Whitehead's theorem, cellular approximation theorem, homotopy excision theorem, Freudenthal
(8) Hurewicz theorems
(9) Eilenberg–Mac Lane spaces and representability of cohomology
(10) Principal G-bundles
(11) Obstruction theory
(12) Steenrod operations
(13) The Leray–Hirsch theorem

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Parts (1)–(4) were covered in Topology I [Lan23], parts (5)–(7) were covered in Topology II [Win24], and parts (8)–(13) were covered in [Lan24]. The lecture notes for these courses are available on the course webpage.

The rough plan for this term is to cover the following (as much as fits into the timeframe).

(1) Thom isomorphism for spherical fibrations, Stiefel–Whitney and Wu classes
(2) Poincare duality complexes and Wu's formula
(3) A survey on manifolds, tangent bundles, Pontryagin–Thom
(4) Serre spectral sequence and the cohomology of BU, BO, remarks on BTop.
(5) The signature theorem (possibly in dimensions $4, 8$ only)
(6) Further Applications to manifolds; geometric interpretation of cup product, existence of manifolds with certain cell structures, spin$^{\mathbb{C}}$-structures + intersection form on 4-manifolds, (obstructions to the) existence of submanifolds representing homology classes, Rokhlin's theorem
(7) Construction of a non-standard homotopy 7-sphere.
(8) Cohomology of Eilenberg–Mac Lane spaces
(9) Homotopy groups of spheres using Serre's method

2. Thom isomorphism for spherical fibrations

We now move towards the Thom isomorphism for spherical fibrations. We begin with all relevant definitions.

2.1. **Definition** Let $\pi\colon E \to B$ be a fibration with typical fibre S^{d-1} and B connected. Then π is called oriented if for all γ in $\pi_1(B)$, the induced homotopy self-equivalence of S^{d-1} is orientation preserving, that is, induces the identity on $H_{d-1}(S^{d-1};\mathbb{Z})$.

2.2. **Definition** Let $\pi\colon E \to B$ be a spherical fibration of rank $d-1$ and B connected. We call its mapping cone $\mathrm{C}(\pi)$ the Thom space of π and also write $\mathrm{Th}(\pi)$ for it.

2.3. **Lemma** *Given a pullback diagram*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r] & B \end{tikzcd}$$

of spherical fibrations of rank $d-1$, there is a canonically defined map $\mathrm{Th}(\pi') \to \mathrm{Th}(\pi)$. In particular, for $B' = \{b\}$ there is a map $S^d \to \mathrm{Th}(\pi)$ and the maps $\mathrm{Th}(\pi') \to \mathrm{Th}(\pi)$ are compatible with this map.

Proof. This follows by passing to vertical (homotopy) cofibres in the pullback diagram of the statement of the lemma, together with the observation that $\mathrm{Th}(S^{d-1} \to *) \simeq S^d$. $\square$

2.4. **Definition** For $d \geq 0$, let $\mathrm{Top}(d) = \mathrm{Homeo}(\mathbb{R}^d)$ and $\mathrm{Top}_0(d)$ be the subgroup of homeomorphisms preserving the origin.

2.5. **Remark** The inclusion $\mathrm{Top}_0(d) \to \mathrm{Top}(d)$ is a homotopy equivalence with homotopy inverse given by $f \mapsto f - f(0)$. It therefore induces a homotopy equivalence $\mathrm{BTop}_0(d) \to \mathrm{BTop}(d)$.

2.6. **Remark** A fibration $\pi\colon E \to B$ with typical fibre S^{d-1} is called a spherical fibration of rank $d-1$ over B. By [Lan24, Theorem 5.9] it is classified by a map $B \to \mathrm{BhAut}(S^{d-1})$. The group $\mathrm{hAut}(S^{d-1})$ is classically denoted by $\mathrm{F}(d)$, so a rank $d-1$-spherical fibration over B is classified by a map $B \to \mathrm{BF}(d)$. Being oriented means that the classifying map lifts to $\mathrm{BhAut}^+(S^{d-1}) =: \mathrm{BSF}(d)$.

There is related concept, that of a pointed spherical fibration of rank d which is a fibration $\pi\colon E \to B$ with typical fibre S^d which is equipped with a section $s\colon B \to E$, that is, such that $\pi s = \mathrm{id}_B$. These are classified by maps $B \to \mathrm{BhAut}_*(S^d) =: \mathrm{BG}(d)$. Likewise, there is an oriented version classified by maps to $\mathrm{BhAut}_*^+(S^d) =: \mathrm{BSG}(d)$. Suspending induces group homomorphisms $\mathrm{F}(d) \to \mathrm{G}(d)$ and $\mathrm{SF}(d) \to \mathrm{SG}(d)$ and therefore maps $\mathrm{BF}(d) \to \mathrm{BG}(d)$ and $\mathrm{BSF}(d) \to \mathrm{BSG}(d)$; we will discuss in the proof of the next lemma what these maps do concretely to a spherical fibration $E \to B$.

For a pointed spherical fibration $\pi\colon E \to B$ with section $s\colon B \to E$, we can define its pointed Thom space $\mathrm{Th}_*(\pi) = \mathrm{C}(s)$ as the mapping cone of the section s.

2.7. **Lemma** *Let $\pi\colon E \to B$ be a spherical fibration of rank $d-1$ and $\Sigma(\pi)$ its associated pointed spherical fibration of rank d. Then there is a canonical equivalence* $\mathrm{Th}(\pi) \simeq \mathrm{Th}_*(\Sigma(\pi))$.

Proof. The total space $\Sigma^{\mathrm{fw}}(E)$ of $\Sigma(\pi)$ is given by the following homotopy pushout.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & \Sigma^{\mathrm{fw}}(E) \end{tikzcd}$$

Indeed, to see this, we consider the square as a homotopy pushout diagram in spaces over B. Then one uses the fact that the homotopy pullback functor from spaces over B to spaces over B', along a map $f\colon B' \to B$, preserves (homotopy) pushout diagrams. In particular, passing to fibres, one obtains a pushout diagram, showing that the homotopy fibres of the map $\Sigma(\pi)\colon \Sigma^{\mathrm{fw}}(E) \to B$ are indeed given by the suspension of the homotopy fibres of π, as is to be expected. It follows that the horizontal mapping cones in the above square are equivalent. The top horizontal cone is $\mathrm{Th}(\pi)$ and the lower one is $\mathrm{Th}_*(\Sigma(\pi))$, giving the claim. □

2.8. **Remark** In case π is a fibre bundle, we note that $\Sigma^{\mathrm{fw}}(E) = \mathrm{C}^{\mathrm{fw}}(E) \cup_E \mathrm{C}^{\mathrm{fw}}(E)$ where $\mathrm{C}^{\mathrm{fw}}(E)$ is the disk bundle classified by the composite $B \to \mathrm{BHomeo}(S^{d-1}) \to \mathrm{BHomeo}(D^d)$; its boundary is then given by π. Then the projection map $\mathrm{C}^{\mathrm{fw}}(E) \to B$ is a homotopy equivalence and $E \to \mathrm{C}^{\mathrm{fw}}(E)$ is a cofibration, showing the above lemma in the special case of fibre bundles.

2.9. **Remark** Taking the join and the suspension of homotopy equivalence and pointed homotopy equivalences yields maps

$$(\#) \qquad \mathrm{BF}(d) \times \mathrm{BF}(d') \xrightarrow{\star} \mathrm{BF}(d+d') \quad \text{and} \quad \mathrm{BG}(d) \times \mathrm{BG}(d') \xrightarrow{\wedge} \mathrm{BG}(d+d')$$

which are compatible with the previously mentioned maps $\mathrm{BF}(k) \to \mathrm{BG}(k)$. In addition, each of the above two maps restricts to the oriented versions. Given spherical fibrations of rank $d-1$ and $d'-1$ over B and B', respectively, with associated fibrations $\pi\colon E \to B$ and

$\pi'\colon E' \to B'$, we write $E \star^{\mathrm{fw}} E' \to B \times B'$ for the resulting rank $d + d' - 1$ spherical fibration over $B \times B'$; Likewise we write $E \wedge^{\mathrm{fw}} E'$ in the pointed case. The superscript fw in each case reflects the fact that the join and wedge are formed *fibrewise*. Similarly to earlier, we note that there is a homotopy pushout

$$\begin{tikzcd} E \times B' \cup B \times E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \times B' \arrow[r] & E \wedge^{\mathrm{fw}} E' \end{tikzcd}$$

Indeed, we again consider the square as a homotopy pushout in spaces over $B \times B'$. Then passing to fibres over (b, b') we obtain the pushout

$$\begin{tikzcd} E_b \times \{b'\} \cup \{b\} \times E'_{b'} \arrow[r] \arrow[d] & E_b \times E'_b \arrow[d] \\ \{(b, b')\} \arrow[r] & (E \wedge^{\mathrm{fw}} E')_{(b,b')} \end{tikzcd}$$

but then this pushout is also $E_b \wedge E'_{b'}$ as needed.

In case $B = B'$ one can further pull back these constructions along the diagonal $B \to B \times B$; By abuse of notation the resulting operation will be denoted by $E \oplus E'$.

The basepoints of $\mathrm{BF}(d')$ and $\mathrm{BG}(d')$ then induce *stabilization maps*

$$\sigma^{d'}: \mathrm{BF}(d) \to \mathrm{BF}(d + d') \quad \text{and} \quad \sigma^{d'}: \mathrm{BG}(d) \to \mathrm{BG}(d + d')$$

which concretely send a (pointed) spherical fibration π to $\sigma^{d'}(\pi)$, the fibrewise join (or wedge) with the trivial (pointed) fibration $S^{d'-1} \to *$ (or $S^d \to *$).

2.10. **Remark** Now, we note that there are group homormophisms

$$\mathrm{O}(d) \to \mathrm{GL}_d(\mathbb{R}) \to \mathrm{Top}_0(d) \to \mathrm{F}(d) \to \mathrm{G}(d).$$

The evident maps

- $\mathrm{O}(d) \times \mathrm{O}(d') \to \mathrm{O}(d + d')$,
- $\mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_{d'}(\mathbb{R}) \to \mathrm{GL}_{d+d'}(\mathbb{R})$
- $\mathrm{Top}_0(d) \times \mathrm{Top}_0(d') \to \mathrm{Top}_0(d + d')$

induce maps on classifying spaces which are compatible with each other as well as with the maps (#). Moreover, the composite $\mathrm{Top}_0(d) \to \mathrm{G}(d)$ factors through $\mathrm{Homeo}_\infty(S^d) \to \mathrm{G}(d)$ by sending a homeomorphism to its induced homeomorphism on one-point compactifications; the resulting homeomorphism then preserves the point at infinity and is then in particular a pointed homotopy self-equivalence of S^d.

It turns out that $\mathrm{O}(d) \to \mathrm{F}(d)$ also factors through $\mathrm{Homeo}(S^{d-1})$ but it turns out that there is no map $\mathrm{Top}_0(\mathbb{R}^d) \to \mathrm{Homeo}(S^{d-1})$ such that the composite with $\mathrm{O}(d) \to \mathrm{Top}(d)$ agrees with $\mathrm{O}(d) \to \mathrm{F}(d)$.[1]

[1] This is not obvious, however.

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