TOPOLOGY III

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ABSTRACT. These are lecture notes for my lecture "Topology III" which I taught in the winter term 2024/25 at LMU Munich and from the lecture "Topology IV" which I taught in the summer term 2025 at LMU Munich.

CONTENTS

1. RECOLLECTION/PREREQUISITES

There will be **no lectures** on 18.11. and 20.11. and we will **reschedule** the lecture on 23.12. We will take some time to discuss exercises I pose during the lectures; either in the beginning of each lecture or regularly (roughly) every 3 weeks. If you want to get credits for this course, you can do so under WP37 for 6ECTS. The examination will be an oral exam at the end of the term.

This course will build on the two lectures Topology I (WS 23/24) and Topology II (SS 24) taught at LMU. We briefly recall the main topics that were covered, so a reader has an impression what will be the assumed background knowledge.

(1) Point-set topology
(2) Homotopy theory: homotopy groups, CW complexes, applications of cellular approximation, cofibrations, Seifert-van Kampen's theorem
(3) Covering theory; Fundamental theorem of covering theory
(4) Singular Homology; Definition, Properties, Applications.
(5) Singular Cohomology; Cup product, Universal coefficient theorems, Künneth theorem
(6) Topological Manifolds: Orientability and Poincaré duality, Applications

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Date: June 3, 2025.

(7) Homotopy theory: Fibrations, long exact homotopy sequence, Whitehead's theorem, cellular approximation theorem, homotopy excision theorem, Freudenthal

Parts (1)–(4) were covered in Topology I [Lan23] while parts (5)–(7) were covered in Topology II [Win24]. These lecture notes are available on the course webpage.

The rough plan for this term is to cover the following.

(1) Hurewicz theorems
(2) Eilenberg–Mac Lane spaces and representability of cohomology
(3) Principal G-bundles; classification, characteristic classes (definitions), homotopy orbits of group actions
(4) Obstruction theory
(5) Steenrod operations; Definitions, Cartan formula, Adem relation, $\mathrm{Sq}^0 = \mathrm{id}$.
(6) Vector bundles and the Thom isomorphism, Stiefel–Whitney and Wu classes.
(7) Applications to manifolds; geometric interpretation of cup product, existence of manifolds with certain cell structures, $\mathrm{spin}^{\mathbb{C}}$-structures + intersection form on 4-manifolds, (obstructions to the) existence of submanifolds representing homology classes.

2. Weak equivalences, singular homology, and the Hurewicz theorem

We first record the following fundamental property of singular homology:

2.1. **Theorem** *Let $f\colon X \to Y$ be a weak equivalence of topological spaces. Then f induces an isomorphism in singular homology.*

For the proof, we will use simplicial techniques. Let us recall some things about simplicial sets; see [Lan23, §4.3] and of course any other source on simplicial sets for more details.

2.2. **Recollection** We let Δ be the full subcategory of the category of posets on the objects $[n] = \{0, \dots, n\}$ with its evident linear order. Hence, a morphism $f\colon [n] \to [m]$ in Δ is a map $f\colon \{0, \dots, n\} \to \{0, \dots, m\}$ such that $f(a) \leq f(b)$ if $a \leq b$. We let sSet be the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ of presheaves on Δ. We denote the image of $[n]$ under the Yoneda embedding by Δ^n. Note that $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) = X([n]) =: X_n$ by the Yoneda lemma. We recall that the collection of topological n-simplices $\Delta^n_{\mathrm{top}} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum_i x_i = 1\}$ for a cosimplicial topological space, that is a functor $\Delta^\bullet_{\mathrm{top}}\colon \Delta \to \mathrm{Top}$. We recall that for $0 \leq i \leq n$ there are maps $\delta^n_i\colon [n-1] \to [n]$ uniquely determined by being injective and such that $i \notin \mathrm{Im}(\delta^n_i)$. Likewise, there are map $\sigma^n_i\colon [n] \to [n-1]$, uniquely determined by being surjective and satisfying $\sigma^n_i(i) = \sigma^n_i(i+1)$. These maps satisfy the *simplicial relations*, see [Lan23, 4.15]. Any injective map in Δ is a composite of δ_i's and any surjective map is a composite of σ_i's. Any map in Δ is a composite of an surjection followed by an injection. In particular, giving a simplicial set X is the same data as specifying sets X_n together with induced maps $d^n_i = (\delta^n_i)^*\colon X_n \to X_{n-1}$ and $s^n_i = (\sigma^n_i)^*\colon X_{n-1} \to X_n$ satisfying the simplicial relations. An n-simplex of a simplicial set is called *degenerate* if it is of the form $f^*(y)$ for some surjective map $f\colon [n] \to [m]$ different from the identity and some m-simplex y. Equivalently, if x is of the form $s^n_i(y)$ for some y and i. It is called non-degenerate if it is not degenerate. Every simplex x in a simplicial set X is uniquely the degeneration of a non-degenerate simplex, i.e. there exists a unique map $f\colon [n] \to [m]$ and a unique element $y \in X_m$ such that y is non-degenerate and $f^*(y) = x$.

Further important simplicial sets are:

(1) $\partial\Delta^n \subseteq \Delta^n$: Its k-simplices are precisely those maps $f \in \mathrm{Hom}_\Delta([k], [n])$ which are not surjective.

(2) $\Lambda^n_k \subseteq \partial\Delta^n$ for $0 \leq k \leq n$: Its k-simplices are those maps $f \in \mathrm{Hom}_\Delta([k],[n])$ such that k is not in the image of f.

In both cases, one readily checks that these are in fact sub simplicial sets.

2.3. **Lemma** *There is an adjunction*

$$|-|\colon \mathrm{sSet} \rightleftarrows \mathrm{Top}\colon \mathrm{Sing}$$

whose left adjoint $|-|$ *is uniquely determined by the requirement* $|\Delta^n| = \Delta^n_{\mathrm{Top}}$.

Proof. The universal property of presheaf categories like $\mathrm{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ says that any functor $\Delta \to \mathrm{Top}$, like $[n] \mapsto \Delta^n_{\mathrm{Top}}$ extends to a unique left adjoint $\mathrm{sSet} \to \mathrm{Top}$. This is the definition of $|-|$. In this generality, the right adjoint is then given by sending an object X to the presheaf sending $[n]$ to $\mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, X)$; this means that $\mathrm{Sing}(X)$ is the simplicial set $\mathrm{Hom}_{\mathrm{Top}}(\Delta^\bullet_{\mathrm{top}}, X)$. □

2.4. **Remark** The functor $|-|$ is called the geometric realization of a simplicial set, and the functor $\mathrm{Sing}(-)$ is the singular complex functor. A concrete formula is given as follows:

$$|X| = \Big(\coprod_{n\geq 0} X_n \times \Delta^n_{\mathrm{Top}}\Big)/\sim$$

where for each $f\colon [m] \to [n]$ a morphism in Δ, $x \in X_n$ and $t \in \Delta^m_{\mathrm{Top}}$, we have $(f^*(x), t) \sim (x, f_*(t))$. Exercise: Prove this fact. Note also that the geometric realization of an injection of simplicial sets is a (closed) subspace inclusion.

2.5. **Definition** Let X be a simplicial set. We let $\mathrm{sk}_n(X)$ be the simplicial set $i_!i^*!(X)$, where $i\colon \Delta_{\leq n} \subseteq \Delta$ is the full subcategory inclusion on objects $[k]$ for $k \leq n$. Dually, we let $\mathrm{cosk}_n(X)$ be $i_*i^*(X)$. Here, $i_!$ and i_* denote the left and right adjoints of i^*, respectively. These are also left and right Kan extensions. More concretely, we have that the counit map $\mathrm{sk}_n(X) = i_!i^*(X) \to X$ exhibits $\mathrm{sk}_n(X)$ as the smallest subsimplicial set of X with same k-simplices as X for $k \leq n$; that is, all k-simplices for $k > n$ are degenerate.

2.6. **Example** We have $\mathrm{sk}_n(\Delta^n) = \Delta^n$ and $\mathrm{sk}_{n-1}(\Delta^n) = \partial\Delta^n$.

2.7. **Lemma** *For every simplicial set, there is an isomorphism* $\mathrm{colim}_n\, \mathrm{sk}_n(X) \to X$ *and there are canonical pushouts*

$$\begin{tikzcd} \coprod_{I_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X) \arrow[d] \\ \coprod_{I_n} \Delta^n \arrow[r] & \mathrm{sk}_n(X) \end{tikzcd}$$

of simplicial sets.

Proof. The colimit statement is clear: Every simplex of X has a dimension. So let us discuss the pushout. The lower horizontal map in the putative pushout diagram is defined by Yoneda's lemma: A non-degenerate n-simplex of X determines a map $\Delta^n \to X$ to which we may apply $\mathrm{sk}_n(-)$. Then we may apply sk_{n-1} to the lower map and obtain the upper horizontal map. The natural transformation $\mathrm{sk}_{n-1} \to \mathrm{sk}_n$ then provides the vertical maps and the fact that the resulting square commutes. To see that the diagram is a pushout, we need to argue on the level of k-simplices for all k. If $k \leq n-1$, then both vertical maps are isomorphisms on k-simplices. Now we discuss $\mathrm{sk}_n(X)_n$: Since every n-simplex is either degenerate or not, and the non-degenerate simplices are in the image of the lower horizontal map and the degenerate simplices are in the image of the right vertical map, to see that it is a pushout, we only need to observe for a non-degenerate simplex α in X_n, if $f\colon [n] \to [n]$ is such that $f^*(\alpha) \in \mathrm{sk}_{n-1}(X)_n$, then f is not surjective, i.e. that it is not the identity. But this is clear, because a non-degenerate n-simplex of X is *not* an n-simplex of $\mathrm{sk}_{n-1}(X)$. The argument for $\mathrm{sk}_n(X)_k$ with $k > n$ is similar: First, we note that any element there is degenerate, and hence as discussed above the unique degeneration of a non-degenerate smaller dimensional simplex. Either that non-degenerate simplex is of dimension n or of dimension smaller than n. In the former case, the simplex in question comes from the lower horizontal morphisms, and in the latter case, it comes from the right vertical morphism. Again, to see that the square is in fact a pushout, one argues that if for $\alpha \in X_n$ non-degenerate, and $f\colon [k] \to [n]$ we have $f^*(\alpha) \in \mathrm{sk}_{n-1}(X)_l$, then f is not surjective. We leave this part as an exercise. $\square$

2.8. **Corollary** *Let X be a simplicial set. Then $|X|$ is canonically a CW complex.*

Proof. We define a filtration on $|X|$ by $\mathrm{sk}_n(|X|) := |\mathrm{sk}_n(X)|$ - this is well-defined because geometric realization of the skeletal inclusions are again (closed) inclusions and geometric realization commutes with colimits (as it is a left adjoint). In particular, it also commutes with the pushouts of Lemma 2.7, so we obtain pushouts

$$\begin{tikzcd} \coprod_{I_n} |\partial\Delta^n| \arrow[r] \arrow[d] & |\mathrm{sk}_{n-1}(X)| \arrow[d] \\ \coprod_{I_n} |\Delta^n| \arrow[r] & |\mathrm{sk}_n(X)| \end{tikzcd}$$

Now, by definition we have $|\Delta^n| = \Delta^n_{\mathrm{Top}}$ which is homeomorphic to D^n. From this, one can deduce (exercise) that $|\partial\Delta^n| = \partial\Delta^n_{\mathrm{Top}}$ which is homeomorphic to S^{n-1}. This shows that the geometric realization of the skeletal filtration on X is a CW filtration on $|X|$. $\square$

2.9. **Remark** Conversely, the image of $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$ is also restricted: For any topological space X, we have that $\mathrm{Sing}(X)$ is a Kan complex, i.e. it satisfies the lifting property against the horn inclusions $\Lambda^n_k \to \Delta^n$ for all $0 \leq k \leq n$. Essentially, this follows from the above adjunction and the fact that the inclusion $\Lambda^n_k \subseteq \Delta^n$ geometrically realizes to an inclusion which admits a retraction. Here Λ^n_k denotes the sub simplicial set of $\partial\Delta^n$ in which also the face opposite to the 0-simplex $\{k\}$ is removed.

Even more, if $f\colon X \to Y$ is a Serre fibration, the $\mathrm{Sing}(f)\colon \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a Kan fibration. This again follows from the above adjunction, the observation that $\Lambda^n_k \to \Delta^n$ geometrically realizes to a cofibration which is also a homotopy equivalence, and [Win24, Cor. 5.2.6].

2.10. **Definition** Let X and Y be simplicial sets and $f, g\colon X \to Y$ maps of simplicial sets. A simplicial homotopy from f to g is a map $h\colon X \times \Delta^1 \to Y$ such that $h_{|X\times\{0\}} = f$ and $h_{|X\times\{1\}} = g$. We denote by $[X, Y]$ the quotient of $\mathrm{Hom}_{\mathrm{sSet}}(X, Y)$ be the equivalence relation generated by the relation homotopy.

2.11. **Remark** The relation of homotopy on $\mathrm{Hom}_{\mathrm{sSet}}(X,Y)$ is typically neither symmetric nor transitiv. (Exercise: proof this). Nevertheless, there is the following coequalizer diagram in sets:

$$\mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta^1, Y) \rightrightarrows \mathrm{Hom}_{\mathrm{sSet}}(X,Y) \longrightarrow [X,Y]$$

where the two maps are induced by the inclusions $\{0\},\{1\} \subseteq \Delta^1$.

If you know what a Kan complex is, then you can try to prove that if Y is a Kan complex, then the homotopy relation on $\mathrm{Hom}_{\mathrm{sSet}}(X,Y)$ is in fact an equivalence relation. You may want to prove or use that when Y is Kan, so is $\mathrm{Hom}(X,Y)$, the internal hom in simplicial sets.

It is elementary to check that composition and the identity map descend to a composition map

$$[Y,Z] \times [X,Y] \to [X,Z]$$

and to elements $[\mathrm{id}_X] \in [X,X]$, satisfying the axioms of a category.

2.12. **Definition** We denote the category with objects the simplicial sets and with morphism sets given by $[X,Y]$ by hsSet; the homotopy category of simplicial sets.

Coming back to our aim to prove weak-homotopy invariance of singular homology, we could have observed the following lemma already in Topology I.

2.13. **Lemma** *For $n \geq 0$, the singular homology groups $H_n(-)$ viewed as functors* Top $\to$ Ab *factor as follows:*

$$\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \to \mathrm{hsSet} \to \mathrm{Ab}.$$

Proof. Since hsSet is obtained from sSet by dividing out an equivalence relation on morphism sets, it suffices to construct a functor sSet $\to$ Ab which is compatible with the generating relation and whose restriction along Sing: Top $\to$ sSet is given by $H_n(-)$. This is easy: Given a simplicial set X, we first construct a chain complex $C_\bullet(X)$ by the same formulas as in [Lan23]: $C_n(X) = \mathbb{Z}[X_n]$, the free abelian group on the set of n-simplices of X. The differential in the chain complex is given by

$$\partial_n = \sum_{i=0}^{n} d_i \colon \mathbb{Z}[X_n] \to \mathbb{Z}[X_{n-1}].$$

That this is in fact a chain complex and the fact that $C_\bullet(\mathrm{Sing}(X)) = C_\bullet^{\mathrm{sing}}(X)$ was verified again in [Lan23]. Now, the claim is that given a simplicial homotopy $h \colon X \times \Delta^1 \to Y$ from f to g, then f and g induce the same map on $H_n(-)$. In fact, they induce homotopic maps $C_\bullet(X) \to C_\bullet(Y)$, and hence equal maps on homology, just as utilized in [Lan23]. For this, we can either reprove homotopy invariance of this chain complex using an version of the acyclic models result we have used in the case of topological spaces, or we can also just use the Prism operator as described in [Lan23, Remark 4.66] is obtained by triangulating the prism $\Delta^n \times \Delta^1$, which in fact gives elements $h_n \in C_{n+1}(\Delta^n \times \Delta^1)$ – recall that such elements give rise to natural maps $h_n^X : C_n(X) \to C_{n+1}(X \times \Delta^1)$ – and that these elements satisfy relations which imply that the maps h_n^X form a chain homotopy from the map $X \times \delta_1$ to $X \times \delta_0$, where $\delta^1 \colon \{0\} \to \Delta^1$ and $\delta^0 \colon \{1\} \to \Delta^1$ are the evident inclusions. This proves the lemma. $\square$

The decisive fact about weak equivalences is now the following proposition, giving a proof of Theorem 2.1.

2.14. **Proposition** *Let $f\colon X \to Y$ be a weak equivalence of topological spaces. Then the map* $\mathrm{Sing}(f)\colon \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ *is sent to an isomorphism under the functor* $\mathrm{sSet} \to \mathrm{hsSet}$. *In particular, a weak equivalence induces an isomorphism on singular homology.*

Proof. By Yoneda's lemma, we aim to show that for any simplicial set Z, the map

$$[Z, \mathrm{Sing}(f)]\colon [Z, \mathrm{Sing}(X)] \to [Z, \mathrm{Sing}(Y)]$$

is a bijection. Exercise: The above adjunction induces bijections making this map isomorphic to the map

$$[|Z|, f]\colon [|Z|, X] \to [|Z|, Y].$$

Now recall that $|Z|$ is a CW complex by Corollary 2.8, and that we have shown in Topology II that this map is therefore a bijection [Win24, Theorem 5.2.18]. $\square$

2.15. **Remark** In the above exercise, it will be crucial to know that for all simplicial sets X, the canonical map $|X \times \Delta^1| \to |X| \times |\Delta^1| = |X| \times \Delta^1_{\mathrm{top}}$ is a homeomorphism. One way to prove this is to regard this canonical map as the component of a natural transformation of functors (in the variable X). Then we observe that source and target both commute with colimits in X. Therefore, it suffices to show that this map is a homeomorphism in case $X = \Delta^n$ in which case it is a direct argument.

2.1. **The Hurewicz theorem.** Next, we will discuss the theorem of Hurewicz, which is about a comparison between homotopy and singular homology groups. To state it, we first recall the Hurewicz homomorphism. To that end, recall that we have fixed generators $[S^n]$ of $H_n(S^n; \mathbb{Z})$ inductively via the suspension isomorphism and a once in life chosen generator of $\tilde{H}_0(S^0; \mathbb{Z})$.

2.16. **Definition** Let (X, x) be a pointed topological space and $n \geq 1$. The *Hurewicz homomorphism* is the map

$$h_n\colon \pi_n(X, x) \to H_n(X; \mathbb{Z}), \quad [f] \mapsto f_*([S^n]).$$

This is well-defined because singular homology is homotopy invariant.

2.17. **Lemma** *The Hurewicz homomorphism is a natural group homomorphism, compatible with the suspension operation and is an isomorphism for S^n.*

Proof. The Hurewicz homomorphism h_n sends the constant map to 0 since $H_n(*; \mathbb{Z}) = 0$ (recall that we assume $n \geq 1$). Moreover, given $f, g \in \pi_n(X, x)$, its sum is represented by the map

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f+g} X$$

so it suffices to recall that $H_n(S^n \vee S^n; \mathbb{Z}) \cong H_n(S^n; \mathbb{Z}) \oplus H_n(S^n; \mathbb{Z})$, that the pinch map sends $[S^n]$ to $([S^n], [S^n])$ and that therefore, the composite sends $[S^n]$ to $f_*[S^n] + g_*[S^n]$, showing that h_n is a group homomorphism. Naturality of h_n means that for any pointed map of spaces $\varphi\colon X \to Y$, the diagram

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{h_n} & H_n(X; \mathbb{Z}) \\ \big\downarrow{\scriptstyle \varphi_*} & & \big\downarrow{\scriptstyle \varphi_*} \\ \pi_n(Y, y) & \xrightarrow{h_n} & H_n(Y; \mathbb{Z}) \end{array}$$

commutes. This is immediate from the functoriality of singular homology. The compatibility with the suspension operation is the claim is that the following diagram, in which the lower horizontal map is the suspension isomorphism, commutes.

$$\begin{tikzcd} \pi_n(X) \arrow[r] \arrow[d] & \pi_{n+1}(\Sigma X) \arrow[d] \\ H_n(X) \arrow[r] & H_{n+1}(\Sigma X) \end{tikzcd}$$

This follows from the fact that we have chosen the fundamental class of S^{n+1} to be the suspension of the fundamental class of S^n and the naturality of the suspension isomorphism in homology. To see the final claim, by compatibility with the suspension isomorphism, it suffices to show that $h_1 \colon \pi_1(S^1) \to H_1(S^1; \mathbb{Z})$ is an isomorphism.[1] This follows from the fact that the degree map $\deg \colon \pi_1(S^1) \to \mathbb{Z}$ is an isomorphism. □

2.18. **Remark** There is a Hurewicz homomorphism for pairs of spaces: Recall that for a pair of spaces (X, A), i.e. a space X with pointed subspace $a \in A \subseteq X$, there is the relative homotopy set $\pi_n(X, A)$; it is represented e.g. by maps of pairs $(D^n, S^{n-1}) \to (X, A)$ and the relation is relative homotopy as discussed in [Lan23, Def. 2.1]. In general, $\pi_n(X, A)$ is then only a set when $n \leq 1$, is a group when $n = 2$ and an abelian group when $n > 2$. There is then a relative fundamental class $H_n(D^m, S^{n-1}; \mathbb{Z})$ and the relative Hurewicz homomorphism is defined similarly:

$$\pi_n(X, A) \to H_n(X, A), \quad [f] \mapsto f_*([D^n, S^{n-1}]).$$

We now note the map of pairs $(X, A) \to (X/A, [A])$ induces a map $\pi_n(X, A) \to \pi_n(X, A)$. Since homotopy groups and singular homology groups are invariant under weak equivalences, we may assume that (X, A) is a CW pair, and in particular that $A \subseteq X$ is a cofibration. It follows straight from the definitions that the relative Hurewicz homomorphism is then also given by the composite

$$\pi_n(X, A) \to \pi_n(X/A, [A]) \xrightarrow{h} H_n(X/A; \mathbb{Z}) \xleftarrow{\simeq} H_n(X, A; \mathbb{Z}).$$

2.19. **Remark** The relative homotopy groups $\pi_n(X, A)$ fit into an evident long exact sequence

$$\dots \to \pi_{n+1}(X, A) \to \pi_n(A, a) \to \pi_n(X, a) \to \pi_n(X, A) \to \dots$$

where the boundary map simply restricts to the appropriate cube of smaller dimension. Let $F = \mathrm{hofib}_a(A \to X)$. Exercise: Construct a canonical bijection $\pi_n(X, A) \cong \pi_{n-1}(F)$ compatible with the long exact sequences and show that the Hurewicz homomorphism induces a commutative diagram

$$\begin{tikzcd} \dots \pi_{n+1}(A) \arrow[r] \arrow[d] & \pi_{n+1}(X) \arrow[r] \arrow[d] & \pi_{n+1}(X, A) \arrow[r] \arrow[d] & \pi_n(A) \arrow[r] \arrow[d] & \pi_n(X) \dots \arrow[d] \\ \dots H_{n+1}(A) \arrow[r] & H_{n+1}(X) \arrow[r] & H_{n+1}(X, A) \arrow[r] & H_n(A) \arrow[r] & H_n(X) \dots \end{tikzcd}$$

[1]Indeed, it was argued in [Lan23, Win24] that the iterated suspension map $\pi_1(S^1) \to \pi_n(S^n)$ is an isomorphism.

2.20. **Remark** Let $F \to E \to B$ be a fibration sequence. Using the above exercise, one also obtains a commutative diagram

$$\begin{tikzcd} \dots\pi_{n+1}(E) \arrow[r] \arrow[d] & \pi_{n+1}(B) \arrow[r] \arrow[d] & \pi_n(F) \arrow[r] \arrow[d] & \pi_n(E) \arrow[r] \arrow[d] & \pi_n(B)\dots \arrow[d] \\ \dots H_{n+1}(E) \arrow[r] & H_{n+1}(B) \arrow[r] & H_{n+1}(\mathrm{C}(p)) \arrow[r] & H_n(E) \arrow[r] & H_n(B)\dots \end{tikzcd}$$

where the middle vertical map is induced by the map suspension map $\pi_n(F) \to \pi_{n+1}(\Sigma F)$, the map $\Sigma F \to \mathrm{C}(p)$ as well as the Hurewicz homomorphism for $\mathrm{C}(p)$.

Finally, we recall the notion of n-connected maps of spaces.

2.21. **Definition** Let $f\colon X \to Y$ be a map of spaces and $n \geq 0$. It is called *n-connected* if it induces a surjection on π_n and a bijection on π_k for $k < n$ (at every basepoint of X, respectively).

2.22. **Remark** A pair of spaces (X, A) is called n-connected if the inclusion map $i\colon A \to X$ is n-connected. This in turn is equivalent to the condition that $\pi_k(X, A) = \{*\}$ for $k \leq n$. In general, a map f is n-connected if and only if for all basepoints $y \in Y$, the space $\mathrm{hofib}_y(f)$ is $(n-1)$-connected, compatible with the Exercise in Remark 2.19.

We now recall the theorem of Blakers–Massey, i.e. the homotopy excision theorem, since we will derive the Hurewicz theorem from it.

2.23. **Theorem** *Consider a homotopy pushout square*[2]

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X \end{tikzcd}$$

in which the map j is n-connected and the map i is m-connected. Then, for all $a \in A$, the map $\mathrm{hofib}_a(j) \to \mathrm{hofib}_{f(a)}(g)$ is $(n+m-1)$-connected.

Proof. See [Win24, Thm. 5.4.1] and use that we may replace C by a homotopy equivalent open neighborhood in A and B. $\square$

As a consequence we record the following corollary which we might also be using a number of times:

2.24. **Corollary** *Let $f\colon X \to Y$ be a map and $X \to Y \to \mathrm{C}(f)$ its associated homotopy cofibration sequence. If f is n-connected and X is m-connected, then the map $X \to \mathrm{hofib}(Y \to \mathrm{C}(f))$ is $(n+m-1)$-connected. In particular, there is a along exact sequence*

$$\pi_{n+m}(X) \to \pi_{n+m}(Y) \to \pi_{n+m}(\mathrm{C}(f)) \to \pi_{n+m-1}(X) \to \dots$$

In general, this sequence does not *extend further to the left.*

[2] That is, a pushout square in which the maps i and j are relative CW inclusions

Proof. Apply Theorem 2.23 to the homotopy pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathrm{C}(X) \arrow[r] & \mathrm{C}(f) \end{tikzcd}$$

and use that the map $X \to \mathrm{C}(X) \simeq *$ is precisely as connected as X is. □

2.25. **Remark** Considering the map $S^n \to *$, we find that the map $S^n \to \Omega S^{n+1}$ is $(2n-1)$-connected. In this case, to extend the sequence appearing in Corollary 2.24 one further to the left says that this map induces an isomorphism on π_{2n-1}, not only a surjection. This is incorrect for $n = 2$, in which this map is isomorphic to the canonical projection map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, see [Win24, Remark 5.4.9].

2.26. **Example** Consider a map $f\colon \bigvee_\beta S^n \to \bigvee_\alpha S^n$ and write $X = \mathrm{C}(f)$. Then the map itself is $(n-1)$-connected and for all $k \geq 0$,we obtain the following exact sequence.

$$\pi_{2n-2-k}(\bigvee_\beta S^n) \to \pi_{2n-2-k}(\bigvee_\alpha S^n) \to \pi_{2n-2-k}(X) \to \pi_{2n-3-k}(\bigvee_\beta S^n) \to \pi_{2n-3-k}(\bigvee_\alpha S^n)$$

When $n \geq 2$, we find $2n-2 \geq 0$ and hence in particular obtain that $\pi_n(X) \cong \mathrm{coker}(\pi_n(f))$. This isomorphism in fact also holds when $n = 1$ by the theorem of Seifert–van Kampen.

We come to the absolute Hurewicz theorem.

2.27. **Theorem** (1) *Let X be a connected space. Then $\pi_1(X,x) \to H_1(X;\mathbb{Z})$ is surjective and induces an isomorphism $\pi_1(X,x)^{\mathrm{ab}} \to H_1(X;\mathbb{Z})$.*

(2) *Let X be an $(n-1)$-connected space with $n \geq 2$. Then the map $h_n\colon \pi_n(X,x) \to H_n(X;\mathbb{Z})$ is bijective and the map $h_{n+1}\colon \pi_{n+1}(X,x) \to H_{n+1}(X;\mathbb{Z})$ is surjective.*

Proof. Since homotopy and homology are invariant under weak equivalences Theorem 2.1, we may assume that X is a CW complex, which in case (2) satisfies $X_{n-1} = \mathrm{sk}_{n-1}(X) = *$. We recall that part (1) has been proven in Topology I, see [Lan23, Prop. 4.25]. But the argument for (2) we will now give in fact also applies to case (1) by Example 2.26. Now we prove (2). We observe that $X_n = \bigvee_\alpha S^n$ is a wedge of n-spheres, since we have $X_{n-1} = *$. Moreover, the diagram

$$\begin{tikzcd} \pi_n(X_{n+1}) \arrow[r] \arrow[d] & H_n(X_{n+1};\mathbb{Z}) \arrow[d] \\ \pi_n(X) \arrow[r] & H_n(X;\mathbb{Z}) \end{tikzcd}$$

whose vertical maps are induced by the inclusion $X_{n+1} \subseteq X$ commutes by the naturality of the Hurewicz homomorphism, and the vertical maps are isomorphisms: On homotopy, this follows by cellular approximation, and on homology we have argued this in Topology I, see [Lan23, Cor. 4.76]. We may hence assume that X is replaced by X_{n+1}. Now, we consider the iterated pushout diagram

$$\begin{tikzcd} \coprod_\beta S^n \arrow[r, "p"] \arrow[d] & \bigvee_\beta S^n \arrow[r] \arrow[d] & \bigvee_\alpha S^n \arrow[d] \\ \coprod_\beta D^{n+1} \arrow[r, "p"] & \bigvee_\beta D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

in which the top horizontal composite is homotopic to the attaching map to obtain X_{n+1} from X_n: Indeed, all the attaching maps can be homotoped to be pointed since X_n is connected. Writing X in place of X_{n+1}, this shows that there is a homotopy cofibre sequence

$$\bigvee_\beta S^n \xrightarrow{f} \bigvee_\alpha S^n \to X.$$

By Example 2.26, we obtain an exact sequence

$$\pi_n(\bigvee_\beta S^n) \to \pi_n(\bigvee_\alpha S^n) \to \pi_n(X) \to 0.$$

By naturality of the Hurewicz homomorphism, we obtain that the following diagram commutes:

$$\begin{tikzcd} \pi_n(\bigvee_\beta S^n) \arrow[r] \arrow[d] & \pi_n(\bigvee_\alpha S^n) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & 0 \\ H_n(\bigvee_\beta S^n;\mathbb{Z}) \arrow[r] & H_n(\bigvee_\alpha S^n) \arrow[r] & H_n(X;\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

Now, the canonical map $\bigoplus_\beta \pi_k(S^n) \to \pi_k(\bigvee_\beta S^n)$ is an isomorphism for $k < 2n$ (Exercise, or see [Win24, 5.3.4]). The same is true for singular homology. We deduce from Lemma 2.17 that the left and middle vertical maps are isomorphisms. Consequently, so is the right vertical map.

To show that the map $h_{n+1}\colon \pi_{n+1}(X) \to H_{n+1}(X;\mathbb{Z})$ is surjective, we may again compare with X_{n+1}: On both $\pi_{n+1}(-)$ and $H_{n+1}(-;\mathbb{Z})$, the map $X_{n+1} \to X$ induces a surjection, so it suffices to prove the result for X_{n+1}. One can always choose a CW structure on X such that X_{n+1} is homotopy equivalent to $M(A,n) \vee \bigvee_\alpha S^{n+1}$: Here, A is an abelian group (namely $\pi_n(X)$), for which we choose a presentation $A = \mathrm{coker}(f\colon \bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z})$ where we may assume that f is injective! The matrix which represents the map f determines a map $\bigvee_I S^n \to \bigvee_J S^n$ whose homotopy cofibre is $M(A,n)$: This is a space with a single (reduced) homology group, namely A in degree n. Clearly, we have a map $M(A,n) \to X$ which induces an isomorphism on $\pi_n(-)$. For an appropriate set S we may therefore build $M(A,n)\vee\bigvee_S S^{n+1}$ and a map to X which is now in addition surjective on π_{n+1}. Then we continue to attach cells of higher dimension to make the map $M(A,n) \vee \bigvee_S S^{n+1} \to X$ a homotopy equivalence, showing that the $(n+1)$-skeleton of X can be chosen as claimed.[3] Hence, we are reduced to proving that the Hurewicz map

$$h_{n+1}\colon \pi_{n+1}(M(A,n)\bigvee_S S^{n+1}) \to H_{n+1}(M(A,n) \vee \bigvee_S S^{n+1})$$

[3]Alternatively, we know that X_{n+1} is given by $\mathrm{C}(f)$ for *some* map $f\colon \bigvee_\beta S^n \to \bigvee_\alpha S^n$, and homotopy classes of such maps are equivalent to their effect on π_n, in particular, f is determined up to homotopy by an integral matrix. This can be put into Smith normal form, i.e. is zero outside of the diagonal. The cofibre of such a map is also as claimed.

is surjective. Since homology of a wedge is the sum of homology, this reduces to the case of S^{n+1}, where we have already argued that the Hurewicz is an isomorphism, and $M(A,n)$ in which case $H_{n+1}(M(A,n);\mathbb{Z}) = 0$ – precisely because we have chosen A to be constructed as a cokernel of an *injective* map. □

To state the relative Hurewicz theorems, we need to recall some things about π_1-actions in fibrations. In the following lemma, we assume that for all points $e \in E$, the inclusion $\{e\} \to E_{p(e)} = p^{-1}(p(e))$ is a cofibration. We also use the analog of [Win24, 5.2.6] for fibrations not merely Serre fibrations).

2.28. **Lemma** *Let $p\colon E \to B$ be a fibration, $e \in E$ and $b = p(e)$ and write E_b for $p^{-1}(b)$. Then are associated canonical fibre functors $\Phi_p\colon \tau_{\le 1}(B) \to \mathrm{hTop}$ and $\Psi_p\colon \tau_{\le 1}(E) \to \mathrm{hTop}_*$ fitting into the commutative square*

$$\begin{tikzcd}
\tau_{\le 1}(E) \arrow[r] \arrow[d] & \mathrm{hTop}_* \arrow[d] \\
\tau_{\le 1}(B) \arrow[r] & \mathrm{hTop}
\end{tikzcd}$$

The constructions of Φ_p and Ψ_p are compatible with pullbacks of fibrations.

Proof. We define Φ_p as follows: On objects, it sends $b \in B$ to the image of E_b under the tautological functor $\mathrm{Top} \to \mathrm{hTop}$. To describe the effect on morphisms, we let $\gamma\colon [0,1] \to B$ be a path from b to b'. We then consider the diagram:

$$\begin{tikzcd}
E_b \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
E_b \times [0,1] \arrow[r, "\gamma \mathrm{pr}_2"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

Then $h(-,1)$ defines a map $\Phi_p(\gamma)\colon E_b \to E_{b'}$. We now show that the homotopy class of $\Phi_p(\gamma)$ is independent of the choice of h and depends only on the homotopy rel endpoint class of γ. Indeed, assume that γ is homotopic rel endpoint to γ' and that h is a choice of a dashed arrow for γ and h' a choice of a dashed arrow for γ'. Pick a map homotopy rel endpoints $\Gamma\colon [0,1] \times [0,1] \to B$ such that $\Gamma(-,0) = \gamma$ and $\Gamma(-,1) = \gamma'$. Consider then the diagram

$$\begin{tikzcd}
(E_b \times [0,1] \times \{0,1\}) \cup (E_b \times \{0\} \times [0,1]) \arrow[r, "h \cup h' \cup \mathrm{const}_{i_b}"] \arrow[d] & E \arrow[d] \\
E_b \times [0,1] \times [0,1] \arrow[r, "\Gamma \mathrm{pr}_{2,3}"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

One readily checks that this diagram commutes and since p is a fibration, a dotted arrow exists. Then we find

(1) $pH(e,1,t) = \Gamma(1,t) = b'$ for all $t \in [0,1]$ since Γ is a homotopy rel endpoints. In particular, $H(e,1,t) \in E_{b'}$ for all $t \in [0,1]$.
(2) $H(e,1,0) = h(e,1)$ and $H(e,1,1) = h'(e,1)$.

Hence, $H(-,1,-)\colon E_b \times [0,1] \to E_{b'}$ is a homotopy from $\Phi_p(\gamma)$ to $\Phi_p(\gamma')$. Exercise: Use this to show that the so-defined Φ_p is a functor.

We now argue that we can similarly define a functor $\Psi_p\colon \tau_{\leq 1}(E) \to \mathrm{hTop}_*$: On objects, it sends e to $E_{p(e)}$. Given a path $\gamma\colon e \to e'$ in E, we consider the diagram

$$\begin{tikzcd} E_{p(e)} \times \{0\} \cup \{e\} \times [0,1] \arrow[r] \arrow[d] & E \arrow[d] \\ E_{p(e)} \times [0,1] \arrow[r] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

and again wish to define $\Psi_p(\gamma) = h(-,1)$. The argument that this is independent of the choice of h and only depends on the homotopy rel endpoints class of γ is similar as above. It follows again from the independence of all choices involved that the diagram

$$\begin{tikzcd} \tau_{\leq 1}(E) \arrow[r] \arrow[d] & \mathrm{hTop}_* \arrow[d] \\ \tau_{\leq 1}(B) \arrow[r] & \mathrm{hTop} \end{tikzcd}$$

commutes. Finally, it is a direct check, again using the independence of all choices, that if

$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback diagram with p a fibration, the composites

$$\tau_{\leq 1}(B') \xrightarrow{f_*} \tau_{\leq 1}(B) \xrightarrow{\Phi_p} \mathrm{hTop} \quad \text{and} \quad \tau_{\leq 1}(E') \xrightarrow{\bar{f}_*} \tau_{\leq 1}(E) \xrightarrow{\Psi_p} \mathrm{hTop}_*$$

are given by $\Phi_{p'}$ and $\Psi_{p'}$, respectively. □

We collect some further properties of the above constructions.

2.29. **Remark** (1) We continue to consider a fibration $p\colon E \to B$. The above implies, in particular, that for each point $b \in B$, there are group homomorphism $\pi_1(B,b) \to \pi_0(\mathrm{hAut}(E_b))$ and $\pi_1(E,e) \to \pi_0(\mathrm{hAut}_*(E_b), e)$ fitting into a commutative diagram.

$$\begin{tikzcd} \pi_1(E,e) \arrow[r] \arrow[d] & \pi_0(\mathrm{hAut}_*(E_b)) \arrow[d] \\ \pi_1(B,b) \arrow[r] & \pi_0(\mathrm{hAut}(E_b)) \end{tikzcd}$$

(2) One may apply these observations in particular to the fibration $Y \to *$. In this case, we obtain a map $\pi_1(Y,y) \to \pi_0(\mathrm{hAut}_*(Y,y)) =: [Y,Y]_*^\times$. Since $\pi_n(-)\colon \mathrm{hTop}_* \to \mathrm{Grp}$ is a functor, one obtains an action of $\pi_1(Y,y)$ on $\pi_n(Y,y)$ for all $n \geq 1$. For $n = 1$, this is really just the evident action by conjugation. We say that Y is *simple* if the resulting action is trivial for all $n \geq 1$ and *nilpotent* if the resulting action is nilpotent for every $n \geq 1$, that is, there is a finite $\pi_1(Y,y)$-equivariant filtration

$$\pi_n(Y,y) = F_0(\pi_n(Y,y)) \supseteq \cdots \supseteq F_k(\pi_n(Y,y)) \supseteq F_{k+1}(\pi_n(Y,y)) \supseteq \cdots \supseteq F_N(\pi_n(Y,y)) = 0$$

by normal subgroups on $\pi_n(Y,y)$ such that the action on F_k/F_{k+1} is trivial for all $k \geq 0$.[4] In particular, π_1 of a simple space is abelian and π_1 of a nilpotent space is a nilpotent group. These notions will also play an important role in our discussion of obstruction theory later.

(3) Let us also note that it follows from the above that there is an action of $\pi_1(B)$ on $\pi_n(F) \otimes_{\mathbb{Z}\pi_1(F)} \mathbb{Z}$ compatible with the $\pi_1(E)$-action on $\pi_n(F)$. Indeed, the obstruction to making an unpointed homotopy equivalence of F act on homotopy groups is precisely the action of $\pi_1(F)$ on $\pi_n(F)$. The compatibility with the $\pi_1(E)$-action is then valid by construction.

(4) Let us finally note that for a fibration $E \to B$ with $F = p^{-1}(B)$ and $e \in F$, the long exact sequence

$$\dots \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \pi_{n-1}(B) \to \dots$$

is one of $\mathbb{Z}[\pi_1(E)]$-modules: Here the action of $\pi_1(E)$ on $\pi_n(F)$ and $\pi_n(E)$ are as desribed above, and the action on $\pi_n(X)$ is via the morphism $\pi_1(E) \to \pi_1(X)$.

2.30. **Remark** In a similar vein, for pointed spaces (X,x) and (Y,y), there is an exact sequence

$$[S^1 \times X, Y]_f \xrightarrow{\partial} \pi_1(Y,y) \to [X,Y]_* \to [X,Y] \to \pi_0(Y)$$

where the first term denotes the set (in fact group) of homotopy classes of pairs $(S^1 \times X, \{1\} \times X) \to (Y,Y)$ where the map $\{1\} \times X \to Y$ is given by any pointed map $f\colon X \to Y$. The map ∂ takes $[h]$ to $h[-,x]$. We recall that exactness at $[X,Y]_*$ means that $\pi_1(Y,y)$ acts on $[X,Y]_*$ and that two elements in $[X,Y]_*$ go to the same element in $[X,Y]$ if and only if they differ by (the action of) an element of $\pi_1(Y,y)$. Here, the action is given as follows: Let $\gamma\colon [0,1] \to Y$ represent an element of $\pi_1(Y,y)$ and let $f\colon (X,x) \to (Y,y)$ be a pointed map. Then consider the diagram

$$\begin{tikzcd} X \times \{0\} \cup \{x\} \times [0,1] \arrow[r, "f \cup \gamma"] \arrow[d] & Y \\ X \times [0,1] \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

which admits a lift h as indicated. Then $h(-,1)$ is a new pointed map which we define to be $\gamma \cdot f$. That this construction is well-defined is done similarly as above. Exactness at $[X,Y]_*$ is then the following argument: Clearly, $\gamma \cdot f$ and f go to the same element in $[X,Y]$ as h is a homotopy between them. Conversely, suppose given two pointed maps $f,g\colon (X,x) \to (Y,y)$ and assume that there exists a homotopy $h\colon X \times [0,1] \to Y$. Then by construction $h(x,-)$ represents an element of $\pi_1(Y)$ and the original homotopy is a witness that $g = h(x,-) \cdot f$. It follows that if Y is simply connected, then the forgetful map $[X,Y]_* \to [X,Y]$ is a bijection. Moreover, $\gamma \in \pi_1(Y,y)$ acts trivially on $f \in [X,Y]_*$ if and only if γ is in the image of the map $[S^1 \times X, Y]_f \to \pi_1(Y,y)$. In particular, if this map is surjective for every $f \in [X,Y]_*$, then $\pi_1(Y,y)$ acts trivially on $[X,Y]_*$ and we find that the map $[X,Y]_* \to [X,Y]$ is injective. This is the case, for instance, when Y is an h-space with basepoint $u \in Y$ the unit of the multiplication. We give the argument in case the multiplication is strictly left-unital, i.e.

[4] When π_1 acts on itself by conjugation, this means that $F_k/F_{k+1} \subseteq C(F_0/F_{k+1})$, where $C(-)$ denotes the centre of a group, and therefore such filtrations are also called *central series*.

if $u \cdot - \colon Y \to Y$ is equal to the identity (not only homotopic to the identity)[5]: In that case, consider $\gamma \in \pi_1(Y, u)$ and $f \in [X, Y]_*$ and define the map $H \colon S^1 \times X \to Y$ given by $(t, x) \mapsto \gamma(t) \cdot f(y)$. Then H indeed represents an element of $[S^1 \times X, Y]_f$ and $\partial(H) = \gamma$ as needed. In general, one has to use a homotopy extension property to correct the problem that the multiplication in Y is only unital up to homotopy. We note that for f the constant map c at the basepoint of y, the map $[S^1 \times X, Y]_c \to \pi_1(Y, y)$ is *always* surjective: Given γ, simply consider the map $(t, x) \mapsto \gamma(t)$. In particular, the action of $\pi_1(Y, y)$ on the constant map is always trivial, and therefore, the constant map is null homotopic if and only if it is pointed null homotopic (a fact we have seen already in topology I).

All of the above is reminiscent of a putative fibration sequence $\mathrm{map}_*(X, Y) \to \mathrm{map}(X, Y) \to Y$, see also Remark 3.11. Indeed, if X is a locally compact CW complex with basepoint x, then $\mathrm{map}(X, Y) \to Y$, given by evaluating at x is a fibration with fibre $\mathrm{map}_*(X, Y)$. We have then constructed an action of $\pi_1(Y)$ on the homotopy type of $\mathrm{map}_*(X, Y)$ and in particular on its π_0, which identifies with $[X, Y]_*$. The two $\pi_1(Y)$-actions then agree as one readily shows, and the slightly cumbersome relative homotopy group appearing as first term in the above exact sequence translates simply to $\pi_1(\mathrm{map}(X, Y), f)$ and the above is the usual long exact sequence in homotopy groups of a fibration.

2.31. **Addendum** For a group G, we may define the following filtration, called the *lower central series* of G: Set $\Gamma^1(G) = G$ and inductively, for $n \geq 2$, set $\Gamma^n(G) = [G, \Gamma^{n-1}G]$, so that we obtain a descending filtration

$$G = \Gamma^1 G \supseteq \Gamma^2 G \supseteq \dots \supseteq \Gamma^n G \supseteq \dots$$

A group G is nilpotent if and only if there exists an $N \geq 1$ such that $\Gamma^N G = 1$ (Indeed, one shows inductively that any central series contains the lower central series). The smallest such N is called the nilpotency class of G. Note that $G/\Gamma^2 G = G^{\mathrm{ab}}$. Moreover, the association $g_1, \dots, g_n \mapsto [\dots[[g_1, g_2], g_3], \dots, g_n]$ taking iterated commutators[6] defines a surjective homomorphism

$$\bigotimes_n G^{\mathrm{ab}} \to \Gamma^n G/\Gamma^{n+1} G$$

see [War76, Thm. 3.1]. As a consequence, given a morphism $G \to H$ of groups which induces a surjection $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$, it also induces a surjection of the associated graded of the respective lower central series. In particular, in this situation, if G is nilpotent of class N, then H is also nilpotent and is of class at most N. Moreover, the map $G \to H$ is itself surjective.

We now come to the relative Hurewicz theorem.

2.32. **Theorem** *Let (X, A) be an $(n-1)$-connected pair with $n \geq 2$ and X connected.*

1. *If A is 1-connected, then $\pi_n(X, A) \to H_n(X, A; \mathbb{Z})$ is an isomorphism and $\pi_{n+1}(X, A) \to H_{n+1}(X, A; \mathbb{Z})$ is surjective.*
2. *If A is connected, then $\pi_n(X, A) \to H_n(X, A; \mathbb{Z})$ is surjective and induces an isomorphism $\pi_n(X, A) \otimes_{\mathbb{Z}[\pi_1(A,a)]} \mathbb{Z} \to H_n(X, A)$.*

[5] If Y is a CW complex, one can make $Y \cup \{u\}[0, 1]$ into a homotopy equivalent and strictly left-unital h-space (Exercise)

[6] Recall that $[g, h] = ghg^{-1}h^{-1}$ is the commutator of g and h.

Proof. Again we may assume that (X, A) is a CW inclusion. To prove (1), we apply Theorem 2.23 to the (almost) homotopy pushout square[7]

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd}$$

Since $A \to X$ is $(n-1)$-connected and $A \to *$ is 2-connected, we conclude that the induced map $F \to \Omega X/A$ is n-connected. Hence, $\pi_n(X, A) \to \pi_n(X/A)$ is an isomorphism and that $\pi_{n+1}(X, A) \to \pi_{n+1}(X/A)$ is surjective. Now, X/A is itself $(n-1)$-connected, so we deduce that $\pi_n(X/A) \to H_n(X/A)$ is an isomorphism and that $\pi_{n+1}(X/A) \to H_{n+1}(X/A)$ is surjective from Theorem 2.27. In total this proves (1).

We prove (2). We denote by F the homotopy fibre of the inclusion $A \to X$. Then we claim that the relative Hurewicz homomorphism, under the isomorphism $\pi_n(X, A) \cong \pi_{n-1}(F)$ is given by the following composite, see Remark 2.20:

$$\pi_{n-1}(F) \to \pi_n(\Sigma F) \to \pi_n(\widetilde{X}/\bar{A}) \to \pi_n(X/A). \tag{1}$$

Here, the first map is induced by the unit map $F \to \Omega\Sigma F$, the second and third maps are induced by the commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \bar{A} \arrow[r] \arrow[d] & A \arrow[d] \\ * \arrow[r] & \widetilde{X} \arrow[r] & X \end{tikzcd} \tag{2}$$

by passing to vertical (homotopy) cofibres. Now, if $n \geq 3$ then two situations simplify: Since F is $(n-2)$-connected, we find from Freudenthal that the map $\pi_{n-1}(F) \to \pi_n(\Sigma F)$ is an isomorphism if $n \geq 3$. Moreover, the composite of the first two maps in (1) identifies with the relative Hurewicz homomorphism for the pair $(\widetilde{X}, \bar{A})$ which is an isomorphism by part (1) since $\bar{A}$ is simply connected (again this uses $n \geq 3$). It then remains to prove that $\pi_n(\widetilde{X}/\bar{A}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \to \pi_n(X/A)$ is an isomorphism. Again, note that since $n \geq 3$ we have $\pi_1(A) \cong \pi_1(X)$. In fact, this statement is also correct for $n = 2$ and we prove it in this generality.

Indeed, we use the absolute Hurewicz theorem to translate this into the statement that, equivalently, the map

$$H_n(\widetilde{X}, \bar{A}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \to H_n(X, A; \mathbb{Z})$$

is an isomorphism. Now, we use several results from Exercise sheet 14 from the Topology I course: Note that $\widetilde{X} \to X$ and $\bar{A} \to A$ are $\pi_1(X)$-Galois covering spaces. Hence we find that $C^{\text{sing}}(\widetilde{X}; \mathbb{Z})$ as well as $C^{\text{sing}}(\bar{A}; \mathbb{Z})$ are $\mathbb{Z}[\pi_1(X)]$-chain complexes, which are levelwise free, and that $C^{\text{sing}}(X; \mathbb{Z}) \cong C^{\text{sing}}(\widetilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}$ as well as $C^{\text{sing}}(A; \mathbb{Z}) \cong C^{\text{sing}}(\bar{A}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}$. We deduce that the canonical map

$$C^{\text{sing}}(\widetilde{X}, \bar{A}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \to C^{\text{sing}}(X, A; \mathbb{Z})$$

is an isomorphism as well. It hence remains to show that the canonical map

$$H_n(\widetilde{X}, \bar{A}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \to H_n(C^{\text{sing}}(\widetilde{X}, \bar{A}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z})$$

[7] To actually apply this, replace $*$ by $\mathrm{C}(A)$. Then X/A and $\mathrm{C}(i)$ are homotopy equivalent and all can be phrased using $\mathrm{C}(i)$ instead of X/A.

is an isomorphism. This is true more generally: Assume R is an associative ring, C is a chain complex of free (right) R-modules, and M is a (left) R-module. Suppose that C is n-connected, that is $H_k(C) = 0$ for $k < n$. Then we may construct a map $C' \to C$ such that $C'_k = 0$ for $k < n$, C' also consists of free right R-modules, and the map $C' \to C$ is a quasi-isomorphism – this is similar to a CW approximation of spaces and does *not* use that C consists of free right R-modules. However, if C also consists of free right R-modules, we see that the map $C' \to C$ is in fact a chain homotopy equivalence. Consequently, $C' \otimes_R M \to C \otimes_R M$ is also a chain homotopy equivalence. It hence suffices to prove that the comparison map (exchanging the tensor product and the homology) is an isomorphism in case the chain complex satisfies $C_k = 0$ for $k < n$ in which case it simply follows from the right exactness of the functor $- \otimes_R M$. This finishes the proof in case $n \geq 3$.

For $n = 2$, we will now prove that

(a) the map $\pi_1(F) \to \pi_2(\Sigma F)$ identifies with $\pi_1(F) \to \pi_1(F) \otimes_{\mathbb{Z}[\pi_1(F)]} \mathbb{Z} \cong \pi_1(F)^{\mathrm{ab}}$, and
(b) the map $\pi_2(\Sigma F) \to \pi_2(\widetilde{X}/\bar{A})$ is an isomorphism.

In total, this shows that for all $n \geq 2$, the composite (1) identifies with the composite

$$\pi_{n-1}(F) \to \pi_{n-1}(F) \otimes_{\mathbb{Z}[\pi_1(F)]} \mathbb{Z} \to \big[\pi_{n-1}(F) \otimes_{\mathbb{Z}[\pi_1(F)]} \mathbb{Z}\big] \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}.$$

Here, the $\pi_1(X)$-action on $\pi_{n-1}(F) \otimes_{\mathbb{Z}[\pi_1(F)]} \mathbb{Z}$ in the final term is the tautological action indicated in Remark 2.29. Now, recall that F is connected, so that we have an exact sequence

$$\pi_1(F) \to \pi_1(A) \to \pi_1(X) \to 1$$

showing that the above composite is indeed isomorphic to the map

$$\pi_{n-1}(F) \to \pi_{n-1}(F) \otimes_{\mathbb{Z}[\pi_1(A)]} \mathbb{Z}$$

as claimed. To see (a) above, we consider the commutative diagram

$$\begin{tikzcd} \pi_1(F) \arrow[r] \arrow[d] & \pi_2(\Sigma F) \arrow[d, "\cong"] \\ H_1(F) \arrow[r, "\cong"] & H_2(\Sigma F) \end{tikzcd}$$

whose lower horizontal arrow is the suspension isomorphism and the right vertical map is the Hurewicz isomorphism for the 1-connected space ΣF. Finally, to see (b), to ease notation, we assume that (X, A) is a 1-connected pair with $\pi_1(X) = 1$ and need then to show that $\pi_2(\Sigma F) \to \pi_2(X/A)$ is an isomorphism. Note that this map is isomorphic to the map $H_1(F) \cong H_2(\Sigma F) \to H_2(X/A)$ by the absolute Hurewicz theorem. Let us then consider the homotopy pushout diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & A \arrow[d] \\ * \simeq \mathrm{C}(F) \arrow[r] & \mathrm{C}(i) \end{tikzcd}$$

Now the map j is 1-connected, as is the map $F \to \mathrm{C}(F) \simeq *$, in particular, from Seifert–van Kampen, we see that $\pi_1(\mathrm{C}(j)) = 1$. From the homotopy excision theorem, we deduce that the map $\Omega X \simeq \mathrm{hofib}(j) \to \Omega\mathrm{C}(j)$ is 1-connected, i.e. surjective on π_1. Note that the equivalence

$\Omega_x X \simeq \mathrm{hofib}(j)$ is obtained from the commutative diagram

$$\begin{tikzcd}
\Omega_x X \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & P_x X \arrow[d] \\
* \arrow[r] & A \arrow[r] & X
\end{tikzcd}$$

in which both squares are pullback squares. Here $P_x X$ denotes the space of paths in X which start at x and the map $P_x X \to X$ is given by taking the endpoint of the path – this map is a fibration and hence F really is the homotopy fibre of $A \to X$.

Now we observe that the map $\Omega X \to \Omega \mathrm{C}(j)$ admits a splitting on the level of spaces induced by the tautological map $\mathrm{C}(j) \to X$: Indeed, consider the refinement of the right upper pullback square

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \mathrm{C}(F) \arrow[r] \arrow[d] & P_x X \arrow[d] \\
A \arrow[r] & \mathrm{C}(j) \arrow[r] & X
\end{tikzcd}$$

which is obtained by showing that the map $F \to P_x X$ may be extended over the inclusion $F \to \mathrm{C}(F)$ by choosing a null-homotopy of the map $F \to P_x X$ (which exists certainly as $P_x X$ is contractible). Passing to vertical fibres, we obtain that the composite $\Omega X \to \Omega\mathrm{C}(j) \to \Omega X$ is a homeomorphism since big square is a pullback.

Consequently, the map $\Omega\mathrm{C}(j) \to \Omega X$ induces an isomorphism on π_1. In particular, $\pi_2(X) \simeq \pi_2(\mathrm{C}(j))$ and by the absolute Hurewicz theorem, also $H_2(X) \cong H_2(\mathrm{C}(i))$. In particular, we deduce that the upper horizontal sequence in the diagram

$$\begin{tikzcd}
H_2(A) \arrow[r] \arrow[d, equal] & H_2(\mathrm{C}(j)) \arrow[r] \arrow[d] & H_1(F) \arrow[r] \arrow[d] & H_1(A) \arrow[r] \arrow[d, equal] & H_1(\mathrm{C}(i)) = 0 \arrow[d] \\
H_2(A) \arrow[r] & H_2(X) \arrow[r] & H_2(X/A) \arrow[r] & H_1(A) \arrow[r] & H_1(X) = 0
\end{tikzcd}$$

is exact (and the lower one is exact by definition). We claim that all squares commute (one perhaps only up to sign). For the left most and right most square, this follows immediately from the factorization of the map $A \to X$ through $\mathrm{C}(i)$. For the other two squares it is convenient to rewrite these using the suspension isomorphism as the diagram

$$\begin{tikzcd}
H_2(\mathrm{C}(j)) \arrow[r] \arrow[d] & H_2(\Sigma F) \arrow[r] \arrow[d] & H_2(\Sigma A) \arrow[d, equal] \\
H_2(X) \arrow[r] & H_2(\mathrm{C}(i)) \arrow[r] & H_2(\Sigma A)
\end{tikzcd}$$

For the right hand side, it is a similar argument as before that $\Sigma F \to \Sigma A$ canonically factors through $\mathrm{C}(i)$. Indeed, the following diagram commutes up to natural homotopy:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "j"] & * \arrow[r] \arrow[d] & \Sigma F \arrow[r, equal] \arrow[d] & \Sigma F \arrow[d] \\ A \arrow[r, "i"] \arrow[d] & X \arrow[r] \arrow[d, equal] & \mathrm{C}(i) \arrow[r] & \Sigma A \\ \mathrm{C}(j) \arrow[r] \arrow[d] & X \arrow[d] & & \\ \Sigma F \arrow[r] & * & & \end{tikzcd}$$

The upper right most square then shows that the right of the above two squares of homology groups commutes. To see that also the left one commutes, we have to observe that, up to homotopy, the composite $\mathrm{C}(j) \to X = X \to \mathrm{C}(i)$ factors as $\mathrm{C}(j) \to \Sigma F \to \mathrm{C}(i)$, giving a diagram (commutative up to homotopy):[8]

$$\begin{tikzcd} \mathrm{C}(j) \arrow[r] \arrow[d] & \Sigma F \arrow[d] \\ X \arrow[r] & \mathrm{C}(i) \end{tikzcd}$$

showing that also the left of the two squares of homology groups commute. The 5-lemma finally finishes the proof of the theorem. $\square$

2.33. **Remark** An equivalent formulation of Theorem 2.32 is the following. Let $f\colon X \to Y$ be an $(n-1)$-connected map, $n \geq 2$ with Y connected. Let $F = \mathrm{hofib}(f)$ be its homotopy fibre and $\mathrm{C}(f)$ its homotopy cofibre. Then the Hurewicz homomorphism induces

(1) if X is 1-connected, an isomorphism $\pi_{n-1}(F) \to H_n(\mathrm{C}(f);\mathbb{Z})$ and a surjection $\pi_n(F) \to H_{n+1}(\mathrm{C}(f);\mathbb{Z})$.
(2) in general, an isomorphism $\pi_{n-1}(F) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \to H_n(\mathrm{C}(f);\mathbb{Z})$.

As a consequence, we deduce the following version of Whiteheads theorem.

2.34. **Corollary** *Let $f\colon X \to Y$ be a map between connected nilpotent spaces which induces an isomorphism on singular homology. Then f is a weak equivalence.*

Proof. First, we claim that f is 1-connected, i.e. surjective on π_1. To see this, we employ the argument from Addendum 2.31, so it suffices to note that the map $\pi_1(X) \to \pi_1(Y)$ induces a surjection on abelianizations. This is true since these abelianizations are isomorphic, by the Hurewicz homomorphism, to $H_1(X)$ and $H_1(Y)$, respectively, and on homology, f induces an isomorphism (in particular a surjection) by assumption. We now conclude that the homotopy fibre F of f is connected, so it remains to show that F satisfies $\pi_{n-1}(F) = 0$ for all $n \geq 2$. We will show this by induction on n, but before doing so, we claim that the $\pi_1(X)$-action on

[8]That this result is true on the level of homology is more directly implied by inspecting the top left most square on singular chain complexes, and then expanding out cone sequences.

$\pi_{n-1}(F)$ is nilpotent for all $n \geq 2$. To see this, we recall that the long exact sequence of the homotopy fibre sequence $F \to X \to Y$

$$\pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(F) \to \pi_{n-1}(X) \to \pi_{n-1}(Y)$$

is one of $\mathbb{Z}[\pi_1(X)]$-modules. Moreover, by assumption, $\pi_n(Y)$ and $\pi_{n-1}(X)$ are nilpotent $\mathbb{Z}[\pi_1(X)]$-modules. As submodules, quotients, and extensions of nilpotent modules are again nilpotent, the claim follows. Next, we show that if M is equipped with a nilpotent G-action and $M \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$, then $M = 0$.[9] To see this, we again argue by induction over the length of a filtration as in the definition of nilpotent actions. The base case is the case where M itself is equipped with a trivial G-action. But then we have $0 = M \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong M$. Inductively, we may then assume that there is a short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

of $\mathbb{Z}[G]$-modules with M/M_1 having a trivial action and where M_1 has a shorter filtration witnessing the nilpotency of the G-action. From $M \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$ and the right exactness of $- \otimes_{\mathbb{Z}[G]} \mathbb{Z}$,[10] we deduce that $(M/M_1) \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$ and hence that $M/M_1 = 0$ and thus that $M_1 = M$. Hence we conclude that $M_1 \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$ and by induction that $0 = M_1 = M$. Finally, we prove that $\pi_{n-1}(F) = 0$ by induction over n. Indeed, inductively, the relative Hurewicz theorem provides the following isomorphism.

$$\pi_{n-1}(F) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \cong H_n(\mathrm{C}(f); \mathbb{Z}) = 0$$

so we conclude that $\pi_{n-1}(F) = 0$ by the above argument and hence f is a weak equivalence. □

3. Eilenberg-Mac Lane spaces and cohomology

We begin with the definition of Eilenberg-Mac Lane spaces.

3.1. **Definition** Let A be an abelian group and $n \geq 0$. A space X is called an Eilenberg-Mac Lane space of type (A, n) if it is equipped with a specified isomorphism $\theta\colon \pi_n(X) \cong A$ and $\pi_k(X) = 0$ for $k \neq n$.

We will see later that any two Eilenberg-Mac Lane spaces are homotopy equivalent via a unique homotopy class of equivalence. Therefore, Eilenberg-Mac Lane spaces of type (A, n) are often denoted $K(A, n)$, but it is important to remember that the identification $\pi_n(K(A, n)) \cong A$ is part of the datum of an Eilenberg-Mac Lane space.

We begin by showing that Eilenberg-Mac Lane spaces exist:

3.2. **Lemma** *Let A be an abelian group an $n \geq 0$. There exists an Eilenberg-Mac Lane space $K(A, n)$.*

Proof. When $n = 0$ we may take the set A with discrete topology as $K(A, 0)$. When $n = 1$, we have argued in Topology I that associated to the group A, we may consider the presentation complex $X(A)$ which comes with a specified isomorphism $\pi_1(X(A)) \cong A$. For $n \geq 2$, we

[9] In case $n = 2$, this is to be read as the coinvariants of the G-action on $\pi_1(F)$ instead of the displayed tensor product.

[10] In case $n = 2$, the right exactness of the functor of coinvariants of the G-action.

may perform a similar construction to obtain general *Moore spaces* for A: Choose short exact sequence

$$0 \longrightarrow \bigoplus_J \mathbb{Z} \xrightarrow{f} \bigoplus_I \mathbb{Z} \longrightarrow A \longrightarrow 0$$

Since $n \geq 2$, we may realise the map f as the map induced on π_n from a map $\bigvee_J S^n \to \bigvee_I S^n$ and consider its homotopy cofibre $M(A,n)$ which now has a single homology group, namely A in degree n. By the Hurewicz theorem, we then obtain a specified isomorphism $\pi_n(M(A,n)) \cong A$.

We have now argued that for all pairs (A,n) with $n \geq 1$, there exists an $(n-1)$-connected space $M(A,n)$ equipped with an isomorphism $\pi_n(M(A,n)) \cong A$. We may now attach cells of dimension $\geq n+2$ to $M(A,n)$ to keep π_n unchanged and to set π_k equal to zero for $k > n$. The resulting space is then a choice of a $K(A,n)$. □

3.3. **Example** We have seen that $\mathbb{Z} \to \pi_n(S^n)$, given by $1 \mapsto [\mathrm{id}_{S^n}]$ is an isomorphism. Moreover, $\pi_k(S^1) = 0$ for $k \neq 1$ and hence S^1, together with the just mentioned isomorphism, is a $K(\mathbb{Z},1)$. Likewise, the map $\mathbb{Z} \cong \pi_1(S^1) \cong \pi_1(\mathbb{RP}^1) \to \pi_1(\mathbb{RP}^n)$ induces an isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^n)$ for all $n \geq 2$, and $\mathbb{Z} \to \pi_2(S^2) \cong \pi_2(\mathbb{CP}^1) \to \pi_2(\mathbb{CP}^n)$ induces an isomorphism for all $n \geq 1$. Since $\pi_k(\mathbb{RP}^\infty) = 0$ for $k \neq 1$ and $\pi_k(\mathbb{CP}^\infty) = 0$ for $k \neq 2$, we find that $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z},1)$ and $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$. Here is another model for $K(\mathbb{Z},2)$: Consider an infinite dimensional separable complex Hilbert space $\mathcal{H}$. Then its unitary group $U(\mathcal{H})$ is contractible in the norm topology, this is a theorem of Kuiper. Scalar multiplication by elements of $U(1) = S^1$ defines a subgroup inclusion $U(1) \to U(\mathcal{H})$. The quotient group $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$, called the projective unitary group sits in a fibration (in fact in a principal $U(1)$-bundle, we will discuss principal bundles in the next section) $U(1) \to U(\mathcal{H}) \to PU(\mathcal{H})$, showing that $PU(\mathcal{H})$ can be made into a $K(\mathbb{Z},2)$. Note that the space $PU(\mathcal{H})$ is a topological group, whereas it can be shown that $\mathbb{CP}^\infty$ cannot be made into a topological group.[11]

We have also discussed that $\mathbb{Z} \cong \pi_4(S^4) \cong \pi_4(\mathbb{HP}^1) \to \pi_4(\mathbb{HP}^\infty)$ is an isomorphism, but it is not true that $\mathbb{HP}^\infty$ is a $K(\mathbb{Z},4)$: Indeed, for every $n \geq 1$, there is a fibration sequence $S^3 \to S^{4n+3} \to \mathbb{HP}^n$ showing that $\pi_k(\mathbb{HP}^\infty) \cong \pi_{k-1}(S^3)$. We have not yet seen that the latter groups are not trivial in general when $k \neq 4$, but this is true: $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, as we will prove in the later stages of this course.

Finally, we discussed that infinite dimensional lens spaces provide models for $K(\mathbb{Z}/n\mathbb{Z},1)$. We have not found "easy" models for other Eilenberg–Mac Lane spaces.

3.4. **Lemma** *Let A be an abelian group and $n \geq 0$. Then the canonical map*

$$H^n(K(A,n);A) \to \mathrm{Hom}(A,A)$$

induced by the universal coefficient theorem is an isomorphism. The unique preimage under id_A *will be denoted* ι_n^A.

Proof. The universal coefficient theorem gives an isomorphism

$$H^n(K(A,n);A) \cong \mathrm{Hom}(H_n(K(A,n);\mathbb{Z}),A)$$

since $K(A,n)$ is $(n-1)$-connected and hence the Ext-term vanishes. The Hurewicz theorem provides a specified isomorphism $\pi_n(K(A,n)) \cong H_n(K(A,n);\mathbb{Z})$ and the source is identified with A by definition of $K(A,n)$. □

[11] It can be made into a topological monoid, but not into a topological group, I believe.

3.5. **Lemma** *Let A be an abelian group an $n \geq 0$. There is a map $s_n\colon \Sigma K(A,n) \to K(A,n{+}1)$ such that under the suspension isomorphism, $s_n^*(\iota_{n+1}^A)$ corresponds to ι_n^A. The map adjoint to s_n is an equivalence $\hat{s}_n\colon K(A,n) \to \Omega K(A,n+1)$.*

Proof. The Hurewicz theorem (together with the homological suspension isomorphism) implies that $\pi_{n+1}(\Sigma K(A,n))$ comes with a specified isomorphism to A. By attaching higher cells, we obtain the map $s_n\colon \Sigma K(A,n) \to K(A,n+1)$, which under the so chosen identifications of π_{n+1} with A induces the identity on π_{n+1}. Then we consider the diagram

$$\begin{tikzcd}
H^{n+1}(K(A,n+1);A) \arrow[r] \arrow[d] & \mathrm{Hom}(H_{n+1}(K(A,n+1);\mathbb{Z}),A) \arrow[d] \\
H^{n+1}(\Sigma K(A,n);A) \arrow[r] \arrow[d] & \mathrm{Hom}(H_{n+1}(\Sigma K(A,n);\mathbb{Z}),A) \arrow[d] \\
H^{n}(K(A,n);A) \arrow[r] & \mathrm{Hom}(H_{n}(K(A,n);\mathbb{Z}),A)
\end{tikzcd}$$

which commutes by naturality of the UCT and by compatibility of UCT with the (co)homological suspension isomorphisms. In each case, the horizontal maps are isomorphisms. The claim then follows from the construction of the map s_n: All homology groups in the right column are identified with A in a way making the right vertical composite the identity map. Hence, i_{n+1}^A is sent to i_n^A under the left vertical composite as claimed. The map $\hat{s}_n$ is then given by the composite

$$K(A,n) \to \Omega\Sigma K(A,n) \to \Omega K(A,n+1).$$

Hence to see that it is an equivalence, one only needs to check that it is an isomorphism on π_n. The first map is $(2n-1)$-connected by Freudenthal and so induces an isomorphism on π_n if $n > 1$ and a surjection for $n = 1$. The second map is an isomorphism on π_n since we have shown that s_n induces an isomorphism on π_{n+1}. It remains to show that the unit map $K(A,1) \to \Omega\Sigma K(A,1)$ is also injective on π_1. So suppose it is not and pick $a \in A \cong \pi_1(K(A,1))$ which lies in the kernel. The element a gives rise to a commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & K(A,1) \arrow[d] \\
\Omega S^2 \arrow[r] & \Omega\Sigma K(A,1)
\end{tikzcd}$$

The lower horizontal map, under the abstract identification of $\pi_1(\Omega\Sigma K(A,1))$ with A still classifies the element a. Hence, it suffices to know that the left vertical map is injective on π_1. But this map is a surjection between finitely generated groups (namely $\mathbb{Z}$) and hence bijective.[12] $\square$

3.6. **Definition** Let X be a space, A an abelian group and $n \geq 0$. We define a map $\chi_n^A\colon [X, K(A,n)] \to H^n(X;A)$ by sending $[f]$ to $f^*(\iota_n^A)$.

Our goal is to prove the following theorem:

[12] In case A is finitely generated, any surjective self-map is an isomorphism. In general, A is a filtered colimit of its finitely generated subgroups, and the map $K(A,1) \to \Omega\Sigma K(A,1)$ is compatible with this filtered colimit. Since filtered colimits of isomorphisms are isomorphisms, this also gives the claimed bijectivity of the map under investigation.

3.7. **Theorem** *For a CW complex X, the map $\chi_n^A\colon [X, K(A,n)] \to H^n(X;A)$ is a bijection.*

Exercise. Convince yourself that the above theorem is true for $n = 0$.

For the proof of Theorem 3.7, we first record a couple of properties of the map χ_n^A.

3.8. **Lemma** *Let X be a space. The diagram*

$$\begin{tikzcd} {[\Sigma X, K(A,n+1)]} \arrow[r, "\chi_{n+1}^A"] & H^{n+1}(\Sigma X;A) \\ {[X, K(A,n)]} \arrow[r, "\chi_n^A"] \arrow[u] & H^n(X;A) \arrow[u] \end{tikzcd}$$

commutes. Here, the right vertical map is the suspension isomorphism and the left vertical map is the adjunction bijection followed by the equivalence $\hat{s}_n\colon \Omega K(A,n+1) \simeq K(A,n)$ constructed above.

Proof. Let $f\colon X \to K(A,n)$ represent an element in the left lower corner. Its image in the upper left corner is represented by the composite $\Sigma X \xrightarrow{\Sigma f} \Sigma K(A,n) \xrightarrow{s_n}$. Consider the commutative diagram

$$\begin{tikzcd} H^{n+1}(\Sigma K(A,n);A) \arrow[r, "\Sigma(f)^*"] & H^{n+1}(\Sigma X;A) \\ H^n(K(A,n);A) \arrow[r, "f^*"] \arrow[u] & H^n(X;A) \arrow[u] \end{tikzcd}$$

induced by the map f and the suspension functor and suspension isomorphism. By Lemma 3.5, $s_n^*(\iota_{n+1}^A)$, an element of the top left corner, corresponds under the left vertical bijection to ι_n^A. Hence $\Sigma(f)^*(s_n^*(\iota_{n+1}^A))$ corresponds under the right vertical bijection to $f^*(\iota_n^A)$. This proves the corollary. $\square$

3.9. **Lemma** *The set $[X, K(A,n)]$ is a group (in fact an abelian group) and the map χ_n^A is a group homomorphism.*

Proof. Since $K(A,n) \simeq \Omega K(A,n+1)$, we find that $[X, K(A,n+1)]$ is π_0 of a loop space and hence a group. In fact, since $K(A,n) \simeq \Omega^2 K(A,n+2)$, $[X, K(A,n)]$ is also π_1 of a loop space and hence an abelian group. Let us denote the multiplication map of $K(A,n)$ by $m\colon K(A,n) \times K(A,n) \to K(A,n)$. Then we find that $m^*(\iota_n^A)$ is an element of

$$H^n(K(A,n) \times K(A,n);A) \cong \mathrm{Hom}(A \oplus A, A).$$

The fact that the multiplication is left and right unital implies that the resulting homomorphism $A \oplus A \to A$ is the fold map, i.e. the identity on each summand of the source. This implies the equation

$$m^*(\iota_n^A) = pr_1^*(\iota_n^A) + pr_2^*(\iota_n^A).$$

Now, the 0-element in $[X, K(A,n)]$ is represented by the constant map which pulls ι_n^A back to 0 as needed. Given two maps $f, g\colon X \to K(A,n)$, their sum in $[X, K(A,n)]$ is represented by the composite

$$X \xrightarrow{(f,g)} K(A,n) \times K(A,n) \xrightarrow{m} K(A,n).$$

Therefore we find

$$\begin{aligned}\chi_n^A(f+g) &= (f,g)^*(m^*(\iota_n^A)) \\ &= (f,g)^*(pr_1^*(\iota_n^A)) + (f,g)^* pr_2^*(\iota_n^A)) \\ &= f^*(\iota_n^A) + g^*(\iota_n^A) \\ &= \chi_n^A(f) + \chi_n^A(g)\end{aligned}$$

as needed. □

We now treat a first basic case of Theorem 3.7.

3.10. **Lemma** *Let A be an abelian group and $n \geq 0$. The map $\chi_n^A\colon [S^k, K(A,n)] \to H^n(S^k;A)$ is an isomorphism for all $k \geq 0$.*

Proof. Since $K(A,n)$ is a loopspace, we find that $\pi_k(K(A,n)) \cong [S^k, K(A,n)]_* \cong [S^k, K(A,n)]$, see Remark 2.30. In particular, the statement of the lemma holds for trivial reasons when $k \neq n$ since then, both sides vanish. So let us consider the case $k = n$. We claim that the following square commutes.

$$\begin{tikzcd} {[S^n, K(A,n)]} \arrow[r] \arrow[d] & H^n(S^n;A) \arrow[d] \\ \mathrm{Hom}(H_n(S^n), H_n(K(A,n))) \arrow[r, "\cong"] & \mathrm{Hom}(H_n(S^n), A) \end{tikzcd}$$

where the left vertical map is induced by the functor $H_n(-)$, the right vertical map is the canonical map appearing in the universal coefficient theorem, and the lower horizontal map is induced by the chosen identification $H_n(K(A,n);A)$. Indeed, to see this, consider $f\colon S^n \to K(A,n)$. Then the composite over the left lower corner is given by the composite $H_n(S^n) \to H_n(K(A,n)) \to A$ where the latter map is induced the chosen identification. The image of f in the upper right corner is given by $f^*(\iota_n^A)$. Since the right vertical map is natural, we see that it sends $f^*(\iota_n^A)$ to the image under $f^*\colon \mathrm{Hom}(H_n(K(A,n)),A) \to \mathrm{Hom}(H_n(S^n),A)$ of the chosen identification of $H_n(K(A,n))$ with A, by definition of ι_n^A. This shows that the upper square indeed commutes. It then suffices to note that both vertical maps are isomorphisms. For the right vertical map this is what the universal coefficient theorem implies. Then we note that the left vertical map, up to identifying the target with $H_n(K(A,n))$ via evaluation at $1 \in \mathbb{Z}$ and identifying the source with $\pi_n(K(A,n))$ is the Hurewicz homomorphism, which we have seen to be an isomorphism. □

3.11. **Remark** We will need the following observation, which implicitly has appeared in topology II. We first observe: Namely, let (B,A) be a relative CW complex with A, B, and B/A locally compact and consider a basepoint $a \in A$. Then the pushout square of pointed spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B/A \end{tikzcd}$$

induces, for any pointed space (X,x) a commutative diagram of pointed mapping spaces

$$\begin{tikzcd} \mathrm{Map}_*(B/A,X) \arrow[r] \arrow[d] & \mathrm{Map}_*(B,X) \arrow[d] \\ * \arrow[r, "\mathrm{const}_x"] & \mathrm{Map}_*(A,X) \end{tikzcd}$$

in which the right most vertical map is a Serre fibration: One directly checks that the square

$$\begin{tikzcd} \mathrm{Map}_*(B,X) \arrow[r] \arrow[d] & \mathrm{Map}(B,X) \arrow[d] \\ \mathrm{Map}_*(A,X) \arrow[r] & \mathrm{Map}(A,X) \end{tikzcd}$$

is a pullback. Hence, it suffices to show that the right vertical map is a Serre fibration. By adjunction, we wish to show that the left hand lifting problem

$$\begin{tikzcd} {[0,1]^{n-1} \times \{0\}} \arrow[r] \arrow[d] & \mathrm{Map}(B,X) \arrow[d] \\ {[0,1]^n} \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(A,X) \end{tikzcd} \qquad \begin{tikzcd} {[0,1]^{n-1} \times B \cup [0,1]^n \times A} \arrow[r] \arrow[d] & X \\ {[0,1]^n \times B} \arrow[ur, dashed] & \end{tikzcd}$$

can be solved. By adjunction yet again, this lifting problem is equivalent to the right hand extension problem, which can be solved since the vertical map is a CW inclusion which is a homotopy equivalence and [Win24, Cor. 5.2.6]. Now we claim that the above square of pointed (or equivalently unpointed) mapping spaces is a homotopy pullback diagram, that is, the map from $\mathrm{Map}_*(B/A,X)$ to the pullback is a weak equivalence. To do so, we investigate whether this map is actually a homeomorphism: To do so, we consider a pointed test space (T,t) and the left commutative diagram of pointed spaces

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathrm{Map}_*(B,X) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_*(A,X) \end{tikzcd} \qquad \begin{tikzcd} T \wedge A \arrow[r] \arrow[d] & T \wedge B \arrow[d] \\ T \wedge * = * \arrow[r] & X \end{tikzcd}$$

which by adjunction and the fact that A and B are locally compact is equivalent to the right commutative diagram. The right diagram is the same datum as a map from the pushout to X, and if T is locally compact, e.g. X is a sphere or a disk, then this pushout is given by $T \wedge B/A$. Using that B/A is locally compact, we adjoin again to see that this is the same as a map $T \to \mathrm{Map}_*(B/A,X)$. It follows that the square under investigation looks like a pullback when tested only against locally compact spaces. It then follows that it is a homotopy pullback diagram as claimed. In particular, we deduce that there is a long exact sequence in homotopy groups for the homotopy fibration sequence

$$\mathrm{Map}_*(B/A,X) \to \mathrm{Map}_*(B,X) \to \mathrm{Map}_*(A,X)$$

which looks like

$$[A,X]_* \leftarrow [B,X]_* \leftarrow [B/A,X]_* \leftarrow [A,\Omega X]_* \leftarrow [B,\Omega X]_* \leftarrow \dots$$

In fact, for any map $f\colon A \to B$, we may alternatively consider the sequence of maps

$$A \to B \simeq \mathrm{Cyl}(f) \to \mathrm{C}(f) \to \Sigma A \to \Sigma B \to \Sigma(\mathrm{C}(f)) \to \Sigma^2 A \to \dots$$

which we have also considered in topology I when discussing the long exact sequence of homology groups of a map of spaces, see [Lan23, Lemma 4.75]. In particular, we have shown that applying $H_n(-)$ or $H^n(-)$ to this sequence, we obtain an exact sequence of homology groups, which, together with the suspension isomorphism is the long exact sequence associated to the mapping cone sequence $A \to B \to \mathrm{C}(f)$ (or rather its portion starting at H_n or H^n). Using arguments entirely dual to the arguments involved in showing that a fibration induces a long exact sequence in homotopy groups, one can show that the above sequence also induces an exact sequence, called the *Puppe sequence*

$$[A, X]_* \leftarrow [B, X]_* \leftarrow [\mathrm{C}(f), X]_* \leftarrow [\Sigma A, X]_* \leftarrow [\Sigma B, X]_* \leftarrow \dots$$

see [tD08, Theorem 4.6.4] for a proof. This sequence is isomorphic to the one above under the adjunction bijections $[\Sigma Y, X]_* \cong [Y, \Omega X]_*$.

With these preliminaries out of the way, we can prove Theorem 3.7:

Proof of Theorem 3.7. First, we note that both source and target of the map χ_n^A send disjoint unions of spaces to products. We may hence assume that X is connected and we may also assume that $n \geq 1$ in which case we note by Remark 2.30 that the map $[X, K(A,n)]_* \to [X, K(A,n)]$ is a bijection, so we may freely used pointed homotopy classes if we wish. Observe that for $k > n$, the inclusion $X_k \to X$ induces an isomorphism on $[-, K(A,n)]_{(*)}$ as well as on $H^n(-; A)$: For the latter, we have done this explicitly in [Lan23, Cor. 4.76] and for the former, it follows inductively from the fact that $K(A,n)$ has trivial homotopy groups in dimension bigger than n: Indeed, the Puppe sequence discussed in Remark 3.11 gives for each $l \geq k$ an exact sequence

$$\prod_J [S^l, K(A,n)]_* \leftarrow [X_l, K(A,n)]_* \leftarrow [X_{l+1}, K(A,n)]_* \leftarrow \prod_J [S^{l+1}, K(A,n)] \leftarrow \dots$$

showing that the middle map is an isomorphism. It then follows (exercise) that also the map $[X_k, K(A,n)] \to [X, K(A,n)]$ is an isomorphism. Hence, it suffices to prove the theorem for finite dimensional and connected X. We will do this by induction over the dimension. The case $\dim(X) = 0$ is the case $X = *$ where the claim is obvious (both sides vanish for $n > 0$ and the case $n = 0$ has been argued earlier). For the inductive step, say from dimension $k-1$ to dimension k, we consider the Puppe sequence as above and note that, by naturality of the map under investigation, the following diagram commutes and consists of exact rows.

$$\begin{array}{ccccccccc}
[\Sigma X_{k-1}, K(A,n)] & \to & [\bigvee S^k, K(A,n)] & \to & [X_k, K(A,n)] & \to & [X_{k-1}, K(A,n)] & \to & [\bigvee S^{k-1}, K(A,n)] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^n(\Sigma X_{k-1}; A) & \longrightarrow & H^n(\bigvee S^k; A) & \longrightarrow & H^n(X_k; A) & \longrightarrow & H^n(X_{k-1}; A) & \longrightarrow & H^n(\bigvee S^{k-1}; A)
\end{array}$$

We observe that since $n \geq 1$, the wedge sum's may be commuted to a direct sum outside of homotopy classes and cohomology. By Lemma 3.10 the second vertical map from left and the final vertical map are isomorphisms. Moreover, by induction, the second vertical map from the right is an isomorphism, and by Lemma 3.8, the same is true for the left most vertical map. Hence, the 5-lemma implies that the middle vertical map is also an isomorphism. $\square$

3.12. **Remark** Given an abelian group A and $n \geq 0$, we may view $H^n(-; A)$ as a functor $\mathrm{hTop}^{\mathrm{op}} \to \mathrm{Set}$. Given a space X equipped with a class $x \in H^n(X; A)$, the Yoneda lemma provides a canonical natural transformation $[-, X] \to H^n(-; A)$; it sends a map $f\colon X' \to X$

to $f^*(x)$. Theorem 3.7 implies that this transformation, for $K(A,n)$ and ι_n^A is an isomorphism on CW complexes, so one says that the functor $H^n(-;A)$ is *representable on CW complexes*, and a representing object is $(K(A,n),\iota_n^A)$. Since the Yoneda lemma is fully faithful, any two representing objects are uniquely isomorphic. For us this means that any two Eilenberg–Mac Lane spaces (say we require them to be CW complexes) are homotopy equivalent via a unique homotopy class (which is specified by being compatible with the respective tautological classes ι_n^A).

From Example 3.3 we deduce:

3.13. **Corollary** *There are specified isomorphisms* $[X,S^1] \cong H^1(X;\mathbb{Z})$ *and* $[X,\mathbb{CP}^\infty] \cong H^2(X;\mathbb{Z})$.

3.14. **Corollary** *The canonical map* $[K(A,n),K(B,n)] \to \mathrm{Hom}(A,B)$ *obtained by applying* $H_n(-)$ *is an isomorphism. For* $0 < n < k$*, we have* $[K(A,k),K(B,n)] = 0$ *and* $[K(A,k),K(B,0)] = B$ *when* $k > 0$.

Proof. By representability and the Hurewicz isomorphism, we have isomorphisms

$$[K(A,k),K(B,n)] \cong H^n(K(A,k);B) \cong \mathrm{Hom}(H_n(K(A,k)),B)$$

where the last isomorphism holds, by the universal coefficient theorem, for $n \leq k$. All claims then follow from the fact that $K(A,k)$ is k-connected and the Hurewicz theorem. □

3.15. **Corollary** *If* $K(A,n)$ *is a locally finite CW complex and* $n \geq 1$*, we have*

$$\pi_k(\mathrm{Map}(K(A,n),K(B,n),\mathrm{const}) = \begin{cases} \mathrm{Hom}(A,B) & \textit{for } k=0 \\ B & \textit{for } k=n \\ 0 & \textit{else} \end{cases}$$

and therefore, $\mathrm{Map}_*(K(A,n),K(B,n)) = K(\mathrm{Hom}(A,B),0)$.

Proof. Indeed, under the locally finite assumption, we have $\pi_k(\mathrm{Map}(K(A,n),K(B,n),\mathrm{const}) = [K(A,n),K(B,n-k)]$ so the above computation yields the result. Moreover, the evaluation map $\mathrm{Map}(K(A,n),K(B,n)) \to K(B,n)$ induces an isomorphism on π_n, since the map $* \to K(A,n)$ induces an isomorphism on $H^0(-;B)$. Hence, the result follows from the fibre sequence

$$\mathrm{Map}_*(K(A,n),K(B,n)) \to \mathrm{Map}(K(A,n),K(B,n)) \to K(B,n)$$

discussed in Remark 3.11. □

3.16. **Remark** The groups $H^n(K(A,k);B) \cong [K(A,k),K(B,n)]$ are bijective, by Yoneda's lemma (applied in the homotopy category of CW complexes), to the set of natural transformations $H^k(-;A) \to H^n(-;B)$. The above results therefore say that there are no non-trivial natural transformations between cohomology functors which *lower* the degree, and that all natural transformations which preserve the degree are given simply by the ones induced from group homomorphisms. We have already seen that there are in fact natural transformations which raise the degree: The Bockstein map is a natural map

$$H^n(-;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(-;\mathbb{Z})$$

and hence corresponds to a unique homotopy class $\beta\colon K(\mathbb{Z}/p\mathbb{Z},n) \to K(\mathbb{Z},n+1)$. Exercise: Recall that this map is non-trivial and think about whether you know this map for $p=2$ and

$n = 1$ (or more generally for any p and $n = 1$) In general, for $n > k$, the groups

$$[K(A, k), K(B, n)] \cong H^n(K(A, k); B)$$

are non-trivial in many cases, but it is not an easy argument to see this:[13] The simplest case might be where A is finitely generated and $B = \mathbb{Z}$. By the UCT and by considering A as the filtered colimit over its finitely generated subgroups, one may hope to evaluate the general case, although we remark that this filtered colimit translates into a derived inverse limit; these involve higher derived functors of the inverse limit, and such higher derived functors (unlike in the colimit case) can be non-trivial in arbitrarily high degrees...! Perhaps even simpler might be the cases $B = \mathbb{F}_p$ and $B = \mathbb{Q}$, i.e. where B runs through the prime fields. In any event, we observe that for A finitely generated, $K(A, k)$ is a finite product of spaces of the form $K(C, k)$ where C is cyclic, so one may appeal to the Künneth theorem to argue that it "suffices" to compute the case where A is cyclic. Later, we will prove some things about $H^*(K(\mathbb{F}_p, k); \mathbb{F}_p)$ (perhaps only when $p = 2$) in particular, that this is non-trivial.

The fact that by Yoneda's lemma, it is "easy" to describe natural transformations from a representable functor to an arbitrary functor is something we will come back to in the next section.

We end with some sample applications one gets from representability and knowing something about representing objects (you might try to prove the first part of the following corollary by hand):

3.17. **Corollary** *Let X be a topological space and $x \in H^1(X; \mathbb{Z})$. Then $0 = x^2 \in H^2(X; \mathbb{Z})$. Moreover, there every natural transformation $H^1(-; \mathbb{Z}) \to H^k(-; A)$ for $k > 1$ is trivial.*

Proof. Indeed, the first claim is true in the universal case $K(\mathbb{Z}, 1)$ since $H^2(K(\mathbb{Z}, 1); \mathbb{Z}) \cong H^2(S^1; \mathbb{Z}) = 0$, and $x^2 = f^*(\iota_1^{\mathbb{Z}})^2$ with $\iota_1^{\mathbb{Z}} \in H^1(S^1; \mathbb{Z})$ for the map $f\colon X \to K(\mathbb{Z}, 1)$ classifying x. To see the second, we recall that the set of such natural transformations is given by $H^k(K(\mathbb{Z}, 1); A) = 0$. $\square$

In a similar vein, we have (I invite you to think of more immediate such consequences):

3.18. **Corollary** *Let k be an odd number and n an even number. Then every natural transformation $H^2(-; \mathbb{Z}) \to H^k(-; A)$ and every natural transformation $H^1(-; \mathbb{F}_2) \to H^k(-; \mathbb{Z})$ is trivial.*

Proof. Again, such natural transformations are given by $H^k(\mathbb{CP}^\infty; A)$ and $H^k(\mathbb{RP}^\infty; \mathbb{Z})$, respectively. Both vanish since k is odd. $\square$

Exercise. Describe as explicitly as you can the natural transformations $H^2(-; \mathbb{Z}) \to H^{2n}(-; A)$ and $H^1(-; \mathbb{F}_2) \to H^{2n-1}(-; A)$ for abelian groups A and natural numbers $n \geq 1$.

As indicated above, later in this course we will be very interested in studying natural transformations $H^k(-; \mathbb{F}_2) \to H^{k+i}(-; \mathbb{F}_2)$ for $i > 0$. We will see that there are many such operations, called Steenrod operations, and they will be extremely helpful in further studying obstructions, e.g. for certain spaces to be homotopy equivalent to closed manifolds (the only obstructions we really know so far are consequences of Poincaré duality, but it turns out that there are more).

[13] An exception: If B is a ring, then the cup product gives a natural map $H^n(-; B) \to H^{2n}(-; B)$ – this map will in fact be the source of many non-trivial classes.

4. Principal bundles, fibre bundles, and fibrations

For the purpose of this section, we will consider a topological group G. For a more thorough treatment of principal bundles (we try to be quick here) one can consult [tD08, §14].

4.1. **Definition** Let G be a topological group and E a topological space. A continuous right action of G on E, i.e. a continuous map $E \times G \to E$, $(e, g) \mapsto eg$, is called *principal* if the projection map $p\colon E \to E/G$ satisfies the following condition: There exists an open cover $\{U_i\}_{i\in I}$ of E/G and G-equivariant homeomorphisms $U_i \times G \to p^{-1}(U_i)$ compatible with the respective projections to U_i.

A map $p\colon E \to B$ is called a principal G-bundle, if there exists a principal G-action on E for which p is equivariant (with respect to the trivial G-action on B) and such that p induces a homeomorphism $E/G \to B$. A map of principal G-bundles $p\colon E \to B$ and $p'\colon E' \to B$ is a G-equivariant map $\varphi\colon E \to E'$. It is called an isomorphism if it admits an inverse (or equivalently, if φ is a homeomorphism).[14]

4.2. **Remark** We note that the definition of a principal G-bundle implies that the projection map $p\colon E \to B$ is surjective, and a quotient map. Moreover, a principal G-bundle $p\colon E \to B$ is obviously a fibre bundle (with typical fibre G), and hence also a Serre fibration, see [Win24, Lemma 5.2.2]. In fact, a numerable fibre bundle is even a Hurewicz fibration. Being numerable means that there exists a trivializing open cover which admits a subordinate partition of unity. This is automatic if the base space is paracompact, but not in general.

4.3. **Remark** Given a principal G-bundle $p\colon E \to B$, consider a trivializing open cover $\{U_i\}_{i\in I}$. Denote by $U_{ij} = U_i \cap U_j$ the intersection. Then there are G-homeomorphisms $U_{ij} \times G \cong p^{-1}(U_{ij}) \cong U_{ij} \times G$, where the first homeomorphism comes from the trivialization over U_i and the second comes from the trivialization over U_j. This G-equivariant homeomorphism is, since it is compatible with the projection p, equivalently given by a continuous map $s_{ij}\colon U_{ij} \to G$. These maps satisfy *cocyle conditions*:

(1) $s_{ii}(u) = 1$ for all $u \in U_i$,
(2) $s_{ji}(u) = s_{ij}(u)^{-1}$ for all $u \in U_{ij}$, and
(3) $s_{ik}(u) = s_{jk}(u) \cdot s_{ij}(u)$ for all $u \in U_{ijk} = U_i \cap U_j \cap U_k$

and are called the associated cocycles of $E \to B$. Given functions $s_{ij}\colon U_{ij} \to G$ satisfying these conditions, one can construct a principal G-bundle whose associated cocylces are given by s_{ij}. (Exercise. Hint: Use these functions to form a suitable quotient of $\coprod_{i\in I} U_i \times G$).

4.4. **Remark** The fibres of a principal G-action on E over $p(e)$ are homeomorphic to G: Indeed, the map $\{e\} \times G \to p^{-1}(\{e\})$, $g \mapsto eg$ is a homeomorphism. In particular, a principal action of G on E is free. For any free action, consider the set $t_G(E) = \{(e, eg) \mid e \in E, g \in G\} \subseteq E \times E$ of all G-translates of points of E, and the *translation* map $t_G(E) \to G$, sending (e, eg) to g.[15] For a principal G-action, this translation map is continuous (exercise). Tom Dieck calls free actions with continuous translation map *weakly proper* actions. Note that if the G-action on E is weakly proper, and $U \subseteq E/G$ is open and E_U is the preimage of U under the projection $E \to E/G$, then E_U is a G-invariant open subset of E and the G-action on E_U is again weakly proper: Indeed, the translation map is simply the composite

[14] Indeed, any inverse of φ is automatically a bundle map.
[15] This is well-defined since the action is free.

$t_G(E_U) \subseteq t_G(E) \to G$. In Remark 4.14 below, we note under what assumptions free and weakly proper actions are principal.

4.5. **Example** Suppose G is a discrete group. Then a principal action is the same as a covering-like action in the sense of Topology I [Lan23]. In particular, when G is discrete, principal G-bundles are the same datum as Galois covering spaces whose Deck transformation group is identified with G.

4.6. **Example** For any space B, the map $B \times G \to B$ is evidently a principal G-bundle and is called the *trivial principal* G-bundle; in fact any bundle isomorphic to $B \times G \to B$ will be called trivial. By definition, any principal G-bundle is locally trivial (on the target) but not necessarily globally trivial, see the next example.

4.7. **Example** The projection maps $S^{2n+1} \to \mathbb{CP}^n$ are principal S^1-bundles. Indeed, S^1 acts diagonally on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$, this action is free. For $0 \leq i \leq n$, consider the subset U_i of S^{2n+1} consisting of points $(x_0, \dots, x_n) \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$ with $x_i \neq 0$. Consider the map $U_i \to U_i/S^1 \times S^1$ given by the projection in the first product factor and by the map sending $(x_0, \dots, x_n)$ to $\frac{x_i}{\|x_i\|}$ in the second factor. One checks that this is an S^1-equivariant homeomorphism and concludes that $S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1-bundle. These are all examples of bundles which are not trivial: S^{2n+1} is not homeomorphic to $\mathbb{CP}^n \times S^1$ (Exercise: find all arguments you can for this).

The same arguments apply for $\mathbb{RP}^n$, which participates in a principal $C_2 \cong O(1)$-bundle $S^n \to \mathbb{RP}^n$, see Example 4.5, and $\mathbb{HP}^n$, which participates in a principal $S^3 \cong SU(2) \cong Sp(1)$-bundle $S^{4n+3} \to \mathbb{HP}^n$. None of these examples are trivial bundles (same exercise as above).

Exercise. Show that $\Omega\mathbb{RP}^n \simeq C_2 \times \Omega S^n$, $\Omega\mathbb{CP}^n \simeq S^1 \times \Omega S^{2n+1}$ and $\Omega\mathbb{HP}^n \simeq S^3 \times \Omega S^{4n+3}$.

4.8. **Example** Let $p\colon E \to B$ be a principal G-bundle and let $f\colon B' \to B$ be a map of spaces. Let $p'\colon E' = E \times_B B' \to B'$ be the pullback of p along f. Then p' is again a principal G-bundle. We often write $f^*(E)$ for the pullback principal G-bundle.

4.9. **Definition** For a topological space B, we denote by $\mathrm{Bun}_G(B)$ the collection of isomorphism classes of principal G-bundles.

4.10. **Lemma** *The association* $B \mapsto \mathrm{Bun}_G(B)$ *refines to a functor* $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$.

Proof. First, we have to see that $\mathrm{Bun}_G(B)$ is a set: To do so, note that any two principal G-bundles over B have the same cardinality, namely that of $B \times G$. In particular, isomorphism classes of principal G-bundles over B can be thought of as certain topologies on the set $B \times G$, and there is a set worth of such topologies. Now, for $f\colon B \to B'$ we need to define an induced map $f^*\colon \mathrm{Bun}_G(B') \to \mathrm{Bun}_G(B)$. We do so by setting $f^*[E' \to B'] = [f^*(E') \to B]$. This indeed provides the claimed functor, since $g^*(f^*(E'))$ is isomorphic to $(gf)^*(E')$ and $\mathrm{id}^*(E)$ is isomorphic to E. $\square$

The following is an important concept not only in algebraic topology, but also in (differential) geometry, and will become crucial in our later investigation of manifolds:

4.11. **Definition** Let G be a topological group, A an abelian group and $n \geq 0$. A characteristic class θ of type (A, n) for principal G-bundles is a natural transformation $\mathrm{Bun}_G(-) \to H^n(-; A)$ of functors $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$.

Concretely, a characteristic class θ of type (A, n) therefore gives, for each principal G-bundle $p\colon E \to B$, a cohomology class $\theta(p) \in H^n(B; A)$ satisfying the following compatibility constraint. For a map $f\colon B' \to B$ with pullback bundle $f^*(p)\colon f^*(E) \to B'$, we have the equation $\theta(f^*(p)) = f^*(\theta(p))$. We will study characteristic classes to some extend later in this lecture course, though only for very specific groups G.

We now come to some basic properties of principal bundles and bundle maps.

4.12. **Lemma** *Let $p\colon E \to B$ and $p'\colon E' \to B$ be principal G-bundles and let $\varphi\colon E \to E'$ be a map of principal G-bundles. Then φ is an isomorphism.*

Proof. Let us first argue that φ is a bijection. To see surjectivity, for $e' \in E'$ we can choose an element $e \in E$ such that $p(e) = p'(e')$. Then $\varphi(e)$ and e' lie in the same fibre of p'. Since G acts transitively on the fibres of p', we see that there exists a $g \in G$ such that $g\varphi(e) = e'$. Since φ is G-equivariant, we deduce that $\varphi(ge) = e'$ so φ is surjective. To see injectivity, we argue similarly. If e_0 and e_1 satisfy $\varphi(e_0) = \varphi(e_1)$, then we conclude that $p(e_0) = p(e_1)$ so e_0 and e_1 lie in the same fibre of p. But $\varphi_{|p^{-1}(b)}$ is a G-equivariant between free G-spaces, and is hence injective. Consequently, φ is also injective. To see that φ is open, we may choose an open cover $\{U_i\}_{i\in I}$ such that p and p' are trivial over each U_i. The restriction of φ to $p^{-1}(U_i)$ is then a map $U_i \times G \to U_i \times G$ over U_i which is G-equivariant (in particular bijective by what we have argued before). Such maps are equivalently given by non-equivariant maps $U_i \to U_i \times G$, and being over U_i corresponds to the first map being the identity. Hence the above map is given by $(u, g) \mapsto (u, q(u)g)$. This map has inverse given by $(u, h) \mapsto (u, q(u)^{-1}h)$ which is continuous since the inverse map on G is continuous. Hence, the map φ is locally open and thus open, and consequently a homeomorphism. □

The following is a basic, but important and useful property of principal bundles.

4.13. **Lemma** *Let $p\colon E \to B$ be a principal G-bundle. Then p admits a section if and only if p is trivial.*

Proof. The if assertion is immediate. So let $s\colon B \to E$ be a section to p. We obtain a bundle map $B \times G \to E$ by sending (b, g) to $g \cdot s(b)$. By Lemma 4.12 this map is an isomorphism. □

4.14. **Remark** In fact, there is the following strengthening of Lemma 4.13 Let G act (freely and) weakly properly on E. Then E is G-equivariantly isomorphic to $B \times G$ if and only if $E \to E/G$ admits a continuous section. (Exercise. Hint construct an explicit inverse of the map $B \times G \to E$ considered in the above proof using the fact that the action is weakly proper).

In particular, a weakly proper action is a principal action if and only if the projection map $E \to E/G$ has locally on the target continuous sections (recall that the restriction of the G-action to preimages of opens of E/G is again weakly proper).

4.15. **Example** We add an example which uses some notions from smooth manifolds (which we will not address here). Let G be a Lie group and H a closed subgroup. Then H acts by right multiplication on G and this action is weakly proper. Moreover, G/H is naturally a smooth manifold and the projection map $G \to G/H$ is a submersion. Since any submersion between smooth manifolds admits local sections (as a consequence of the implicit function theorem), we deduce that the right multiplication action of H on G is principal and $G \to G/H$ is a principal H-bundle.

In fact, more generally, if G is a topological group and $H \subseteq G$ is a subgroup, then the right multiplication action of H on G is weakly proper. Moreover, the projection map $p\colon G \to G/H$ is a principal H-bundle if p has a section in a neighborhood of $[1] \in G/H$ – in the Lie group case and when H is closed, this local section exists automatically, but in general it does not.

We will next address the following two important theorems:

4.16. **Theorem** (Homotopy theorem) *The functor* $\mathrm{Bun}_G\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ *is a homotopy functor on CW complexes, i.e. its restriction to spaces that admit a CW structure factors through* $\mathrm{Top} \to \mathrm{hTop}$.

4.17. **Theorem** (Classification theorem) *The functor* $\mathrm{Bun}_G(-)\colon \mathrm{hTop}^{\mathrm{op}} \to \mathrm{Set}$ *is representable on CW complexes.*

4.18. **Remark** One can avoid the restriction to CW complexes if one replaces $\mathrm{Bun}_G(-)$ by $\mathrm{Bun}_G^{\mathrm{num}}(-)$ which takes isomorphism classes of *numerable* principal G-bundles. Then both the homotopy and the classification theorem hold true for all spaces, as we will indicate below.

A representing object is therefore in particular principal G-bundle which we denote by $EG \to BG$. That this principal G-bundle represents $\mathrm{Bun}_G(-)$ on CW complexes is the statement that for any CW complex B, the map $[B, BG] \to \mathrm{Bun}_G(B)$, $f \mapsto f^*(EG)$, is a bijection. Again, Yoneda's lemma implies that a universal principal G-bundle is unique up to unique homotopy classes of homotopy equivalences. We refer to BG as a *classifying space for G* or more precisely, for principal G-bundles. A consequence of the above and Yoneda's lemma is also given by the following characterization of characteristic classes:

4.19. **Corollary** *A characteristic class of type (A, n) for for principal G-bundles is the same datum as an element $\theta \in H^n(BG; A)$.*

Therefore, computing the cohomology of classifying spaces is one way to describe characteristic classes.

To establish the above theorems, we will use the following construction which is interesting in its own right:

4.20. **Construction** Let $E \to B$ be a principal G-bundle and let F be a space equipped with a continuous left G-action. The *associated bundle* is the map $E \times_G F \to B$. Here the map is induced by the G-equivariant map $F \to *$, and $E \times_G F$ refers to the quotient of the diagonal right G-action on $E \times F$ given by $g \cdot (e, f) = (eg, g^{-1}f)$. This associated bundle is now a fibre bundle with typical fibre F (Exercise. Hint: think about the case where $E \to B$ is trivial, and reduce to this case).

Conversely, given a fibre bundle $p\colon E \to B$ with typical fibre a locally compact space F, we construct a principal $\mathrm{Homeo}(F)$-bundle using Remark 4.3 as follows. Pick a trivializing open cover $\{U_i\}_{i \in I}$. Then there are homeomorphisms $U_{ij} \times F \cong p^{-1}(U_{ij}) \cong U_{ij} \times F$ over U_{ij}. These are equivalently given by a continuous map $U_{ij} \to \mathrm{Homeo}(F)$ and one checks that they satisfy the cocycle relations. Therefore one obtains a principal $\mathrm{Homeo}(F)$-bundle.

One can show that these constructions yields a one-to-one correspondence between isomorphism classes of fibre bundles with typical fibre a locally compact space F and isomorphism classes of principal $\mathrm{Homeo}(F)$-bundles.

4.21. **Example** Let $G \to G'$ be a continuous group homomorphism and let $E \to B$ be a principal G-bundle. We may view G' as a left G-space via the map f and form the associated bundle $E \times_G G' \to B$. This turns out to be a principal G'-bundle and in fact, this construction gives rise to a natural transformation of functors $\mathrm{Bun}_G(-) \to \mathrm{Bun}_{G'}(-)$.

Consequently, given a locally compact left G-space F, the associated bundle $E \times_G F$ can be factored as follows: From the continuous group homomorphism $G \to \mathrm{Homeo}(F)$, we can construct the principal $\mathrm{Homeo}(F)$-bundle $E \times_G \mathrm{Homeo}(F)$ and then form the tautologically associated bundle $[E \times_G \mathrm{Homeo}(F)] \times_{\mathrm{Homeo}(F)} F$ which is isomorphic to $E \times_G F$.

For a continuous group homomorphism $G \to \mathrm{Homeo}(F)$, one then says that a fibre bundle $X \to B$ with fibre F has *structure group reduced to* G if this bundle is equipped with an isomorphism to an associated bundle $E \times_G F$ for some principal G-bundle $E \to B$, or equivalently, in hopefully evident notation, if the cocycles $s_{ij} \colon U_{ij} \to \mathrm{Homeo}(F)$ are equipped with cocycle lifts $\hat{s}_{ij} \colon U_{ij} \to G$ along the map $G \to \mathrm{Homeo}(F)$.

4.22. **Example** A vector bundle $E \to B$ is a fibre bundle with typical fibre $\mathbb{R}^n$ together with a factorization of the maps $U_{ij} \to \mathrm{Homeo}(\mathbb{R}^n) =: \mathrm{Top}(n)$ through the inclusion $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathrm{Top}(n)$. In particular, by the above, a vector bundle is the associated bundle of a principal $\mathrm{GL}_n(\mathbb{R})$-bundle via the left $\mathrm{GL}_n(\mathbb{R})$-space $\mathbb{R}^n$. The same is true for complex or quaternionic vector bundles; these are equivalently given by $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{H})$-principal bundles, respectively.

The principal $\mathrm{GL}_n(\mathbb{K})$-bundle corresponding to a $\mathbb{K}$-vector bundle is called the *frame bundle* or *bundle of frames*: In each fibre, it is simply given the space of basis on each fibre, and $\mathrm{GL}_n(\mathbb{K})$ acts freely and transitively on the space of bases of a fixed n-dimensional vector space.

Equipping a real vector bundle with a Riemannian metric amounts to reducing the structure group from $\mathrm{GL}_n(\mathbb{R})$-bundle to $\mathrm{O}(n)$, and likewise equipping a complex vector bundle with a hermitian metric amounts to reducing the structure group from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{U}(n)$ (Exercise).

4.23. **Corollary** *Theorem 4.17 implies that every fibre bundle (over a CW complex) with typical fibre F is pulled back from the universal such bundle, which is given by $E\mathrm{Homeo}(F) \times_{\mathrm{Homeo}(F)} F \to B\mathrm{Homeo}(F)$. In particular, fibre bundles over a contractible space are globally trivial. The same is true for the functor taking isomorphism classes of $\mathbb{K}$-vector bundles for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.*

Later, we will be particularly interested in characteristic classes of vector bundles, by which we mean, characteristic classes of $\mathrm{GL}_n(\mathbb{R})$-principal bundles.

4.24. **Definition** Let $p \colon E \to B$ and $p' \colon E' \to B'$ be principal G-bundles. A bundle map from $p \colon E \to B$ to $p' \colon E' \to B$ is a G-equivariant continuous map $E \to E'$, or equivalently, a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

where $\bar{f}$ is G-equivariant.

4.25. **Remark** The diagram appearing in the definition of bundle maps is a pullback: Indeed, the commutative diagram induces a map $E \to f^*(E')$ which is a bundle map over the identity and hence an isomorphism by Lemma 4.12.

The following is a basic, but very useful observation.

4.26. **Lemma** *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be principal G-bundles. Then bundle maps from p to p' correspond bijectively to sections of the associated bundle $E \times_G E' \to B$.*

Proof. Let $f\colon E \to E'$ be a bundle map, i.e. a G-equivariant map. Consider then the composite $E \to E \times E' \to E \times_G E'$ where the first map is the pair (id_E, f) and the second is the canonical projection. Concretely, this composite is given by $e \mapsto [e, f(e)]$. Observe that for all $g \in G$, we have $eg \mapsto [eg, f(eg)] = [eg, f(e)g] = [e, f(e)]$. Consequently, as p is isomorphic to the projection map $E \to E/G$, the just described map descends to a continuous map $s\colon B \to E \times_G E'$. The composite of this map with the projection $E \times_G E'$, which is given by $[e, e'] \mapsto p(e)$ is then the identity by construction, so s is a section of the associated bundle as needed.

Conversely, let $s\colon B \to E \times_G E'$ be a section of the associated bundle. Pick a trivializing open cover $\{U_i\}_{i\in I}$ of $E \to B$. Then $E \times_G E'$ is also trivial over this cover since $E_{|U_i} \times_G E' \cong (U_i \times G) \times_G E' = U_i \times E$. The section s is equivalently described by maps $s_i = s_{|U_i}$ which can be regarded as continuous maps $U_i \to E'$. One can then form the maps $\bar{s}_i\colon E_{|U_i} \cong U_i \times G \to E'$, induced by sending $(x, g) \in U_i \times G$ to $s_i(x) \cdot g$. All the maps $\bar{s}_i$ are continuous and they agree on intersections (exercise), hence they determine a unique continuous map $f\colon E \to E'$ which is G-equivariant by construction; concretely, this is given by sending an $e \in E$ to the following point of E': write $s(b) = [s_1(b), s_2(b)]$ and observe that $p(s_1(b)) = b$ since s is a section. Since the G-action on E is free, we may represent $[s_1(b), s_2(b)]$ uniquely by the tuple (e, e') and doing so, we have $f(e) = e'$.

It is then readily checked that these two constructions are inverse to each other. $\square$

We can now prove the homotopy theorem for bundles.

Proof of Theorem 4.16. Let us consider a principal G-bundle $E \to B \times [0,1]$ and let us denote by E_0 its pullback to $B \times \{0\}$ and likewise by E_1 its pullback to $B \times \{1\}$. We will construct a bundle map appearing in the right square of the following diagram

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] & E \arrow[r, dashed] \arrow[d] & E_0 \arrow[d] \\
B \times \{0\} \arrow[r] & B \times [0,1] \arrow[r, "\mathrm{pr}"] & B
\end{tikzcd}$$

making both horizontal composites the identity map. By Lemma 4.26, this is equivalent to constructing an appropriate section of the associated bundle $E \times_G E_0 \to B \times [0,1]$. The condition that the above composites ought to be the identity imply that we wish to solve the following lifting problem, in which the top arrow is given by the tautological section of $E_0 \times_G E_0 \to B$, followed by the map $E_0 \times_G E_0 \to E \times_G E_0$ induced by the inclusion $E_0 \subseteq E$.

$$\begin{tikzcd}
B \times \{0\} \arrow[r] \arrow[d] & E \times_G E_0 \arrow[d] \\
B \times [0,1] \arrow[r, equal] \arrow[ur, dashed] & B \times [0,1]
\end{tikzcd}$$

Since the associated bundle is a fibre bundle, it is a Serre fibration, and hence, if B is a CW complex, a dashed arrow exists.[16] Hence, there exists a bundle map $E \to E_0$ over $\mathrm{pr}\colon B \times [0,1] \to B$, and consequently, we find an isomorphism of principal G-bundles $E \cong E_0 \times [0,1] = \mathrm{pr}_*(E_0)$ over $B \times [0,1]$ as a consequence of Lemma 4.12, and this map restricts to the identity upon pullback to $B \times \{0\}$. This isomorphism however also induces an isomorphism $E_1 \cong E_0$ by restricting along $B \times \{1\} \to B$.

We finish by arguing that this implies the theorem: Let $f, g\colon B \to B'$ be homotopic maps and $E' \to B'$ a principal G-bundle. We wish to show that $f^*(E')$ and $g^*(E')$ are isomorphic. To see this, let $h\colon B \times [0,1] \to B'$ be a homotopy between f and g and let $E = h^*(E')$. Then, on the one hand, $f^*(E') = E_0$ and $g^*(E') = E_1$, and on the other we have just constructed an isomorphism $E_1 \cong E_0$ of G-bundles over B, finishing the proof of the theorem. □

To go towards the proof of the classification theorem for principal G-bundles, we will to this is a two step process: First, we will prove a recognition principle for universality of a principal G-bundle, and then we will show that such a bundle exists. The recognition principle is the following analog of a result in covering theory which we have proven in [Lan23].

4.27. **Proposition** *Let $p\colon E \to B$ be a principal G-bundle with E weakly contractible. Then p is universal on spaces which admit a CW structure.*

Proof. We need to show that for all CW complexes B', the map $[B', B] \to \mathrm{Bun}_G(B')$ given by $f \mapsto f^*(p)$ is a bijection. To prove surjectivity, let $p'\colon E' \to B'$ be a principal G-bundle. By Lemma 4.12, it suffices to construct a bundle map $E' \to E$. By Lemma 4.26 this amounts to constructing a section of the associated bundle $E \times_G E' \to B$. This is a fibre bundle with weakly contractible fibre E', and hence this map is a weak equivalence. Hence, the lifting problem

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & E \times_G E' \arrow[d] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a solution as we have shown in topology II [Win24, Prop. 5.2.15]. To see injectivity, consider two maps $f, g\colon B' \to B$ and an isomorphism $f^*(p) \cong g^*(p)$. Equivalently, we may consider a principal G-bundle $p'\colon E' \to B'$ with two bundle maps $E' \to E$ lying over f and g respectively. We need to show that these bundle maps are G-equivariantly homotopic, so that we find that f and g are homotopic in the usual sense. Now one observes that a G-equivariant homotopy is equivalently a bundle map $E' \times [0,1] \to E$, where the former is the principal G-bundle $E' \times [0,1] \to B \times [0,1]$. By the same argument as in the surjectivity and translating such a bundle map to suitable sections, it suffices to solve the lifting problem

$$\begin{tikzcd} B' \times \{0,1\} \arrow[r] \arrow[d] & (E' \times [0,1]) \times_G E \arrow[d] \\ B' \times [0,1] \arrow[r, equal] & B' \times [0,1] \end{tikzcd}$$

which can again be solved by [Win24, Prop. 5.2.15] as the right vertical is still a weak equivalence and Serre fibration, and $B' \times \{0,1\} \to B \times [0,1]$ is a relative CW complex. □

[16] The same argument shows the general form of the homotopy theorem: If all bundles are numerable, then the associated bundle is also numerable, and hence a Hurewicz fibration. The lifting problem can therefore be solved for all spaces B.

4.28. **Remark** Suppose the $p\colon E \to B$ is a numerable principal G-bundle with E contractible. Then p is in fact universal on all spaces, and the exact same proof strategy as above works: In the case at hand, $E \times_G E'$ is a numerable fibre bundle with contractible fibre, such bundle projections are homotopy equivalences, see e.g. [Dol63, Corollary 3.2][17], and Hurewicz fibrations as we have claimed earlier. Moreover, homotopy equivalences that are also Hurewicz fibrations have the RLP against all closed cofibrations such as $\varnothing \to B'$ and $B' \times \{0,1\} \to B' \times [0,1]$, see e.g. [Str66, Str68].

Consequently, to finish the proof of the classification theorem, we need to show the existence of principal G-bundles with (weakly) contractible total space. We will outline two approaches for this, one based on Milnor's infinite join construction, and one based on the Bar construction. In fact, these two turn out to be quite related; see [Seg68].

4.29. **Theorem** *Let G be a topological group. There exists a principal G-bundle $EG \to BG$ with EG contractible.*

4.30. **Corollary** *Let G be a topological group. Then there exists a principal G-bundle $\tilde{EG} \to \tilde{BG}$ with $\tilde{EG}$ weakly contractible and $\tilde{BG}$ a CW complex.*

Proof. Pick a CW approximation $\tilde{BG} \to BG$ and let $\tilde{EG} = EG \times_{BG} \tilde{BG}$. Exercise: Show that $\tilde{EG} \to EG$ is a weak equivalence. □

Before coming the constructions of $EG \to BG$, let us contemplate the situation.

4.31. **Remark** Suppose that $EG \to BG$ is any principal G-bundle with EG contractible. Then BG is canonically pointed and $\Omega BG \simeq G$ are weakly equivalent (Exercise). In particular, we have $\pi_n(BG) \cong \pi_{n-1}(G)$ As indicated above, the operation $G \mapsto BG$ can be chosen functorially. In particular, we may think of B as a functor which takes a topological group G as input and gives as output a (pointed) space BG which is a *delooping* of G, i.e. a space such that Ω of it is equivalent to G. One can show that such deloopings exist more generally than for topological groups (a grouplike E_1-space in the ∞-categorical sense is sufficient). We will later make use of this fact later, but we will not prove it in this generality in this course.

Finally, we note that if A is a discrete abelian group, and we assume that the functor B preserves finite products, then it follows formally that BA is again a topological abelian group (since both the inversion map and the addition are group homomorphisms) and in particular, we may iterate this construction: The above argument then shows that $B^n(A)$ is a functorial construction of $K(A,n)$. In general, the functorial constructions of B we indicate below do not preserve products, but they do if the topological group is *nice enough*, and inductively, it turns out that for discrete abelian groups A and any $n \geq 1$, the group $B^{n-1}(A)$ is nice enough.

4.32. **Remark** Let $\varphi\colon G \to G'$ be a group homomorphism. As indicated in Example 4.21, forming the associated bundle provides a natural map $\mathrm{Bun}_G(-) \to \mathrm{Bun}_{G'}(-)$, which by the classification theorem is induced by a unique homotopy class $B\varphi\colon BG \to BG'$. One can show that this provides the association $B(-)$ with a functor from topological groups to hTop_*, the homotopy category of pointed spaces; In fact, we will show that one can find a

[17] Warning: It is not true that Hurewicz fibrations with contractible fibre are homotopy equivalences, see `https://math.stackexchange.com/questions/2802883/fibration-with-contractible-fibre` for a counterexample using the Warsaw circle.

model for B which is strictly functorial, i.e. gives rise to a functor from topological groups to pointed topological spaces. As a consequence of this and Remark 4.31, we deduce that the isomorphism $\pi_n(BG) \cong \pi_{n-1}(G)$ is natural with respect to group homomorphisms. Indeed, we need to construct only a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[r] \arrow[d] & BG \arrow[d] \\ G' \arrow[r] & EG' \arrow[r] & BG' \end{tikzcd}$$

whose vertical maps are induced by $G \to G'$. To see this, simply observe that there is a canonical map $EG \to EG \times_G G'$ and that the latter is the pullback of the span $BG \to BG' \leftarrow EG'$.

In particular, suppose $G \to G'$ is a group homomorphism which is a weak homotopy equivalence. We then conclude that the induced map $BG \to BG'$ is again a weak homotopy equivalence. Consequently, there is a canonical bijection $\mathrm{Bun}_G(B) \cong \mathrm{Bun}_{G'}(B)$ between isomorphism classes of principal G- and principal G'-bundles over a CW complex B.

4.33. **Example** Let H be a subgroup of a topological group G such that $G \to G/H$ is a principal G-bundle, e.g. G a Lie group and H a closed subgroup. We obtain a commutative diagram of pullback squares

$$\begin{tikzcd} H \arrow[r] \arrow[d] & EH \arrow[d] & \\ G \arrow[r] \arrow[d] & EH \times_H G \arrow[r] \arrow[d] & EG \arrow[d] \\ * \arrow[r] & BH \arrow[r] & BG \end{tikzcd}$$

in which the map $H \to G$ is the canonical inclusion and the map $BH \to BG$ is the induced map as in Remark 4.32. Moreover, the map $EH \to *$ induces a homotopy equivalence $EH \times_H G \to G/H$ (Exercise), so we obtain a homotopy fibration sequence $G/H \to BH \to BG$ in which the map $G/H \to BH$ classifies the principal H-bundle $G \to G/H$.

4.34. **Example** Suppose G is a discrete group. Then we may consider the space $K(G,1)$ considered earlier. Choosing a CW model, we may assume that it has a universal cover, which is then weakly contractible and hence contractible by Whitehead's theorem. In other words, we have already proven Theorem 4.29 in case G is discrete; $K(G,1)$ is indeed a choice of a BG.

We add here the following lemma, which is new only for non-abelian groups.

4.35. **Lemma** *Let G and H be discrete groups. Then the canonical map $[BG, BH]_* \to \mathrm{Hom}_{\mathrm{Grp}}(G,H)$ is bijective. Moreover, $[BG, BH] \cong \mathrm{Hom}_{\mathrm{Grp}}(G,H)/H$, where H acts by conjugation on group homomorphisms. If BG is a locally finite CW complex, then $\mathrm{Map}_*(BG, BH) \simeq \mathrm{Hom}_{\mathrm{Grp}}(G,H)$, that is, all components of this pointed mapping space are contractible.*

Proof. Let $X(G)$ be a presentation complex for G. Then BG can be built from $X(G)$ by attaching cells of dimension ≥ 3. The Puppe sequences from Remark 3.11 imply that therefore the map $[BG, BH]_* \to [X(G), BH]_*$ is bijective. Moreover, another application of the

Puppe sequence, together with the fact that $X(G) = \mathrm{C}(f\colon \bigvee_J S^1 \to \bigvee_I S^1)$ gives the upper horizontal exact sequence in the diagram

$$\begin{tikzcd}
{[\bigvee_J S^1, BH]_*} \arrow[d] & {[\bigvee_I S^1, BH]_*} \arrow[l] \arrow[d] & {[X(G), BH]_*} \arrow[l] \arrow[d] & {[\bigvee_J S^2, BH]_* = 0} \arrow[l] \arrow[d] \\
\mathrm{Hom}_{\mathrm{Grp}}(\mathcal{F}_J, H) & \mathrm{Hom}_{\mathrm{Grp}}(\mathcal{F}_I, H) \arrow[l] & \mathrm{Hom}_{\mathrm{Grp}}(G, H) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the vertical maps are given by applying the functor $\pi_1(-)$ and the lower horizontal map is also exact since $G = \mathrm{coker}(\mathcal{F}_J \to \mathcal{F}_I)$. The left two vertical maps are bijective, and hence so is the map $[X(G), BH]_* \to \mathrm{Hom}_{\mathrm{Grp}}(G, H)$.

Then we recall from Remark 2.30 the exact sequence

$$\cdots \to \pi_1(BH) \to [BG, BH]_* \to [BG, BH] \to 0$$

and use that $\pi_1(BH) = H$, and the resulting H-action on $[BG, BH]_* \cong \mathrm{Hom}_{\mathrm{Grp}}(BG, BH)$ is given by conjugation of group homomorphisms. This requires unravelling of the definitions, I leave it as an exercise.

The final claim can be proven as follows. Pick a pointed map $f\colon BG \to BH$. To calculate $\pi_n(\mathrm{Map}_*(BG, BH), f)$ for $n \geq 1$, consider a representative as a map $S^n \times BG \to BH$ with the property that its restriction to $\{1\} \times BG \to BH$ is f and that its restriction to $S^n \times \{x\} \to BH$ is constant at the basepoint y of BH, x being the basepoint of BG. Consider the diagram

$$\begin{tikzcd}
S^n \times BG \cup D^{n+1} \times \{x\} \arrow[r] \arrow[d] & BH \\
D^{n+1} \times BG \arrow[ur, dashed] &
\end{tikzcd}$$

and observe that the vertical map is an inclusion, where the codomain is obtained from the domain by attaching cells of dimension ≥ 3. In particular, since BH has only π_1 non-trivial, an extension exists, showing the triviality of $\pi_n(\mathrm{Map}_*(BG, BH), f)$ as needed. $\square$

4.36. **Remark** In particular, this result and the sequence from Remark 2.30 implies that $\pi_n(\mathrm{Map}(BG, BH), f) = 0$ for $n \geq 2$, and all (pointed) maps $f\colon BG \to BH$. Moreover, $\pi_1(\mathrm{Map}(BG, BH), f)$ identifies with the stabiliser of the H-action on f, i.e. all those $h \in H$ such that $c_h(f) = f$, where $c_h(f)(g) = hf(g)h^{-1}$.

4.37. **Example** We have that $\mathbb{CP}^\infty$ is a choice of BS^1 where $S^1 \subseteq \mathbb{C}$ is the unit circle. Indeed, from Example 4.7 and by passing to the colimit over n, we find that $S^1 \to S^{2\infty+1} \to \mathbb{CP}^\infty$ is indeed a principal S^1-bundle, and $S^{2\infty+1}$ is contractible. Since $S^1 = \mathrm{U}(1)$, we find that $\mathbb{CP}^\infty$ equivalently classifies complex vector bundles of rank 1 (such vector bundles are called *line bundles*) which are equipped with a hermitian metric. In particular, $\mathbb{CP}^\infty$ carries itself such a line bundle, it is called the tautological line bundle: We may think of a point of $\mathbb{CP}^\infty$ as a complex line $L \subseteq \mathbb{C}^\infty$ and may then form the set

$$\gamma = \{(L, x) \mid L \in \mathbb{CP}^\infty, x \in L\} \to \mathbb{CP}^\infty$$

which one checks to be the vector bundle associated bundle $S^{2\infty+1} \times_{\mathrm{U}(1)} \mathbb{C} \to \mathbb{CP}^\infty$.

Recall that $H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong H^2(\mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}$ via a specific isomorphism. The element corresponding to $1 \in \mathbb{Z}$ is often written $c_1 = c_1(\gamma)$, the first Chern class of the tautological bundle γ.[18] Given a complex line bundle $E \to X$, define $c_1(E) := f^*(c_1(\gamma))$ for $f\colon X \to \mathbb{CP}^\infty$ the map classifying E. This is then a characteristic class of line bundles in the sense of Definition 4.11. Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$, we conclude that complex line bundles are equivalently classified by their first Chern class: That is, two line bundles are isomorphic if and only if their first Chern classes agree, and every element in $H^2(X;\mathbb{Z})$ arises as the first Chern class of a complex line bundle on X.

4.38. **Example** We have that $\mathbb{RP}^\infty$ is a choice of $B\mathrm{O}(1)$, which carries the universal (tautological) real line bundle $\gamma_\mathbb{R} \to \mathbb{RP}^\infty$ (same formulas as in the complex case). Denoting the non-trivial element of $H^1(\mathbb{RP}^\infty;\mathbb{F}_2)$ by $w_1 = w_1(\gamma_\mathbb{R})$ we arrive at the same results as above: Associating to a real line bundle $E \to X$ the element $w_1(E) = f^*(w_1)$ where $f\colon X \to \mathbb{RP}^\infty$ classifies E defines a characteristic class $w_1(-)$ called the first Stiefel–Whitney class. Again, since $\mathbb{RP}^\infty \simeq K(\mathbb{F}_2,1)$, we see that two real line bundles are isomorphic if and only if their first Stiefel–Whitney classes agree, and that every element in $H^1(X;\mathbb{F}_2)$ arises as the first Stiefel–Whitney class of a real line bundle on X.

4.39. **Example** We have that $\mathbb{HP}^\infty$ is a choice of $B\mathrm{Sp}(1)$, so that it again carries the universal tautological quaternionic line bundle $\gamma_\mathbb{H} \to \mathbb{HP}^\infty$ (same formulas as above). Denoting a generator of $H^4(\mathbb{HP}^\infty;\mathbb{Z})$ by q_1, we obtain the following: Associating to a quaternionic line bundle $E \to X$ the element $q_1(E) = f^*(q_1)$ where $f\colon X \to \mathbb{HP}^\infty$ classifies E defines a characteristic class q_1.[19] However, since $\mathbb{HP}^\infty$ is not equivalent to $K(\mathbb{Z},4)$, we do not arrive at the statement that two quaternionic line bundles are isomorphic if and only if their characteristic class q_1 agrees, and also not that every element in $H^4(X;\mathbb{Z})$ arises as $q_1(E)$ for some quaternionic line bundle E.

4.40. **Remark** Let G be a topological group and $EG \to BG$ a principal G-bundle with EG contractible. Let H be a subgroup of G such that $G \to G/H$ is a principal H-bundle, i.e. such that the projection $G \to G/H$ has a section in a neighborhood of the $[1] \in G/H$. Then the restricted H-action on EG is principal (exercise), so that EG/H is a valid choice for BH.

This can be used efficiently for compact Lie groups as follows: Any such group is a subgroup of $\mathrm{U}(n)$ for some n as a consequence of the Peter–Weyl theorem (in particular, this says that any compact Lie group admits a faithful and unitary finite dimensional complex representation). Hence it suffices to construct classifying spaces for $\mathrm{U}(n)$, which can be done very geometrically, similar to the case of $\mathrm{U}(1)$ above, see Example 4.41 below. Moreover, any connected Lie group is homotopy equivalent to a compact Lie group (to so-called maximal compact subgroups), so the above remark implies that there are classifying spaces for arbitrary connected Lie groups.

4.41. **Example** We describe geometric models for classifying spaces for $\mathrm{GL}_n(\mathbb{K})$ where $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$. Indeed, similar to $\mathbb{KP}^n$, one may consider the space $\mathrm{Gr}_k(\mathbb{K}^n)$ of k-dimensional subspaces of of $\mathbb{K}^n$, this is called the *Grassmannian of k-planes in $\mathbb{K}^n$*.[20] Passing to the colimit over n, one arrives at $\mathrm{Gr}_k(\mathbb{K}^\infty)$, the Grassmannian of k-planes in $\mathbb{K}^\infty$, topologized as the indicated colimit. This again carries a tautological $\mathbb{K}$-vector bundle $\gamma^k_\mathbb{K}$ of rank k given by

[18] Actually, there is no uniform convention regarding the signs here.

[19] This does not have a name as prominent as in the real and complex cases.

[20] These are smooth manifolds which admit explicit CW structures based on *Schubert cells*.

the colimit over n of the spaces γ_n whose points are pairs (E, x) where $E \in \mathrm{Gr}_k(\mathbb{K}^\infty)$ is a k-dimensional subspace of $\mathbb{K}^\infty$ and $x \in E$. One can then show that the associated frame bundle has contractible total space, so $\gamma^k_{\mathbb{K}} \to \mathrm{Gr}_k(\mathbb{K}^\infty)$ models $E\mathrm{GL}_n(\mathbb{K}) \to B\mathrm{GL}_n(\mathbb{K})$; see below for a possible argument.

There is a description of these Grassmannians by means of homogenous spaces, i.e. quotients of Lie groups: To see this, we first note that if $H \times K \subseteq G$ is a closed subgroup of a Lie group, then K acts principally on G/H and we obtain that $G/H \to G/H \times K$ is a principal K-bundle.

(1) There are homeomorphisms $\mathrm{O}(n)/\mathrm{O}(n-k) \times \mathrm{O}(k) \cong \mathrm{Gr}_k(\mathbb{R}^n)$; the tautological bundle is then associated to the principal $\mathrm{O}(k)$-bundle $\mathrm{O}(n)/\mathrm{O}(n-k) \to \mathrm{O}(n)/\mathrm{O}(n-k) \times \mathrm{O}(k)$. Indeed, $\mathrm{O}(n)$ acts on $\mathrm{Gr}_k(\mathbb{R}^n)$ as by sending $(A, V) \in \mathrm{O}(n) \times \mathrm{Gr}_k(\mathbb{R}^n)$ to $A(V)$. We learn in linear algebra that this action is transitive, that is, for any two k-dimensional subspaces V, V' there exists an orthogonal matrix A with $A(V) = V'$. Now consider the standard k-dimensional subspace $\mathbb{R}^k \times 0 \subseteq \mathbb{R}^n$. The stabilizer of the $\mathrm{O}(n)$-action on $\mathrm{Gr}_k(\mathbb{R}^n)$ at the point $\mathbb{R}^k$ is given by $\mathrm{O}(k) \times \mathrm{O}(n-k)$, giving the claimed description.
(2) There are homeomorphisms $\mathrm{U}(n)/\mathrm{U}(n-k) \times \mathrm{U}(k) \cong \mathrm{Gr}_k(\mathbb{C}^n)$; the tautological bundle is then associated to the principal $\mathrm{U}(k)$-bundle $\mathrm{U}(n)/\mathrm{U}(n-k) \to \mathrm{U}(n)/\mathrm{U}(n-k) \times \mathrm{U}(k)$. As before, $\mathrm{U}(n)$ acts transitively on $\mathrm{Gr}_k(\mathbb{C}^n)$ and the stabilizer of $\mathbb{C}^k \subseteq \mathbb{C}^n$ is $\mathrm{U}(k) \times \mathrm{U}(n-k)$.
(3) There are homeomorphisms $\mathrm{Sp}(n)/\mathrm{Sp}(n-k) \times \mathrm{Sp}(k) \cong \mathrm{Gr}_k(\mathbb{H}^n)$; the tautological bundle is then associated to the principal $\mathrm{Sp}(k)$-bundle $\mathrm{Sp}(n)/\mathrm{Sp}(n-k) \to \mathrm{Sp}(n)/\mathrm{Sp}(n-k) \times \mathrm{Sp}(k)$.

In all of the above examples, we may increase n and obtain comparison maps whose colimit is then $\mathrm{Gr}_k(\mathbb{K}^\infty)$ with its tautological bundle. Exercise: Show that $\mathrm{colim}_n\, \mathrm{O}(n)/\mathrm{O}(n-k)$, $\mathrm{colim}_n\, \mathrm{U}(n)/\mathrm{U}(n-k)$, and $\mathrm{colim}_n\, \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$ are weakly contractible. Hint: If $k = 1$, these spaces are spheres of suitable dimension. This verifies the claim that $\mathrm{Gr}_k(\mathbb{K}^\infty)$ indeed models the classifying spaces of $\mathrm{O}(k)$, $\mathrm{U}(k)$, and $\mathrm{Sp}(k)$ when $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, respectively.

4.42. **Remark** We will argue here that $\mathrm{O}(n)/\mathrm{O}(n-k)$ is $(n-k-1)$-connected. Passing to the colimit over n then shows that $\mathrm{colim}_n\, \mathrm{O}(n)/\mathrm{O}(n-k)$ is ∞-connected, i.e. weakly contractible. To do this, observe that there is a fibre sequence $\mathrm{O}(n-k) \to \mathrm{O}(n) \to \mathrm{O}(n)/\mathrm{O}(n-k)$ and that the first map in this fibre sequence can be written as the composite

$$\mathrm{O}(n-k) \to \mathrm{O}(n-k+1) \to \cdots \to \mathrm{O}(n).$$

Now, for every $l \geq 1$, we have $\mathrm{O}(l)/\mathrm{O}(l-1) \simeq S^{l-1}$. This implies that the first map of the above composite has the lowest connectivity, and this connectivity is given by $(n-k-1)$ since S^{n-k} is $(n-k-1)$-connected as claimed.

Since $\mathrm{U}(l)/\mathrm{U}(l-1) \cong S^{2l-1}$ and $\mathrm{Sp}(l)/\mathrm{Sp}(l-1) \cong S^{4l-1}$, a similar argument applies to show that $\mathrm{colim}_n\, \mathrm{U}(n)/\mathrm{U}(n-k)$ and $\mathrm{colim}_n\, \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$ are weakly contractible.

We finish by proving the existence of contractible principal bundles. We begin with Milnor's construction [Mil56a, Mil56b], since it works for all topological groups and also refer to [Dol63, §8] as well as [tD08, §14.4].

4.43. **Definition** Let I be a set and for all $i \in I$, let X_i be a non-empty topological space. We define a new space, the join $\star_I X_i$ of the X_i's to be a the quotient of the set

$$\{(t_i, x_i) \subseteq ([0,1] \times X)^I \mid \text{ almost all } t_i = 0, x_i \in X_i, \sum_{i \in I} t_i = 1\}$$

where two elements $(t_i, x_i)_{i \in i}$ and $(s_i, y_i)_{i \in I}$ represent the same element of $\star_I X_i$ if $t_i = s_i$ for all $i \in I$ and $x_i = y_i$ whenever t_i (or equivalently s_i) is non-zero. In particular, we have $[0, x_i]_{i \in I} = [0, y_i]_{i \in I}$. For each $j \in I$, we have maps $t_j \colon \star_I X_i \to [0,1]$ induced by the projection to the jth factor of $[0,1] \times X$ and then further projecting to $[0,1]$, as well as a maps $p_j \colon t_j^{-1}(0,1] \to X_j$, sending $(t_i, x_i)_{i \in I}$ to x_j. Define the strong topology on $\star_I X_i$ to be the coarsest (that is the one with fewest open sets) for which the maps t_j and p_j are continuous. Hence, a map $Y \to \star_I X_i$ is continuous if and only if its compositions with t_j and p_j (on the subspace of Y where it is defined) is continuous.

4.44. **Remark** If X_i is T1 for all i, then $\star_I X_i$ is T1 in the strong topology. Indeed, we first note that the map

$$\operatorname*{colim}_{J \subseteq I \text{ finite}} \star_J X_j \to \star_I X_i$$

is a continuous bijection. Unfortunately, I am not sure whether it is a homeomorphism at this point. In particular $\star_I X_i$ also comes with what we shall refer to as the colimit topology, i.e. just the left hand side of the above map.

Regardless of whether the strong and the colimit topologies agree, we see that for any finite subset $J \subseteq I$, the inclusion $\star_J X_j \to \star_I X_i$ is closed in the strong topology: The complement is given by the union of the open sets $t_i^{-1}(0,1]$ where $i \in I \setminus J$.

We will give some perspective on the following result; see [Mil56b] for details and generalisations.

4.45. **Lemma** *Let G be a locally compact Hausdorff topological group. Then $\star_{\mathbb{N}} G$ is weakly contractible in the colimit topology.*[21]

Proof. Note that the inclusion $\star_n G \to \star_{n+1} G$ is closed as its complement is given by $t_{n+1}^{-1}(0,1]$ which is open by definition. As above, we deduce from this that $\star_n G$ is also T1. Furthermore, we claim that the inclusion $\star_n G \to \star_{n+1} G$ is null-homotopic. Indeed, we factor the inclusion as follows:

$$\star_n G = (\star_n G) \star \varnothing \to (\star_n G) \star \{*\} \to (\star_n G) \star G = \star_{n+1} G.$$

It then suffices to observe that for any space X, $X \star \{*\} \cong [0,1] \times X / \{0\} \times X \cong \mathrm{C}(X)$ is contractible.

It now follows that for every compact space Y, the canonical map $\operatorname{colim}_n \mathrm{Hom}_{\mathrm{Top}}(Y, \star_n G) \to \mathrm{Hom}_{\mathrm{Top}}(Y, \star_{\mathbb{N}} G)$ is bijective, and in particular, we deduce that any map $S^k \to \star_{\mathbb{N}} G$ factors through a null-homotopic map and is hence itself null-homotopic, so $\star_{\mathbb{N}} G$ is weakly contractible. $\square$

In fact, Dold shows that $\star_{\mathbb{N}} G$ is contractible in the strong topology, see [Dol63, Proof of Thm. 8.1].

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[21] We should also note that $\star_n G \subseteq \star_{n+1} G$ admits an open thickening, that is, there is an open neighborhood which deformation retracts back to $\star_n G$; this does not use that G is Hausdorff. It is well possible that the same is true for $\star_n G \subseteq \star_{\mathbb{N}} G$, in which case one can conclude that $\star_{\mathbb{N}} G$ is weakly contractible without the Hausdorff assumption.

4.46. **Lemma** *Let $\{X_i\}_{i\in I}$ be a family of spaces equipped with a continuous right G-action. Then there is an induced continuous right G-action on $\star_I X_i$ in the strong topology.*

Proof. Define $(t_i, x_i)\cdot g := (t_i, x_i\cdot g)$, note that it is well-defined. Moreover, the map $\star_I X_i \times G \to \star_I X_i$ is continuous: Its composition with t_j is the projection, and the "composition" with p_j is given by first applying p_j (which is continuous) and then the G-action on X_j (which is also continuous). □

4.47. **Remark** In the above situation, if G is locally compact, then G also acts continuously on $\star_I X_i$ in the colimit topology. Indeed, since G is locally compact, the functor $G \times -$ preserves colimits, so $G \times \star_I X_i$ has the colimit topology along the subspaces $G \times \star_J X_j$ for $J \subseteq I$. Continuity of $G \times \star_I X_i \to \star_I X_i$ therefore follows from the the case of finite I's where the topology is the strong topology and hence the above argument applies.

4.48. **Lemma** *Let G be a topological group. Then the G-action on $\star_{\mathbb{N}} G$ is principal for the strong topology.*

Proof. The functions $t_n\colon \star_{\mathbb{N}} G \to [0,1]$ are G-invariant and hence descend to functions $\bar{t}_n\colon (\star_{\mathbb{N}} G)/G \to [0,1]$. Consider the open subsets $\bar{U}_n = \bar{t}_n^{-1}(0,1] \subseteq (\star_{\mathbb{N}} G)/G$, denote by p the projection $\star_{\mathbb{N}} G \to (\star_{\mathbb{N}} G)/G$, and let $U_n = p^{-1}(\bar{U}_n) = t_n^{-1}(0,1]$, so that $\bar{U}_n = p(U_n)$. Then there is a continuous map

$$(p, p_n)\colon U_n \to \bar{U}_n \times G$$

which we claim is a G-equivariant homeomorphism. To see this, consider the map $\varphi\colon U_n \to U_n$ sending $(t_i, g_i)_{i\geq 0}$ to $(t_i, g_i g_n^{-1})_{i\geq 0}$. It satisfies

$$\varphi[(t_i, g_i)_{i\geq 0}\cdot g] = \varphi(t_i, g_i g)_{i\geq 0} = (t_i, (g_i g)(g_n g)^{-1})_{i\geq 0} = (t_i, g_i g_n^{-1})_{i\geq 0}.$$

Consequently, it induces a continuous map $\bar{\varphi}\colon \bar{U}_n \to U_n$. The continuous map

$$\bar{U}_n \times G \xrightarrow{\bar{\varphi}\times \mathrm{id}} U_n \times G \to U_n$$

whose final map is the G-action is then G-equivariant and an inverse of (p, p_n). Since the open sets $\bar{U}_n$ cover $(\star_{\mathbb{N}} G)/G$, this shows that the G-action on $\star_{\mathbb{N}} G$ is principal as claimed. □

4.49. **Remark** If G is locally compact, so that the G acts continuously on $\star_{\mathbb{N}} G$ in the colimit topology, then this action is again principal. Indeed, there is a continuous bijection

$$[\operatorname*{colim}_{n\geq 0} \star_n G]/G \to (\star_{\mathbb{N}} G)/G$$

so that the subsets $\bar{U}_n$ are also open in the colimit topology, and the restriction of the quotient map to these $\bar{U}_n$ is again trivial by the same argument as above.

Combining Dold's argument and the above shows that $\star_{\mathbb{N}} G$, with strong topology, gives rise to a principal G-bundle with contractible total space, proving Theorem 4.29. Combining our arguments above shows that for G locally compact Hausdorff, $\star_{\mathbb{N}} G$ with the colimit topology, gives rise to a principal G-bundle with weakly contractible total space. This suffices for representability of $\mathrm{Bun}_G(-)$ on CW complexes.

We finally mention another attempt to construct a principal G-bundle with (weakly) contractible total space, and refer to [Seg68] for details. We do so mainly, since in the ∞-categorical world, this construction always provides a delooping of a grouplike monoid in spaces. First, we digress:

4.50. **Definition** A small topological category $\mathcal{C}$ consists of a space of objects $\mathrm{ob}(\mathcal{C})$ and a space of morphisms $\mathrm{mor}(\mathcal{C})$, such that the formation of unit morphisms, as well as source and target, and composition amount to continuous maps

$$\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C}), \quad \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{C}), \text{ and } \mathrm{mor}(\mathcal{C}) \times_{\mathrm{ob}(\mathcal{C})} \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C}).$$

Considering the usual nerve construction, i.e. sending $[n] \in \Delta^{\mathrm{op}}$ to $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ gives a simplicial space $N(\mathcal{C})$.

Now, let G be a topological group and X a topological space equipped with a continuous left action of G.

4.51. **Definition** When G acts continuously on X, we may define a category $G \curvearrowright X$ with

(1) $\mathrm{ob}(G \curvearrowright X) = X$, and
(2) $\mathrm{mor}(G \curvearrowright X) = G \times X$

with identity morphisms given by $(1, x)$, source and target maps are given by $s(g, x) = x$ and $t(g, x) = gx$ and composition given by $(h, gx) \circ (g, x) = (hg, x)$. Its nerve $N(G \curvearrowright X)$ is then a simplicial space, functorial in the G-space X.

4.52. **Remark** In particular, if a left G-space is acted upon continuously and compatibly from the right by H, then $N(G \curvearrowright X)$ is equipped with a continuous H-action from the right. This is the case for $X = G$ and $H = G$, where G acts both from the left and the right on itself by left and right multiplication, respectively.

4.53. **Remark** The formula for geometric realizations of simplicial sets makes sense verbatim for simplicial spaces. This defines a functor $|-| \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Top}) \to \mathrm{Top}$ which is left adjoint to functor sending X to the cosimplicial space $\mathrm{Map}(\Delta^{\bullet}_{\mathrm{top}}, X)$. Likewise, the underlying semi-simplicial space has a geometric realization (by forgetting all non-injective morphisms in Δ).

4.54. **Corollary** *The geometric realization $|N(G \curvearrowright G)|$ admits a continuous G-action from the right, with quotient given by $|N(G \curvearrowright *)|$. Moreover, $|N(G \curvearrowright G)|$ is contractible.*

Proof. The first statement is a direct check. The second follows since the realization commutes with quotients, as does the formation of the category out of a G-space, so the claim is reduced to the obvious fact that $G/G = *$. To see that $|N(G \curvearrowright G)|$ is contractible, it is enough to find a continuous natural transformation between the identity and the constant functor on the topological category $G \curvearrowright G$. Exercise: Show that the morphisms $(g, 1)$ provide such a continuous natural transformation. $\square$

We may therefore try to define $EG = |N(G \curvearrowright G)|$ and $BG = |N(G \curvearrowright *)|$. What remains for this to be a principal G-bundle is the the projection map $EG \to BG$ needs to be locally trivial. This is, however, not always the case, but it is under suitable hypothesis on G, for instance if the underlying topological space of G is an ANR (absolute neighborhood retract), examples of which are discrete groups or Lie groups, again we refer to [Seg68]. Moreover, there is a canonical map $(\star_{\mathbb{N}} G)/G \to |N(G \curvearrowright *)|$ which is a quotient map; it collapses degenerate simplices. Finally, $(\star_{\mathbb{N}} G)/G$ can also be described via simplicial spaces as Segal points out.

We finish this section with the construction of *homotopy orbits* which we will mainly (but not exclusively) use in the case of discrete groups.

4.55. **Definition** Let X be a left G-space. We define the *homotopy orbits* X_{hG} of X to be the associated bundle $EG \times_G X$. In particular, we find that there is a fibration sequence $X \to X_{hG} \to BG$ and $BG = *_{hG}$.

4.56. **Remark** The weak homotopy type of X_{hG} is independent of the choice of EG: Given $EG \to BG$ and $\tilde{EG} \to \tilde{BG}$ be two choices. Then EG and $\tilde{EG}$ are G-equivariantly homotopy equivalent. Such a homotopy equivalence induces a homotopy equivalence $EG \times_G X \to \tilde{EG} \times_G X$, showing that X_{hG}, up to homotopy equivalence, does not depend on the choice of EG.

4.57. **Example** Suppose X is a space with trivial G-action. Then $X_{hG} = X \times BG$. On the other hand, suppose that X has a principal G-action. Then we may view $X_{hG} = EG \times_G X \to X/G$ is a fibre bundle with contractible fibre and hence a homotopy equivalence. In particular, X_{hG} and X/G are homotopy equivalent in the case of principal actions (i.e. free actions in case G is a finite group).

4.58. **Lemma** *The association $X \mapsto X_{hG}$ refines to a functor $G\mathrm{Top} \to \mathrm{Top}$ and it sends weak equivalences to weak equivalences.*

Proof. Let $X \to Y$ be G-equivariant. Then we obtain an induced map $EG \times_G X \to EG \times_G Y$ which gives the necessary functoriality. Observe then that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_{hG} \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ Y \arrow[r] & Y_{hG} \arrow[r] & BG \end{tikzcd}$$

The claim about weak equivalences then follows from the 5-lemma (plus explicit low degree considerations). □

4.59. **Remark** Let G be a discrete group acting on a space X. The projection map $EG \times X \to X$ is then G-equivariant and a homotopy equivalence, and G acts covering-like on the source. Hence, for finite groups acting on X, the formation of homotopy orbits amounts to replacing, up to homotopy equivalence, the action by a free action, and then simply taking strict orbits. In particular, if N is a normal subgroup of G, and the action of G on X is free, the restricted action of N on X is also free, and we have that $X/N \to X/G$ is a G/N-Galois covering. In particular, $X_{hG} \simeq X/G = (X/N)/(G/N) \simeq (X/N)_{hG/N} \simeq (X_{hN})_{hG/N}$.

5. Obstruction theory

The purpose of this section is to treat the following question: Suppose given the solid part of the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "k"] \arrow[ur, dashed] & Y \end{tikzcd} \tag{3}$$

which commutes up to specified homotopy $h\colon A \times [0,1] \to Y$ (this specified homotopy may well be constant if the diagram strictly commutes). We may then wonder whether a dashed arrow exists, making both small triangles commute up to homotopies $h_0\colon A \times [0,1] \to X$ witnessing

$g \simeq \varphi i$, and $h_1 \colon B \times [0,1] \to Y$, witnessing $f\varphi \simeq k$, so that the combined homotopy witnessing $fg \simeq f\varphi i \simeq ki$ is homotopic to the given homotopy h. We call such a lift a *lift up to compatible homotopy.*

5.1. **Lemma** *Suppose that the solid part of* (3) *commutes with constant homotopy witnessing the strict commutativity. If $i \colon A \to B$ is a relative CW complex and $f \colon X \to Y$ a Serre fibration, and one can lift up to compatible homotopy, one can also find a lift making both triangles commute strictly.*

Proof. Let $h_0 \colon A \times [0,1] \to X$ be a homotopy from φi to g, $h_1 \colon B \times [0,1] \to Y$ a homotopy from $f\varphi$ to k, and $H \colon A \times [0,1] \times [0,1] \to Y$ a homotopy from the combined homotopy $h_1 i \star f h_0$ to the constant homotopy $A \times [0,1] \to A \xrightarrow{fg=ik} Y$. First, we consider the extension problem

$$\begin{tikzcd}
B \times \{0\} \cup A \times [0,1] \arrow[r, "\varphi \cup h_0"] \arrow[d] & X \\
B \times [0,1] \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

which can be solved since i is a cofibration. Then h is a homotopy from φ to a map φ' which makes the upper triangle commute strictly. Moreover, the combined homotopy $h_1 \star fh \colon B \times [0,1]$ is a homotopy between $f\varphi'$ and k. Restricting this homotopy along $A \times [0,1] \to B \times [0,1]$ yields $h_1 i \star fhi = h_1 i \star fh_0$ so that we obtain an extension problem

$$\begin{tikzcd}
A \times [0,1] \times [0,1] \cup B \times [0,1] \times \{0\} \arrow[r, "H \cup h_1 \star fh"] \arrow[d] & Y \\
B \times [0,1] \times [0,1] \arrow[ur, dashed, "H'"'] &
\end{tikzcd}$$

for which a dashed arrow H' can be found again since $A \to B$ is a cofibration.

The map $H'(-,-,1)$ is then a map satisfying the following properties: restricted to $A \times [0,1]$ it given by $A \times [0,1] \to A \xrightarrow{fg=ik} Y$, and it is a homotopy from $h(-,0,1) =: \hat{\varphi}$ to $h(-,0,0) = \varphi'$. Therefore, we may replace φ' by $\hat{\varphi}$ and have then achieved that the upper triangle commutes strictly, and the lower triangle commutes up to a homotopy which is constant on $A \times [0,1]$. Finally, we can then consider the lifting problem

$$\begin{tikzcd}
B \times \{0\} \cup A \times [0,1] \arrow[r, "\varphi \cup g\mathrm{pr}"] \arrow[d] & X \arrow[d, "f"] \\
B \times [0,1] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

which can be solved since $A \to B$ is a relative CW complex and f is a Serre fibration, see [Win24, Lemma 5.2.5] $\square$

Obstruction theory, as we will prove in this section, says that this question can be iteratively treated using cohomological invariants. To set the stage we begin with Moore–Postnikov factorizations of maps.

5.2. **Definition** Let $f\colon X \to Y$ be a map between path connected spaces and $n \geq 1$. The nth Moore–Postnikov stage of f consists of a space Z_n together with maps $a_n\colon X \to Z_n$ and $p_n\colon Z_n \to Y$ such that

(1) $p_n a_n = f$,
(2) a_n is an isomorphism on π_k for $k < n$ and surjective on π_n, and
(3) p_n is an isomorphism on π_k for $k > n$ and injective on π_n,

If need be, we may factor the map $p_n\colon Z_n \to Y$ as a homotopy equivalence followed by a fibration, so we may assume that p_n is a fibration.

5.3. **Lemma** *Let $f\colon X \to Y$ be a map between path connected spaces and $n \geq 1$. Then an nth Moore–Postnikov stage Z_n of f exists. Moreover, there are fibrations $q_n\colon Z_n \to Z_{n-1}$ with $q_n a_n = a_{n-1}$ and such that the induced map $X \to \lim_n Z_n$ is a weak equivalence.*

Proof. Fix $n \geq 1$ and let $K = \ker(\pi_n(f)) \subseteq \pi_n(X)$. Choose generators of K, say $\{\alpha_i\}_{i \in I}$. Consider the pushout

$$\begin{array}{ccc} \coprod_I S^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_I D^{n+1} & \longrightarrow & X' \end{array}$$

so that we obtain a factorization of f through X' and induces on π_n the canonical epi-mono factorization as the surjection onto the image and the inclusion of the image. Then pick generators of $\pi_{n+1}(Y)$ and consider the map $X' \vee \bigvee S^{n+1} \to Y$ by extending the map $X' \to Y$ tautologically on the sphere summands. The resulting map is then surjective on π_{n+1}. Again considering generators of the kernel of this map, we may attach $n+2$-cells to make the map injective on π_{n+1}. Continuing in this fashion, we build a factorization $X \to Z_n \to Y$ as claimed, here we use cellular approximation to ensure that in the iterative process, we never change the low homotopy groups. As indicated above, upon replacing Z_n with a homotopy equivalent space, we may assume that $Z_n \to Y$ is a fibration.

Now let Z_1 be the connected covering of Y[22] with characteristic subgroup $\mathrm{Im}(\pi_1(f)) \subseteq \pi_1(Y)$. By covering theory, there exists a factorization $X \xrightarrow{a_1} Z_1 \xrightarrow{q_1} Y$ with all required properties. Consider then the map $a_1\colon X \to Z_1$. Using the above argument, we may factor it as the composite $X \xrightarrow{a_2} Z_2 \xrightarrow{q_2} Z_1$ and we define $p_2 = q_1 q_2$. Then $p_2 a_2 = q_1 q_2 a_2 = q_1 a_1 = f$ so point (1) of Definition 5.2 is valid, as is point (2), by construction. It remains to verify part (3), i.e. that p_2 induces an isomorphism on π_k for $k > 2$ and an injection on π_2. By construction, we know that q_2 has these properties. Moreover, q_1 induces an isomorphism on π_k for $k \geq 2$, so the result follows. We may then construct Z_3 by appropriately factoring $a_2\colon X \to Z_2$ and continue like this iteratively.

Now, considering the inverse limit $Z = \lim_n Z_n$ along the fibrations $Z_{n+1} \to Z_n$ we observe that the induced diagram on π_k eventually becomes one consisting of isomorphisms. This implies that the composite

$$\pi_k(X) \to \pi_k(Z) \to \lim_n \pi_k(Z_n)$$

is an isomorphism. We now show that also the second map is an isomorphism, this implies the remaining claim. We prove by hand that the map $\pi_k(Z) \to \lim_n \pi_k(Z_n)$ is surjective and injective. For surjectivity, take an element in the codomain, given by a compatible family of

[22] We may assume that Y is so nice that covering theory applies.

elements $[\alpha_n] \in \pi_k(Z_n)$. Concretely, compatibility means that $q_n\alpha_n$ is homotopic to α_{n-1}. Inductively, let us assume that we have $q_n\alpha_n = \alpha_{n-1}$. Then we consider the lifting problem

$$\begin{tikzcd} S^k \times \{0\} \cup \{*\} \times [0,1] \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ S^k \times [0,1] \arrow[r] \arrow[ur, dashed, "H"] & X_n \end{tikzcd}$$

where the bottom horizontal map witnesses that $q_{n+1}\alpha_{n+1}$ is (pointed) homotopic to α_n. Since the right hand vertical map is a fibration, this lifting problem can be solved. Denote by α'_{n+1} the map $H(-,1)$. Then $[\alpha'_{n+1}] = [\alpha_{n+1}] \in \pi_k(Z_n)$ and $q_{n+1}\alpha'_{n+1} = \alpha_n$. It follows that we may represent the compatible family $[\alpha_n]$ to satisfy $q_n\alpha_n = \alpha_{n-1}$ for all n. In particular, the maps α_n give rise to a map $\alpha\colon S^k \to \lim_n Z_n$ representing a lift of the compatible family $[\alpha_n]_{n\geq 0}$ along the map $\pi_k(\lim_n Z_n) \to \lim_n \pi_k(Z_n)$.

To prove injectivity, consider a map $\alpha\colon S^k \to \lim_n Z_n$ representing an element in the kernel of the map $\pi_k(Z) \to \lim_n \pi_k(Z_n)$. Then for all $n \geq 0$, the induced map $\alpha_n\colon S^k \to Z_n$ is (pointed) nullhomotopic. Again, we want to construct an extension of α over D^{k+1}. To do so, consider (again inductively) the lifting problem

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & Z_{n+1} \arrow[d] \\ D^{k+1} \arrow[r] \arrow[ur, dashed] & Z_n \end{tikzcd}$$

which can be solved for n sufficiently large (recall that the fibre of the right vertical map is given by $K(\pi_n(F), n)$), again using [Win24, Prop. 5.2.15]. $\square$

We depict a Moore–Postnikov factorization as follows

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & Z_n \arrow[d] \arrow[dddr] & \\ & Z_{n-1} \arrow[d] \arrow[ddr] & \\ & \vdots \arrow[d] \arrow[dr] & \\ X \arrow[r] \arrow[uuur] \arrow[uur] \arrow[ur] & Z_1 \arrow[r] & Y \end{tikzcd}$$

and therefore also call it a Moore–Postnikov tower.

5.4. **Remark** In this remark, we record some things about Milnor's lim-lim^1-sequence which provides a general formula for the homotopy groups of the inverse limit of a tower (of fibrations). So assume for the moment given a tower

$$\cdots \to X_n \xrightarrow{f_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{f_1} X_0$$

of topological spaces. Recall that the limit of this diagram in Top can be described as the pullback of the left hand square

$$
\begin{tikzcd}
\lim\limits_{n\geq 0} X_n \arrow[r] \arrow[d] & \prod\limits_{n\geq 0} X_n \arrow[d, "\Delta"] \\
\prod\limits_{n\geq 0} X_n \arrow[r] & \prod\limits_{n\geq 0} X_n \times X_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname*{holim}\limits_{n\geq 0} X_n \arrow[r] \arrow[d] & \prod\limits_{n\geq 0} \mathrm{Map}([0,1], X_n) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\prod\limits_{n\geq 0} X_n \arrow[r] & \prod\limits_{n\geq 0} X_n \times X_n
\end{tikzcd}
$$

whereas what is called the homotopy limit is given by the pullback of the right hand square above. In both diagrams the lower horizontal map is given by sending a sequence $(x_n)_{n\geq 0}$ of points in X_n's to the sequence $(x_n, f_{n+1}(x_{n+1}))$. Now, if all maps $f_n\colon X_n \to X_{n-1}$ are fibrations, one can show that the map $\lim_n X_n \to \operatorname{holim}_n X_n$ induced by the constant path maps $X_n \to \mathrm{Map}([0,1], X_n)$ is a weak equivalence, so to compute homotopy groups of the limit, we may as well compute homotopy groups of the homotopy limit. Then we observe that the right vertical map in the right above square is a fibration whose fibre is $\prod_n \Omega X_n$. Therefore, there is a fibration sequence

$$\prod_{n\geq 0} \Omega X_n \to \operatorname*{holim}_{n\geq 0} X_n \to \prod_{n\geq 0} X_n$$

and consequently a long exact sequence in homotopy groups

$$\cdots \prod_{n\geq 0} \pi_{k+1}(X_n) \xrightarrow{\partial} \prod_{n\geq 0} \pi_k(\Omega X_n) \to \pi_k(\operatorname*{holim}_{n\geq 0} X_n) \to \prod_{n\geq 0} \pi_k(X_n) \xrightarrow{\partial} \prod_{n\geq 0} \pi_{k-1}(\Omega X_n) \to \dots$$

Under the isomorphisms $\pi_l(\Omega X_n) \cong \pi_{l+1}(X_n)$ the maps ∂ in this sequence are given as follows: They are induced by the pairs $(\mathrm{id}, -(f_n)_*)$. It follows that for $k \geq 2$, there is a short exact sequence (this is in fact one way to define $\lim^1$)

$$0 \to \lim_{n\geq 0}{}^1 \pi_{k+1}(X_n) \to \pi_k(\operatorname*{holim}_{n\geq 0} X_n) \to \lim_{n\geq 0} \pi_k(X_n) \to 0$$

whereas for low degree homotopy, the notation $\lim^1$ is perhaps not quite appropriate.

If for fixed k, there exists an N such that for all $n \geq N$, we have that $\pi_k(X_n) \to \pi_k(X_{n-1})$ is surjective (e.g. an isomorphism), it is not too difficult to show that the term we denoted by $\lim^1$ vanishes; this then recovers our previous result that $X \to \lim_n Z_n$ is a weak equivalence whenever $\{Z_n\}$ is a Moore–Postnikov tower for a map $X \to Y$.

5.5. **Remark** Applied to the map $X \to *$, a Moore–Postnikov tower is simply called a Postnikov tower for X and we write suggestively $X_{\leq n}$ for Z_{n+1}: Indeed, Z_{n+1} has the same homotopy groups as X in degrees $\leq n$ and trivial homotopy groups above degree n. Conversely, applied to the map $* \to X$, a Moore–Postnikov tower is called a Whitehead tower. We write $X_{\geq n+1}$ for Z_n: Indeed, Z_n has trivial homotopy groups in degrees $\leq n$ and the same homotopy groups as X above degree n.

5.6. **Remark** Consider a pullback diagram with f a fibration.

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}
$$

Let $\{Z_n\}$ be a Moore–Postnikov tower for f. Then $\{Y' \times_Y Z_n\}$ forms a Moore–Postnikov tower for f' (Exercise). In particular, applying this to $Y' = *$ in which case $X' = F$ is the homotopy fibre of the map f, we obtain the following (homotopy) pullback diagrams:

$$\begin{tikzcd} K(\pi_n(F), n) \arrow[r] \arrow[d] & F_{\leq n} \arrow[r] \arrow[d] & Z_{n+1} \arrow[d] \\ * \arrow[r] & F_{\leq n-1} \arrow[r] \arrow[d] & Z_n \arrow[d] \\ & * \arrow[r] & Y \end{tikzcd}$$

In particular, the (homotopy) fibres of the maps $Z_{n+1} \to Z_n$ are Eilenberg–Mac Lane spaces of type $(\pi_n(F), n)$.

5.7. **Remark** Moore–Postnikov towers are unique up to weak equivalence. Indeed, assume that $\{Z_n\}_{n\geq 1}$ and $\{Z'_n\}_{n\geq 1}$ are two Moore–Postnikov towers for a map $f\colon X \to Y$. As indicated above, we may assume that the maps $Z'_n \to Z'_{n-1}$ are fibrations for all $n \geq 1$ where we interpret Z'_0 as Y. Similarly, we may assume that the maps $X \to Z_n$ are cofibrations for all $n \geq 1$. Moreover, since $X \to Z_n$ induces an isomorphism on π_k for $k < n$ and a surjection for $k = n$, we may assume that (Z_n, X) is a CW pair with relative cells of dimension $\geq n+1$. We will now inductively show that there are maps $\varphi_n\colon Z_n \to Z'_n$ making all diagrams commute. To do so, we begin with the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z'_1 \arrow[d] \\ Z_1 \arrow[r] \arrow[ur, dashed, "\varphi_1"] & Y \end{tikzcd}$$

We claim that a dashed arrow exists, making both triangles commute. Indeed, as mentioned above, we are in the situation that the relative cells of (Z_1, X) are in dimensions ≥ 2, so in order to apply [Win24, Prop. 5.2.15], we need to know that $\pi_k(\mathrm{hofib}(Z'_1 \to Y))$ vanishes for all $k \geq 2$ and by definition of Moore–Postnikov sections, we even have that these homotopy groups vanish whenever $k \geq 1$.[23] Now let us inductively assume that we have constructed a map $\varphi_{n-1}\colon Z_{n-1} \to Z'_{n-1}$ participating in a commutative diagram with $\varphi_{n-2}\colon Z_{n-2} \to Z'_{n-2}$. Then we consider the lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z'_n \arrow[d] \\ Z_n \arrow[r] \arrow[ur, dashed] & Z'_{n-1} \end{tikzcd}$$

whose lower horizontal map is the composite $Z_n \to Z_{n-1} \to Z'_{n-1}$. Again we want to appeal to [Win24, Prop. 5.2.15] to find a dashed arrow making both triangles commute, and again this amounts to showing that the homotopy fibre of $Z'_n \to Z'_{n-1}$ has trivial homotopy groups in degrees $\geq n+1$. By Remark 5.6 this homotopy fibre is given by $K(\pi_{n-1}(F), n-1)$, where $F = \mathrm{hofib}(f\colon X \to Y)$. We have therefore constructed a map of towers $\varphi\colon \{Z_n\}_{n\geq 1} \to \{Z'_n\}_{n\geq 1}$ and it remains to argue that each of the maps $\varphi_n\colon Z_n \to Z'_n$ is a weak equivalence.

[23]This implies that the dashed arrow is in fact unique up to homotopy.

To prove this, we note that, by construction, these maps participate in the left of the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z_n' \arrow[d] \\ Z_n \arrow[r] \arrow[ur] & Y \end{tikzcd} \qquad \begin{tikzcd} \pi_k(X) \arrow[r] \arrow[d] & \pi_k(Z_n') \arrow[d] \\ \pi_k(Z_n) \arrow[r] \arrow[ur] & \pi_k(Y) \end{tikzcd}$$

Applying π_k to this diagram we now observe the following: If $k < n$, then both maps eminating from $\pi_k(X)$ are isomorphisms, so we deduce that $\pi_k(\varphi)$ is an isomorphism. If $k > n$, then both maps with target $\pi_k(Y)$ are isomorphisms, so again we deduce that $\pi_k(\varphi)$ is an isomorphism. Finally, for $k = n$, we find that $\pi_k(\varphi)$ is surjective because the top horizontal map is surjective, and $\pi_k(\varphi_n)$ is injective because the bottom horizontal map is injective.

In the situation of the beginning of this section, when considering a lifting problem as in (3)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $A \to B$ an inclusion of a sub CW complex and $X \to Y$ a fibration,[24] we may consider the Moore–Postnikov factorization of f and deduce that if this lifting problem can be solved, also the lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Z_{n+1} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

can be solved. Consersely, if we can inductively solve the lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Z_{n+1} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Z_n \end{tikzcd}$$

for all $n \geq 0$, then we can also solve the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \lim_n Z \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Using that A and B are CW complexes and that $X \to \lim_n Z_n$ is a weak equivalence, it then follows that the original lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

[24]We will from now on restrict to these cases.

can be solved up to compatible homotopy (exercise) and hence can be solved according to Lemma 5.1. We will therefore be interested in solving the lifting problems

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & Z_{n+1} \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & Z_n
\end{tikzcd}
\tag{4}
$$

inductively.

5.8. **Example** Let us discuss the first step in the above inductive approach, i.e. the lifting problem

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"] & Z_1 \arrow[d, "p_1"] \\
B \arrow[r, "k"] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

in some detail. In this case, we have arranged the right vertical map is a covering map. We claim that this lifting problem admits a solution if and only if for all basepoints $b \in B$, there exists a point $z \in p_1^{-1}(k(b))$ such that the diagram of fundamental groups

$$
\begin{tikzcd}
 & \pi_1(Z_1, z) \arrow[d] \\
\pi_1(B, b) \arrow[r] \arrow[ur, dashed] & \pi_1(Y, k(b))
\end{tikzcd}
$$

admits a solution (since the right vertical map is an injection, this is really just a property of the given situation, the lift itself either does not exist, or it exists uniquely). The only if part is clear (apply $\pi_1(-)$ to a solution of the lifting problem) and the if part follows from covering theory: Indeed, under the above assumption on $\pi_1(-)$, we obtain a map $B \to Z_1$ making the lower right triangle commute. Then upper triangle then commutes again by covering theory and the fact that the square commutes.

As a conclusion, the first step in our inductive approach is completely determined by studying the induced map on fundamental groups. In all lifting problems that arise next, we will aim to characterize solvability by means of cohomological invariants. In the following special case of the above situation, this is also the case here. Namely, let us first assume that $Z_1 \to Y$ is a Galois covering, that is, that the image of the map $\pi_1(X) \to \pi_1(Y)$ is a normal subgroup. Denoting the quotient group $\pi_1(Y)/\mathrm{Im}(\pi_1(f))$ by G, we obtain that $Z_1 \to Y$ is a G-Galois covering, or in other words, a principal G-bundle. Consequently, there is a map $Y \to BG$ and a pullback diagram

$$
\begin{tikzcd}
Z_1 \arrow[r] \arrow[d] & EG \arrow[d] \\
Y \arrow[r] & BG
\end{tikzcd}
$$

We deduce that the map $B \to Y$ lifts along $Z_1 \to Y$ if and only if the composite

$$B \to Y \to BG$$

is (pointed) null-homotopic. Note that we have argued earlier that the map induced by $\pi_1(-)$

$$[B, BG]_* \to \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(B), G)$$

is a bijection. Now if we in addition assume that G is abelian, then we find that the composite $B \to Y \to BG$ classifies an element θ in $H^1(B;G)$ by representability of cohomology using that BG is a $K(G,1)$. This element θ then vanishes if and only if there exists a lift of $B \to Y$ along $Z_1 \to Y$. Note also that the image of θ in $H^1(A;G)$ vanishes since the map $A \to B \to Y$ does lift along $Z_1 \to Y$. This implies that there exists a refined element $\theta' \in H^1(B/A;G)$ whose image under the map $H^1(B/A;G) \to H^1(B;G)$ is θ. Now, above we have argued that it suffices to find any lift $B \to Z_1$ making the triangle with Y commute; the triangle with A will commute automatically by covering theory. The cohomological interpretation of this result is the following: In general, finding a lift of $B \to Y$ along $Z_1 \to Y$ amounts to the condition that $0 = \theta \in H^1(B;G)$. Finding a lift that also makes the upper triangle commute will amount to the condition the refined element θ' vanishes in $H^1(B/A;G)$. Considering the long exact sequence

$$H^0(B;G) \to H^0(A;G) \to H^1(B/A;G) \to H^1(B;G)$$

and the fact the first map in this sequence is surjective (exercise!) we find that the last map is injective. In particular, θ' vanishes if and only if θ vanishes as needed.

It remains to understand when the lifting problem (4) can be solved for $n \geq 1$. To study this case, we will use the following theorem. It is an analog of the classification theorem of principal G-bundles in the context of fibrations, and is (for instance) a consequence of the straightening-unstraightening equivalence we have discussed in the seminar and is a vast refinement of what we have discussed in Lemma 2.28.

5.9. **Theorem** (Classification of fibrations) *Let $E \to B$ be a fibration with B connected and typical fibre a locally finite connected CW complex F. Then there is a homotopy pullback diagram*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{BhAut}_*(F) \arrow[d] \\ B \arrow[r] & \mathrm{BhAut}(F) \end{tikzcd}$$

Here, we agree on the convention that the right hand vertical map in the above diagram is a fibration. That the square is a homotopy pullback diagram then means that the diagram commutes and the induced map on vertical (homotopy) fibres is a weak equivalence. To see that this is reasonable, let us convince ourselves that the right hand vertical (homotopy) fibre indeed ought to be equivalent to F. To that end, note that the right hand vertical map is a delooping of the map of "groups up to homotopy" $\mathrm{hAut}_*(F) \to \mathrm{hAut}(F)$. Recall that this map sits in a fibration sequence $\mathrm{hAut}_*(F) \to \mathrm{hAut}(F) \to F$ by evaluating a homotopy equivalence on a basepoint of F. As in principal G-bundles, this is in fact induced by a homotopy fibration sequence $F \to \mathrm{BhAut}_*(F) \to \mathrm{BhAut}(F)$, so that the homotopy fibres indeed match up.

Now, we recall that the (homotopy) fibre of the fibration $Z_{n+1} \to Z_n$ is a $K(\pi_n(F), n)$, where F is the (homotopy) fibre of the map $f\colon X \to Y$. Therefore we obtain the following.

5.10. **Corollary** *Let $X \to Y$ be a map with Moore–Postnikov tower $\{Z_n\}_n$. Then for all $n \geq 1$, there are homotopy pullback diagrams*

$$\begin{tikzcd} Z_{n+1} \arrow[r] \arrow[d] & \mathrm{BhAut}_*(K(\pi_n(F), n)) \arrow[d] \\ Z_n \arrow[r] & \mathrm{BhAut}(K(\pi_n(F), n)) \end{tikzcd}$$

5.11. **Lemma** *For every abelian group A and $n \geq 1$, there is a homotopy fibration sequence*

$$K(A, n+1) \to \mathrm{BhAut}(K(A, n)) \to B\mathrm{Aut}(A).$$

Proof. We have calculated the homotopy groups of $\mathrm{hAut}(K(A, n))$ implicitly in Corollary 3.15, they are given by $\mathrm{Aut}(A)$ in degree 0 and A in degree n. Hence, for $\mathrm{BhAut}(K(A, n))$ we have π_1 equal to $\mathrm{Aut}(A)$ and π_{n+1} equal to A. Therefore, the homotopy fibre of the canonical map $\mathrm{BhAut}(K(A, n)) \to B\mathrm{Aut}(A)$ is given by $K(A, n+1)$ as claimed. □

5.12. **Corollary** *For every abelian group A and $n \geq 1$, the diagram*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathrm{BhAut}_*(K(A, n)) \arrow[d] \\ K(A, n+1) \arrow[r] & \mathrm{BhAut}(K(A, n)) \end{tikzcd}$$

is a homotopy pullback diagram.

Proof. Recall that we have already argued that the right hand vertical (homotopy) fibre is given by $K(A, n)$. Therefore, on vertical homotopy fibres, the above diagram induces a map $\Omega K(A, n+1) \to K(A, n)$ which we need to show is a weak equivalence. Equivalently, we need to show that it induces an isomorphism on π_n. To do so, consider the commutative diagram

$$\begin{tikzcd} \pi_{n+1}(K(A, n+1)) \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_{n+1}(\mathrm{BhAut}(K(A, n))) \arrow[d] \\ \pi_n(\Omega K(A, n)) \arrow[r] & \pi_n K(A, n) \end{tikzcd}$$

arising from the boundary maps (vertically) in both vertical long exact homotopy sequences. By Lemma 5.11, the top horizontal map is an isomorphism since $n \geq 1$ and $B\mathrm{Aut}(A)$ only has π_1 non-trivial. It the remains to argue that the right vertical map is also an isomorphism. It is injective since $n \geq 1$ and $B\mathrm{Aut}(A)$ therefore does not have π_{n+1}, and it is surjective because A is abelian, so that in the extreme case $n = 1$, the map $A \to \mathrm{Aut}(A)$, which is given by sending an element to its conjugation endomorphism, is trivial as A is abelian. □

Now, the map $Z_n \to \mathrm{BhAut}(K(\pi_n(F), n))$ factors up to homotopy through as a composite

$$Z_n \to K(\pi_n(F), n+1) \to \mathrm{BhAut}(K(\pi_n(F), n))$$

if and only if the the composite

$$Z_n \to \mathrm{BhAut}(K(\pi_n(F), n)) \to B\mathrm{Aut}(\pi_n(F))$$

is trivial. This is the case if and only if the induced map $\pi_1(Z_n) \to \mathrm{Aut}(\pi_n(F))$ is trivial. Since $X \to Z_n$ is π_1-surjective, this amounts to asking that the $\pi_1(X)$-action on $\pi_n(F)$ discussed in Remark 2.29 is trivial for all $n \geq 1$. We therefore make the following definition.

5.13. **Definition** A fibration $f\colon X \to Y$ between path connected spaces is called *simple*, resp. *nilpotent*, if the $\pi_1(X)$-action on $\pi_n(F)$ is trivial, resp. nilpotent, for all $n \geq 1$.[25]

We arrive at the following corollary:

5.14. **Corollary** *Let $f\colon X \to Y$ be a simple fibration between path connected spaces. Then for all $n \geq 1$, the stages in the Moore–Postnikov tower appear in homotopy pullback diagrams as follows:*

$$\begin{tikzcd} Z_{n+1} \arrow[r] \arrow[d] & * \arrow[d] \\ Z_n \arrow[r, "\theta_n"] & K(\pi_n(F), n+1) \end{tikzcd}$$

The maps θ_n are called the *obstruction classes* for the map $X \to Y$.

Proof. By assumption, there is a map $Z_n \to K(\pi_n(F), n+1)$ such that the composite

$$Z_n \to K(\pi_n(F), n+1) \to \mathrm{BhAut}(K(\pi_n(F), n))$$

is given by the lower horizontal map appearing in Corollary 5.10. To compute the homotopy pullback along the right vertical map in Corollary 5.10, we may use Corollary 5.14 and arrive at the claimed statement. □

5.15. **Remark** Suppose more generally that the $\pi_1(X)$-action on $\pi_n(F)$ is nilpotent. Pick a $\pi_1(X)$-equivariant central series on $\pi_n(F)$ for all $n \geq 1$ witnessing the nilpotency of the action. Then one can construct a refined Moore–Postnikov tower $\{Z_{n,k}\}$ where the map $Z_{n+1} \to Z_n$ is again factored as a sequence of maps $Z_{n+1} \to Z_{n,0} \to Z_{n,1} \to \cdots \to Z_n$ such that the fibres of the map $Z_{n,k} \to Z_{n,k+1}$ are Eilenberg–Mac Lane spaces of type (A, n) with A being the associated graded of the chosen central series on $\pi_n(F)$. In particular, the $\pi_1(X)$-action on A is trivial, and one arrives at a similar conclusion as in Corollary 5.14 but the groups $\pi_n(F)$ have to be replaced by the associated graded of a chosen central series and there are now (possibly) several obstruction classes in a single cohomological degree.

5.16. **Remark** The obstruction classes $\theta_n\colon Z_n \to K(\pi_n(F), n+1)$ are not a priori unique (up to homotopy). Indeed, any two null-homotopies of the composite $Z_n \to B\mathrm{Aut}(\pi_n(F))$ differ by a map $Z_n \to \mathrm{Aut}(\pi_n(F))$ via the action of $[Z_n, \Omega B\mathrm{Aut}(\pi_n(F))]$ on $[Z_n, K(\pi_n(F), n+1)]$. This action is, however, trivial: This is because the map $\Omega\mathrm{BhAut}(K(\pi_n(F), n)) \to B\mathrm{Aut}(\pi_n(F))$ admits a section; see the similar discussion in Remark 2.30.

In conclusion, if the $\pi_1(X)$-action on $\pi_n(F)$ is trivial (or nilpotent) then the obstruction classes $\theta_n \in [Z_n, K(\pi_n(F), n+1)]$ are independent of any further choices.

We come back to our main situation at hand and consider the commutative diagram as in (4). We continue to assume that A and B are CW complexes. By assumption, we may expand it as follows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & Z_{n+1} \arrow[r] \arrow[d] & * \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted, "\varphi"] & Z_n \arrow[r, "\theta_n"] & K(\pi_n(F), n+1) \end{tikzcd}$$

[25] Recall that this implies that F is simple, resp. nilpotent, since the restricted $\pi_1(F)$-action on $\pi_n(F)$ is the one appearing in the definition of simple and nilpotent spaces.

where the right square is a homotopy pullback diagram. That the outer large diagram commutes up to specified homotopy implies that the maps $\varphi^*(\theta_n)$ factors through $\mathrm{C}(i)$, that is, there is a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & Z_n \arrow[d, "\theta_n"] \\ \mathrm{C}(i) \arrow[r, "\varphi_A^*(\theta_n)"] & K(\pi_n(F), n+1) \end{tikzcd}$$

Now $\mathrm{C}(i)$ is again a CW complex, so we may view the elements $\varphi_A^*(\theta_n)$ as elements $H^{n+1}(\mathrm{C}(i); \pi_n(F))$. If i is an inclusion of a subcomplex, then $\mathrm{C}(i) \simeq B/A$. We conclude the main theorem of obstruction theory.

5.17. **Theorem** *Let $f\colon X \to Y$ be a simple map*[26] *between path connected spaces and consider a lifting problem as in* (3) *with i an inclusion of a subcomplex and f a fibration. Then, for $n \geq 1$, there are inductively defined obstruction classes $\varphi_A^*(\theta_n) \in H^{n+1}(B/A; \pi_n(F))$, with the property that*

(1) *$\varphi_A^*(\theta_1)$ is defined provided a solution on the level of π_1 exists.*
(2) *if $\varphi_A^*(\theta_n)$ is defined and vanishes, then $\varphi_A^*(\theta_{n+1})$ is defined.*
(3) *if $\varphi_A^*(\theta_n)$ is defined and vanishes for all $n \geq 1$, then the lifting problem has a solution.*

Proof. We have already argued (1) and (3), so it remains to prove (2). For this, we need to argue that finding lifts up to compatible homotopy in a (homotopy commutative) diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is equivalent to finding null-homotopies of the induced map $\mathrm{C}(i) \to C$; we leave this as an exercise. □

5.18. **Remark** Some textbooks add further assumptions on the map f, e.g. in particular so that $\pi_n(F)$ is naturally an abelian group also for $n = 0$ and so that the first step in our obstruction theory (which we decided to phrase in terms of covering theory, i.e. in terms of fundamental groups) is given by a cohomological obstruction class, namely one in $H^1(B/A; \pi_0(F))$ as we also argued in Example 5.8.

We finish this section with some applications.

5.19. **Corollary** (Whitehead's theorem) *Let $f\colon X \to Y$ be a weak equivalence between connected CW complexes. Then f is a homotopy equivalence.*

Proof. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & E_f \arrow[d] \\ \mathrm{Cyl}(f) \arrow[r, "\simeq"] \arrow[ur, dashed] & Y \end{tikzcd}$$

[26] As in Remark 5.15, a similar conclusion holds for nilpotent maps.

The right hand square is obtained by factoring the map $X \to Y$ both as a homotopy equivalence followed by a fibration (the upper right composite, see e.g. [Win24, Construction 5.2.10]) and by a cofibration followed by a homotopy equivalence (the lower left composite). Note that the map $E_f \to Y$ is a simple map, for the simple reason that it is a weak equivalence (since f is a weak equivalence) and hence its homotopy fibre has trivial homotopy groups.

We are looking to show that we can find a dashed arrow making both small triangles commute, and will use the above developed obstruction theory to show that this is the case. Indeed, it is readily checked that such a map provides a homotopy inverse of f, by considering the composite $Y \simeq \mathrm{Cyl}(f) \to E_f \simeq X$. By Theorem 5.17 the obstructions to finding such a dashed arrow are controlled by π_1 (where it is trivial that a lift exists since a weak equivalence induces an isomorphism on π_1), and by elements in $H^{n+1}(\mathrm{C}(f); \pi_n(F))$, where F is the homotopy fibre of f. These obstruction groups vanish since f is a weak equivalence. □

5.20. **Corollary** *Let $n \geq 1$ and $f\colon X \to Y$ be $(n+1)$-connected nilpotent map and let A be a CW complex with $H^k(A; M) = 0$ for all abelian groups M and all $k \geq n+1$. Then the map $f_*\colon [A, X] \to [A, Y]$ is a bijection.*

Proof. We may assume that f is a fibration. To see injectivity, assume $\varphi, \varphi'\colon A \to X$ are maps such that $f\varphi$ and $f\varphi'$ are homotopic. Choosing a homotopy gives rise to the following commutative diagram.

$$\begin{tikzcd} A \times \{0,1\} \arrow[r] \arrow[d] & X \arrow[d] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Pick a point $a \in A$ and consider the restricted lifting problem with A replaced by $\{a\}$. This can be solved by [Win24, Prop. 5.2.15]. Therefore, we may consider the slightly improved commutative diagram

$$\begin{tikzcd} A \times \{0,1\} \cup \{a\} \times [0,1] \arrow[r] \arrow[d] & X \arrow[d] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

The obstructions to finding a dashed arrow (which would give that φ and φ' are homotopic), making the diagram commute are controlled firstly by π_1 which is no problem as $\pi_1(X) \to \pi_1(Y)$ is an isomorphism (recall that we assume $n \geq 1$). Then, the further obstructions are controlled by elements $H^{k+1}(\Sigma A; \pi_k(F)) \cong H^k(A; \pi_k(F))$, where F is the homotopy fibre of f. Now, since f is $(n+1)$-connected, we find that F is n-connected, so all of these cohomology groups vanish by assumption. Similarly, for surjectivity, we consider the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Again, there is no issue with finding a lift on π_1. For the further obstructions, we use that the groups $H^{k+1}(A; \pi_k(F))$ also vanish (in fact, here, it suffices that the map is n-connected, as long as $n \geq 2$). □

Next, we prove the classical theorem of Hopf (which Hopf initially proved geometrically). We use here that closed manifolds have the homotopy type of CW complexes.[27]

5.21. **Corollary** *Let M be a closed connected n-manifold. Then*

$$\deg\colon [M,S^n] \to \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable} \\ \mathbb{Z}/2 & \text{if } M \text{ is not orientable} \end{cases}$$

are bijective. In both cases, a generator is given by the collapse map $M \to M/(M \setminus \mathring{D}^n) \cong S^n$ for some embedding of $D^n \subseteq M$.

Proof. Recall that the map $S^n \to K(\mathbb{Z},n)$ is $(n+1)$-connected. By Corollary 5.20, it hence induces an isomorphism on $[M,-]$. It follows that the sets are as claimed. That the collapse map are generators is then an explicit computation. □

5.22. **Corollary** *Let M be a connected closed n-dimensional manifold with $n \geq 2$. Then $[M,S^{n-1}]$ sits inside an exact sequence*

$$H^{n-2}(M;\mathbb{Z}) \to H^n(M;\pi_n(S^{n-1})) \to [M,S^{n-1}] \to H^{n-1}(M;\mathbb{Z}) \to 0$$

Proof. If $n=2$, then $S^{n-1} = K(\mathbb{Z},1)$ and the result follows immediately from the representability theorem. So assume that $n \geq 3$ and let F be the homotopy fibre of the map $S^{n-1} \to K(\mathbb{Z},n-1)$. By the final observation in the proof of Corollary 5.20, we deduce that the map $[M,S^{n-1}] \to [M,K(\mathbb{Z},n-1)]$ is surjective. Moreover, from the fibration sequence defining F we obtain an exact sequence (of sets)

$$[M,\Omega K(\mathbb{Z},n-1)] \to [M,F] \to [M,S^{n-1}] \to [M,K(\mathbb{Z},n-1)] \to 0$$

Moreover, notice that $\pi_n(F) \to \pi_n(S^{n-1})$ is an isomorphism, and therefore that the map $F \to K(\pi_n(S^{n-1}),n)$ is $(n+1)$-connected, so the induced map $[M,F] \to [M,K(\pi_n(S^{n-1}),n)]$ is bijective again by Corollary 5.20, giving the exact sequence of the corollary. □

5.23. **Remark** If $n \geq 4$, using the results of the next section, we have $\pi_n(S^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$; so far, we know that there is a surjection $\mathbb{Z}/2\mathbb{Z} \to \pi_n(S^{n-1})$. Hence, if M is simply connected and $n \geq 4$ we have that $[M,S^{n-1}]$ is a quotient of $H^n(M;\mathbb{Z}/2\mathbb{Z})$ and hence has either two elements, or one element. In fact, we will see later in this course, that $[M,S^{n-1}]$ has two elements if and only if a certain characteristic class $w_2(M)$ of M vanishes.

In addition, one can show that the map $[M,S^n] \to [M,S^{n-1}]$ induced by the appropriate suspension of the Hopf map $\eta\colon S^3 \to S^2$ induces a surjection (in case M is simply connected), so that we also find explicit representatives of the non-trivial homotopy classes.

6. Steenrod operations

In this section we aim to construct and study certain cohomology operations called Steenrod squares. They are operations

$$\mathrm{Sq}^i\colon H^n(-;\mathbb{F}_2) \to H^{n+i}(-;\mathbb{F}_2), \text{ for } i \geq 0$$

satisfying various relations and properties. We will see that, essentially, they are induced by the cup square operation $x \mapsto x^2$. Let us contemplate the following description of this

[27] This is not a priori obvious and we will not give a proof in this course.

operation; we write $\iota_n \in H^n(K(\mathbb{F}_2,n);\mathbb{F}_2)$ for the tautological class. Then the cross product $\iota_n \times \iota_n \in H^{2n}(K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n);\mathbb{F}_2)$ is the image of $\iota_n \otimes \iota_n$ under the map

$$H^n(K(\mathbb{F}_2,n);\mathbb{F}_2)\otimes H^n(K(\mathbb{F}_2,n);\mathbb{F}_2) \to H^{2n}(K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n);\mathbb{F}_2)$$

appearing in the Künneth theorem. By the representability theorem, we may think of $\iota_n \times \iota_n$ as a map $K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n) \to K(\mathbb{F}_2,2n)$ which becomes null homotopic when restricted to $K(\mathbb{F}_2,n)\vee K(\mathbb{F}_2,n)$, by naturality of the Künneth map. Hence, we really may think of it as a map

$$\iota_n \times \iota_n \colon K(\mathbb{F}_2,n)\wedge K(\mathbb{F}_2,n) \to K(\mathbb{F}_2,2n).$$

Likewise, we may consider the k-fold cross product $\iota_n^{\times k}$ as a map

$$\iota_n^{\times k}\colon \bigwedge_k K(\mathbb{F}_2,n) \to K(\mathbb{F}_2,kn).$$

6.1. **Remark** There is no reason to believe that this map is equivariant for the Σ_k-action via permutations on the source and the trivial source on the target. In fact, the Steenrod operations we will construct turn out to be obstructions for this. In particular, since we will show that the Steenrod squares are non-trivial, it follows that there is no representative of the homotopy class of $\iota_n^{\times k}$ which is equivariant for the above mentioned Σ_k-action.

However, it is true that for any permutation $\sigma \in \Sigma_k$, thought of as a self-map of $\bigwedge_k K(\mathbb{F}_2,n)$ by permuting the factors, we have that the two maps

$$\iota_n^{\times k}\circ\sigma \quad \text{and} \quad \iota_n^{\times k}$$

are homotopic. Indeed, this is simply because σ induces an isomorphism on cohomology and $H^{nk}(\bigwedge_k K(\mathbb{F}_2,n);\mathbb{F}_2)\cong\mathbb{F}_2$.

It turns out that more is true, and this is the basis for the existence of the Steenrod squares. though. To formulate it, fixing a basepoint in $K(\mathbb{F}_2,n)$, let us recall that $\bigwedge_k K(\mathbb{F}_2,n)$ is the quotient of $\prod_k K(\mathbb{F}_2,n)$ by those tuples where at least one entry is the basepoint. In particular $\bigwedge_k K(\mathbb{F}_2,n)$ is canonically pointed, let us denote this basepoint simply by $*$. Then we observe that the Σ_k-action by permutations preserves this basepoint. In particular, the inclusion of the basepoint is Σ_k-equivariant.

For a general pointed space (X,x) equipped with a basepoint preserving G-action, G a finite group say, let us then set

$$(X,x)_{hG} := \mathrm{C}(\{*\}_{hG}\to X_{hG}).$$

With this notation established, the following lemma is at the heart of the construction of the Steenrod squares we will present here.[28]

6.2. **Lemma** *There is a, unique up to homotopy, extension up to homotopy of the map $\iota_n^{\times k}$ to a map $(\bigwedge_k K(\mathbb{F}_2,n),*)_{h\Sigma_k}\to K(\mathbb{F}_2,nk)$.*

Proof. To ease notation, let us write X for $\bigwedge_k K(\mathbb{F}_2,n)$. We may assume that X is a CW complex and that $E\Sigma_k$ is a CW complex. The task is to show that the canonical projection map $X\to(X,*)_{h\Sigma_k}$ induces an isomorphism on $H^{nk}(-;\mathbb{F}_2)$. Now, the map $X\times E\Sigma_k\to (X\times E\Sigma_k)/\Sigma_k = X_{h\Sigma_k}$, as well as the map $E\Sigma_k\to B\Sigma_k$ are covering maps. In the composite

$$C_\bullet^{\mathrm{sing}}(X\times E\Sigma_k, *\times E\Sigma_k)\otimes_{\mathbb{Z}[\Sigma_k]}\mathbb{Z}\to C_\bullet^{\mathrm{sing}}(X_{h\Sigma_k},\{*\}_{h\Sigma_k})\to C_\bullet^{\mathrm{sing}}((X,*)_{h\Sigma_k})$$

[28]I learned this perspective in a lecture of Stefan Schwede in Bonn.

the first map is an isomorphism and the second one is a chain homotopy equivalence. Indeed, for the latter see [Lan23, Lemma 4.58] together with Exercise 1 (e) of Exercise Sheet 14 from Topology 1. The former is argued similarly as in our proof of the Hurewicz Theorem 2.32. Moreover, again as in the proof of the Hurewicz theorem, since the chain complex $C_\bullet^{\mathrm{sing}}(X \times E\Sigma_k, * \times E\Sigma_k)$ is $(nk-1)$-connected and consists of levelwise free $\mathbb{Z}[\Sigma_k]$-modules, we can find a $\mathbb{Z}[\Sigma_k]$-chain homotopy equivalence to a chain complex M which is levelweise free (over $\mathbb{Z}[\Sigma_k]$) and with $M_l = 0$ for $l < nk$. We deduce that there is also a chain homotopy equivalence

$$M \otimes_{\mathbb{Z}[\Sigma_k]} \mathbb{Z} \simeq C_\bullet^{\mathrm{sing}}((X,*)_{h\Sigma_k}).$$

Therefore, to compute $H^*((X,*)_{h\Sigma_k};\mathbb{F}_2)$, we may compute the homology of the chain complex

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}[\Sigma_k]} \mathbb{Z}, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathbb{Z}[\Sigma_k]}(M, \mathbb{F}_2)$$

where $\mathbb{F}_2$ is viewed as trivial Σ_k-representation (there is no other way, in fact). Writing this complex out gives

$$\cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}[\Sigma_k]}(M_{nk+1}, \mathbb{F}_2) \leftarrow \mathrm{Hom}_{\mathbb{Z}[\Sigma_k]}(M_{nk}, \mathbb{F}_2) \leftarrow 0$$

where the first non-trivial term appears in homological degree $-nk$. Since the Σ_k-action on $\mathbb{F}_2$ is trivial, this complex is isomorphic to

$$\cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}}(M_{nk+1}, \mathbb{F}_2)^{\Sigma_k} \leftarrow \mathrm{Hom}_{\mathbb{Z}}(M_n, \mathbb{F}_2)^{\Sigma_k} \leftarrow 0$$

Since the functor $(-)^{\Sigma_k}$ is a right adjoint, it preserves kernels and hence we obtain that the $-nk$th homology of the above complex is given by

$$\ker[\mathrm{Hom}_{\mathbb{Z}}(M_{nk+1}, \mathbb{F}_2) \to \mathrm{Hom}_{\mathbb{Z}}(M_{nk}, \mathbb{F}_2)]^{\Sigma_k}.$$

Now, before taking Σ_k-fixed points, this kernel is (again using that M is chain homotopy equivalent to $C_\bullet^{\mathrm{sing}}(X,*)$ since $E\Sigma_k$ is contractible) isomorphic to $H^{nk}(X;\mathbb{F}_2)$. By Künneth, this group is isomorphic to $\mathbb{F}_2$, generated precisely on the element $\iota_n^{\otimes k}$. The Σ_k-action on this $\mathbb{F}_2$ is necessarily trivial, and we arrive at the desired result. □

6.3. **Construction** Let X be a topological space and $x \in H^n(X;\mathbb{F}_2)$. Consider the the diagram composite

$$X \to X \times X \to K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n) \to K(\mathbb{F}_2, 2n)$$

and note that this map classifies the element x^2. By Lemma 6.2, the latter map in the above composite factors as a composite

$$[K(\mathbb{F}_2, n) \times K(\mathbb{F}_2)]_{h\Sigma_2} \to [K(\mathbb{F}_2, n) \wedge K(\mathbb{F}_2, n)]_{h\Sigma_2} \to K(\mathbb{F}_2, 2n)$$

since there is a tautological map $X_{h\Sigma_2} \to X_{*h\Sigma_2}$ for any pointed space with Σ_2-action. In particular, the above composite refines to a composite

$$X \times \mathbb{RP}^\infty = X_{h\Sigma_2} \to (X \times X)_{h\Sigma_2} \to (K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n))_{h\Sigma_2} \to K(\mathbb{F}_2, 2n)$$

in other words, gives an element of $H^{2n}(X \times \mathbb{RP}^\infty;\mathbb{F}_2)$. Now recall that by the Künneth theorem, we have an isomorphism

$$H^*(X \times \mathbb{RP}^\infty;\mathbb{F}_2) \cong H^*(X;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = H^*(X;\mathbb{F}_2)[t]$$

where $|t| = 1$, since $H^*(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[t]$. We denote the above composite by $P_t(x)$. Here, the P stands for *Power operation*, the t for the fact that we obtain a polynomial in t.

6.4. **Definition** We define the Steenrod squares $\mathrm{Sq}^i(x) \in H^{n+i}(X;\mathbb{F}_2)$ on $x \in H^n(X;\mathbb{F}_2)$ to be the coefficient of t^{n-i} of $P_t(x)$, that is, we have

$$P_t(x) = \sum_{i=-n}^{n} \mathrm{Sq}^i(x) \cdot t^{n-i}.$$

We now aim to prove several structural results about these elements summarized in the following theorem.

6.5. **Theorem** *For $x \in H^n(X;\mathbb{F}_2)$ and $y \in H^m(X;\mathbb{F}_2)$, the operations $x \mapsto \mathrm{Sq}^i(x)$ satisfy the following properties:*

(1) *Naturality: That is, if $x = f^*(y)$, then $\mathrm{Sq}^i(x) = f^*(\mathrm{Sq}^i(y))$.*
(2) *Triviality: We have $\mathrm{Sq}^i(x) = 0$ for $i > n$ and for $i < 0$.*
(3) *Square-likeness: We have $\mathrm{Sq}^n(x) = x^2$.*
(4) *Cartan formula: $\mathrm{Sq}^i(x \times y) = \sum_{k+l=i} \mathrm{Sq}^k(x) \times \mathrm{Sq}^l(y)$ in $H^{n+m+i}(X \times Y;\mathbb{F}_2)$; same for $\wedge$ in place of $\times$.*
(5) *Stability: We have $\mathrm{Sq}^i(\sigma(x)) = \sigma(\mathrm{Sq}^i(x))$ where σ is the suspension isomorphism.*
(6) *Unitality: We have $\mathrm{Sq}^0(x) = x$.*
(7) *Adem relations: For $i < 2j$, we have*

$$\mathrm{Sq}^i\mathrm{Sq}^j(x) = \sum_{k}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{j+i-k}\mathrm{Sq}^k(x).$$

where the binomial coefficient is to be interpreted in $\mathbb{F}_2$.

6.6. **Remark** By representability and naturality, we may think of the Steenrod squares as follows: For each $n \geq 0$ and $0 \leq i \leq n$, there is a unique map $\widehat{\mathrm{Sq}}^i \colon K(\mathbb{F}_2, n) \to K(\mathbb{F}_2, n+i)$ with the property that for $x \in H^n(X;\mathbb{F}_2)$ with classifying map $x \colon X \to K(\mathbb{F}_2, n)$, we have that $\mathrm{Sq}^i(x)$ is classified by the composite

$$X \xrightarrow{x} K(\mathbb{F}_2, n) \xrightarrow{\widehat{\mathrm{Sq}}^i} K(\mathbb{F}_2, n+i).$$

Exercise: The stability property (5) in the above theorem translates to the statement that the map

$$\Omega\widehat{\mathrm{Sq}}^i \colon \Omega K(\mathbb{F}_2, n+1) \to \Omega K(\mathbb{F}_2, n+1+i)$$

corresponds under the identifications $\Omega K(\mathbb{F}_2, k+1) \simeq K(\mathbb{F}_2, k)$ to the map $\widehat{\mathrm{Sq}}^i \colon K(\mathbb{F}_2, n) \to K(\mathbb{F}_2, n+i)$. In particular, we deduce that the maps $\widehat{\mathrm{Sq}}^i$ are (infinite) loop maps, and hence their effect on cohomology is a group homomorphism. That is, we have $\mathrm{Sq}^i(x + x') = \mathrm{Sq}^i(x) + \mathrm{Sq}^i(x')$.[29] Note that this, together with $\mathrm{Sq}^n(x) = x^2$ looks cumbersome at first glance, until we recall that $x \mapsto x^2$ is indeed additive (as we are working in characteristic 2).

6.7. **Remark** The Cartan formula (4) as above is equivalent to the following statement also often referred to as the Cartan formula (exercise). Let $x, x' \in H^*(X;\mathbb{F}_2)$. Then

$$\mathrm{Sq}^i(x \cdot x') = \sum_{k+l=i} \mathrm{Sq}^k(x) \cdot \mathrm{Sq}^l(x').$$

[29]One can also show this property more directly by definition of $\mathrm{Sq}^i(x + x')$, contemplating the behaviour of $(-)_{h\Sigma_2}$ on wedges of spaces.

6.8. **Remark** The Steenrod squares are uniquely determined by properties (1), (2), (3), (4), and (6). It is worth noting that we will prove (5) as a consequence of these axioms. However, the Adem relations will be shown to be a consequence of the construction, not a consequence of these properties alone. See Remark 6.21 for a sketch of the argument.

We now prove each of these properties step by step. The first ones are rather easy.

6.9. **Lemma** (Naturality) *Let $f\colon X \to Y$ be a map and $y \in H^n(X;\mathbb{F}_2)$. Then* $\mathrm{Sq}^i(f^*(y)) = f^*(\mathrm{Sq}^i(y))$.

Proof. We consider the diagram commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X \times X \arrow[r, "f^*(y)\times f^*(y)"] \arrow[d, "f\times f"] & K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n) \arrow[r] \arrow[d, equal] & K(\mathbb{F}_2,2n) \arrow[d, equal] \\
Y \arrow[r] & Y \times Y \arrow[r, "y\times y"] & K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n) \arrow[r] & K(\mathbb{F}_2,2n)
\end{tikzcd}$$

Here, the lower composite classifies y^2, and the upper composite classifies $f^*(y^2) = f^*(y)^2$. Since the left two squares are equivariant with respect to the evident Σ_2-actions, we obtain the commutative diagram

$$\begin{tikzcd}
X \times \mathbb{RP}^\infty \arrow[r] \arrow[d, "f\times \mathrm{id}"] & (X\times X)_{h\Sigma_2} \arrow[r] \arrow[d] & (K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n)_{h\Sigma_2} \arrow[r] \arrow[d, equal] & K(\mathbb{F}_2,2n) \arrow[d, equal] \\
Y \times \mathbb{RP}^\infty \arrow[r] & (Y\times Y)_{h\Sigma_2} \arrow[r] & (K(\mathbb{F}_2,n)\times K(\mathbb{F}_2,n)_{h\Sigma_2} \arrow[r] & K(\mathbb{F}_2,2n)
\end{tikzcd}$$

The upper horizontal composite is $P_t(f^*(y))$, by definition. The composite going through the lower left corner is $(f\times \mathrm{id})^*P_t(y)$. The commutativity of the diagram hence implies that $P_t(f^*(y)) = (f\times\mathrm{id})^*(P_t(y))$, and in particular the lemma. □

6.10. **Lemma** (Triviality) *For $x \in H^n(X;\mathbb{F}_2)$ we have $\mathrm{Sq}^i(x) = 0$ if $i > n$ or $i < 0$.*

Proof. Recall that $\mathrm{Sq}^i(x)$ is the coefficient of t^{n-i} in $P_t(x) \in H^{2n}(X\times\mathbb{RP}^\infty;\mathbb{F}_2)$. Hence, if $i > n$, then $\mathrm{Sq}^i(x)$ must be zero because $H^*(\mathbb{RP}^\infty) = 0$ for $* < 0$. Now consider the case $i < 0$. We may, by abuse of notation, write $x = x^*(\iota_n)$, where we also denote by x the map $X \to K(\mathbb{F}_2,n)$ which classifies x. By naturality, we have $\mathrm{Sq}^i(x) = x^*(\mathrm{Sq}^i(\iota_n))$, so it suffices to prove the result in case $X = K(\mathbb{F}_2,n)$ and $x = \iota_n$. In this case, there are two options: If $i < 0$ is different from $-n$, then $\mathrm{Sq}^i(\iota_n) \in H^{n+i}(K(\mathbb{F}_2,n);\mathbb{F}_2) = 0$. It remains to show that $\mathrm{Sq}^{-n}(\iota_n) = 0$. In this case, consider the map $c\colon * \to K(\mathbb{F}_2,n)$. Then it suffices to show that $c^*\mathrm{Sq}^{-n}(\iota_n) = 0$. Again, by naturality this is equal to $\mathrm{Sq}^{-n}(c^*(\iota_n)) = 0$ since we have assumed (implicitly) that $n > 0$. □

6.11. **Lemma** (Square-likeness) *For $x \in H^n(X;\mathbb{F}_2)$ we have $\mathrm{Sq}^n(x) = x^2$.*

Proof. By definition, $\mathrm{Sq}^n(x)$ is the coefficient of t^0 of $P_t(x) \in H^{2n}(X\times\mathbb{RP}^\infty)$. This coefficient is equivalently given by the restriction of $P_t(x)$ along the map $X = X\times * \to X\times\mathbb{RP}^\infty$ using some basepoint of $\mathbb{RP}^\infty$. This map is the tautological map $X \to X_{h\Sigma_2}$. Therefore, the result follows from the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times X \arrow[r] \arrow[d] & K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n) \arrow[r] \arrow[d] & K(\mathbb{F}_2, 2n) \\
X_{h\Sigma_2} \arrow[r] & (X \times X)_{h\Sigma_2} \arrow[r] & (K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n))_{h\Sigma_2} \arrow[ur] &
\end{tikzcd}$$

□

We work towards the Cartan fomula next. To do this, we recall that we have constructed a natural map

$$P_t \colon H^*(X; \mathbb{F}_2) \to H^*(X \times \mathbb{RP}^\infty; \mathbb{F}_2) = H^*(X; \mathbb{F}_2)[t]$$

where naturality means that for a map $f \colon X \to Y$, the diagram

$$\begin{tikzcd}
H^*(Y; \mathbb{F}_2) \arrow[r] \arrow[d, "f^*"] & H^*(Y; \mathbb{F}_2)[t] \arrow[d, "f^*"] \\
H^*(X; \mathbb{F}_2) \arrow[r] & H^*(X; \mathbb{F}_2)[t]
\end{tikzcd}$$

commutes (the right hand vertical map sends t to t). Let us denote by δ the canonical map $(X \times Y)_{h\Sigma_2} \to X_{h\Sigma 2} \times Y_{h\Sigma_2}$; concretely, it is the map $X \times Y \times \mathbb{RP}^\infty \to X \times \mathbb{RP}^\infty \times Y \times \mathbb{RP}^\infty$, which uses the symmetric monoidal structure on $\times$ and the diagonal $\mathbb{RP}^\infty \to \mathbb{RP}^\infty \times \mathbb{RP}^\infty$. The following lemma is the crucial relation giving the Cartan formula (we will make this explicit momentarily).

6.12. **Lemma** (Cartan formula) *Let X and Y be spaces, $x \in H^n(X; \mathbb{F}_2)$ and $y \in H^m(Y; \mathbb{F}_2)$. Then we have:*

$$P_t(x \times y) = \delta^*(P_t(x) \times P_t(y)).$$

Proof. First we note that both sides are additive in x and y. Therefore, we may assume that X and Y are connected. Moreover, suppose $n = 0$. Then x is pulled back from the unique map $X \to *$. Exercise: prove the lemma in case $X = *$. The same argument applies when $m = 0$.

We now assume that n and m are positive. As before, we may write $x = x^*(\iota_n)$ and $y = y^*(\iota_m)$. Then we find

$$P_t(x \times y) = P_t((x, y)^*(\iota_n \times \iota_m)) = (x, y)^*(P_t(\iota_n \times \iota_m))$$

Hence, if we can show that

$$P_t(\iota_n \times \iota_m) = \delta^*(P_t(\iota_n) \times P_t(\iota_m))$$

the claim follows. Now note that since n and m are positive, the map classifiying $\iota_n \times \iota_m$ factors as

$$K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, m) \to K(\mathbb{F}_2, n) \wedge K(\mathbb{F}_2, m) \to K(\mathbb{F}_2, n + m)$$

where the latter map classifies the element $\iota_n \wedge \iota_m$. Hence, it suffices to show that

$$P_t(\iota_n \wedge \iota_m) = \delta^*(P_t(\iota_n) \wedge P_t(\iota_m))$$

This amounts to proving the commutativity of the following diagram. Let us shorten notation and write K_n for $K(\mathbb{F}_2, n)$. Also, we will not include basepoints in the notation explicitly, but all homotopy orbits appearing next are to be taken in the pointed sense.

$$\begin{tikzcd} (K_n \wedge K_n)_{h\Sigma_2} \arrow[r] \arrow[d] & \big[(K_n \wedge K_m) \wedge (K_n \wedge K_m)\big]_{h\Sigma_2} \arrow[r] \arrow[d] & (K_{n+m} \wedge K_{n+m})_{h\Sigma_2} \arrow[r] & K_{2(n+m)} \arrow[d, "\simeq"] \\ (K_n)_{h\Sigma_2} \wedge (K_m)_{h\Sigma_2} \arrow[r] & (K_n \wedge K_n)_{h\Sigma_2} \wedge (K_m \wedge K_m)_{h\Sigma_2} \arrow[r] & K_{2n} \wedge K_{2m} \arrow[r] & K_{2n+2m} \end{tikzcd}$$

Here, the left most horizontal maps are again induced by the diagonals and the two left vertical maps are the pointed versions of the tautological maps described earlier, so the left square commutes by naturality of these maps. It remains to argue that the big right square also commutes. To do this, note that both composites are equivalently described by an element of $H^{2(n+m)}(-;\mathbb{F}_2)$ of the space appearing on the top left corner of the square under investigation. Now, the same argument we used in the proof of Lemma 6.2 shows that the map

$$K_n \wedge K_m \wedge K_n \wedge K_m \to \big[(K_n \wedge K_m) \wedge (K_n \wedge K_m)\big]_{h\Sigma_2}$$

induces an isomorphism on $H^{2(n+m)}(-;\mathbb{F}_2)$. It hence suffices to prove that the right square commutes after precomposition with this map. Then, one observes that both composites classify the unique non-trivial element of the corresponding cohomology group (by construction and Künneth). □

We now show that this indeed amounts to the Cartan formula by writing out the two sides of the equality of Lemma 6.12. For the left hand side, we get

$$P_t(x \times y) = \sum_{i=0}^{n+m} \mathrm{Sq}^i(x \times y) \cdot t^{n+m-i}$$

while the left hand side gives

$$\begin{aligned} \delta^*(P_t(x) \times P_t(y)) &= \big[\sum_{k=0}^{n} \mathrm{Sq}^k(x) \cdot t^{n-k}\big] \times \big[\sum_{l=0}^{m} \mathrm{Sq}^l(y) \cdot t^{m-l}\big] \\ &= \sum_{k=0}^{n}\sum_{l=0}^{m} (\mathrm{Sq}^k(x) \times \mathrm{Sq}^l(y)) \cdot t^{n+m-(k+l)} \\ &= \sum_{i=0}^{n+m} \big[\sum_{k+l=i} \mathrm{Sq}^k(x) \times \mathrm{Sq}^l(y)\big] \cdot t^{n+m-(k+l)} \end{aligned}$$

showing the Cartan formula. For the next results, we first need to prove the following lemma by hand:

6.13. **Lemma** *Let $i_1 \in H^1(S^1;\mathbb{F}_2)$ denote the non-trivial element. Then $\mathrm{Sq}^0(i_1) = i_1$.*

Proof. The element $\mathrm{Sq}^0(i_1) \cdot t$ lives in $H^1(S^1;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^1(\mathbb{RP}^\infty;\mathbb{F}_2) \subseteq H^2(S^1 \times \mathbb{RP}^\infty$. This inclusion is equivalently given by map $H^2((S^1,*)_{h\Sigma_2});\mathbb{F}_2) \to H^2(S^1_{h\Sigma_2};\mathbb{F}_2)$, induced from the map from ordinary homotopy orbits to pointed homotopy orbits. It hence suffices to prove that the composite

$$(S^1,*)_{h\Sigma_2} \to (S^1 \wedge S^1,*)_{h\Sigma_2} \to (K_1 \wedge K_1,*)_{h\Sigma_2} \to K_2$$

is non-trivial, or equivalently, induces an isomorphism on $H^2(-;\mathbb{F}_2)$. Note that the final two maps induce isomorphisms on $H^2(-;\mathbb{F}_2)$, the latter by Lemma 6.2 and the former for the same connectivity reasons leading to Lemma 6.2. It hence suffices to prove that also the first map induces an isomorphism on $H^2(-;\mathbb{F}_2)$. To that end, let us consider the following homotopy cofibre sequence of pointed Σ_2-spaces:

$$(\Sigma_2)_+ \to S^0 \to S^\sigma$$

where S^σ is therefore the one-point compactification of the sign Σ_2-representation on $\mathbb{R}$. Exercise: as Σ_2-space, there is a homeomorphism $(S^1 \wedge S^1) \cong S^{1+\sigma} := S^1 \wedge S^\sigma$, the suspension of S^σ, under which the diagonal map corresponds to the suspension of the inclusion $S^0 \to S^\sigma$. Indeed, both terms are the one-point compactifications of representations of Σ_2 on $\mathbb{R}^2$: The former is the flip action $(x,y) \mapsto (y,x)$ and the latter is $\mathbb{R}^{1+\sigma}$, the trivial plus the sign representation $(x,y) \mapsto (x,-y)$. Consider the linear automorphism of $\mathbb{R}^2$ given by $(a,b) \mapsto (a+b, a-b)$. This determines an isomorphism from the $\mathbb{R}^{1+\sigma}$ to $\mathbb{R}^2$ with flip action, and it sends the inclusion $\mathbb{R}^1 \to \mathbb{R}^{1+\sigma}$, $x \mapsto (x,0)$ to the diagonal embedding $x \mapsto (x,x)$ of $\mathbb{R}$ into $\mathbb{R}^2$ with flip action. Passing to one-point compactifications then yields the desired Σ_2-equivariant homeomorphism $(S^1 \wedge S^1) \cong S^{1+\sigma}$. Now, the functor $(-)_{h\Sigma_2}$, in the pointed context, preserves homotopy cofibre sequences and sends $X \wedge (\Sigma_2)_+$ to X, see Remark 6.14 below. Hence, we obtain that the map under investigation participates in a homotopy cofibre sequence

$$S^1 \to (S^1,*)_{h\Sigma_2} \to (S^1 \wedge S^1,*)_{h\Sigma_2} \to S^2 \to S^2_{h\Sigma_2}.$$

This shows that the map

$$H^2((S^1 \wedge S^1,*)_{h\Sigma_2};\mathbb{F}_2) \to H^2((S^1,*)_{h\Sigma_2};\mathbb{F}_2)$$

is a surjection from $\mathbb{F}_2$ to $\mathbb{F}_2$ and hence an isomorphism as needed. $\square$

6.14. **Remark** Let (X,x) be a pointed CW complex and G a finite group. Then $X \wedge G_+$ is a G-space with basepoint preserving G-action. Recall that $X \wedge G_+ = X \times G/\{x\} \times G$. To compute its homotopy orbits, let us then consider the following (pushout) diagrams

$$\begin{tikzcd} \{x\} \times G \arrow[r] \arrow[d] & X \times G \arrow[d] \\ * \arrow[r] & X \wedge G_+ \end{tikzcd} \qquad \begin{tikzcd} (\{x\} \times G)_{hG} \arrow[r] \arrow[d] & (X \times G)_{hG} \arrow[d] \\ *_{hG} \arrow[r] & (X \wedge G_+)_{hG} \end{tikzcd}$$

where the right square is obtained from the left by first crossing with EG and then taking strict orbits with regards to the G-action; both of these operations perserve pushout squares. Let us calculate $(Y \times G)_{hG}$ for a general space Y. Recall that this is, by definition, $(Y \times G \times EG)/G$. Since the G-action is only on $G \times EG$, we obtain that $(Y \times G)_{hG} = Y \times G_{hG}$ and $G_{hG} = (G \times EG)/G$. Viewing this as the associated bundle of the principal G-bundle $G \to *$ with the G-space EG, we find that $G_{hG} = EG$; compare the discussion in Example 4.57. We deduce that the right above pushout sqaure has top horizontal map given by $\{x\} \times EG \to X \times EG$. In particular, this map is a cofibration. Hence, so is its pushout, i.e. the lower map in the above pushout square, so that its mapping cone is equivalent to the strict quotient; this strict quotient is then isomorphic to the strict quotient of the cofibration $\{x\} \times EG \to X \times EG$. We hence obtain a homotopy equivalence

$$(X \wedge G_+,*)_{hG} \simeq (X,x).$$

To prove the compatibility of the Steenrod operations with the suspension isomorphism σ, it will be convenient to characterise σ as follows.

6.15. **Lemma** *The suspension isomorphism σ is given by the composite*

$$H^n(X;\mathbb{F}_2) \to H^{n+1}(X \wedge S^1;\mathbb{F}_2) \cong H^{n+1}(\Sigma X;\mathbb{F}_2).$$

where the first map is given by the exterior product with the unique non-trivial class $\widetilde{H}^1(S^1;\mathbb{F}_2)$.

Proof. This map is indeed a natural isomorphism by the Künneth theorem. Composing with the inverse of the suspension isomorphism, we obtain a natural self-isomorphism of the functor $H^n(-;\mathbb{F}_2)$. The only such self-isomorphism is the identity (by representability, the set of natural self-homorphisms is given by $H^n(K(\mathbb{F}_2,n);\mathbb{F}_2) \cong \mathbb{F}_2$) and hence the map of the lemma indeed agrees with the suspension isomorphism. □

6.16. **Corollary** (Stability) *For $x \in H^n(X;\mathbb{F}_2)$ we have $\mathrm{Sq}^i(\sigma(x)) = \sigma(\mathrm{Sq}^i(x))$.*

Proof. Lemma 6.15 says that $\sigma(x) = x \wedge i_1$. Note that $\mathrm{Sq}^k(i_1) = 0$ unless $k = 0$ since $H^n(S^1;\mathbb{F}_2) = 0$ for $n \neq 1$. Moeover, $\mathrm{Sq}^0(i_1) = i_1$ by the previous lemma. Thus we find from the Cartan formula that

$$\mathrm{Sq}^i(\sigma(x)) = \mathrm{Sq}^i(x \wedge i_1) = \mathrm{Sq}^i(x) \wedge i_1$$

and therefore that $\mathrm{Sq}^i(\sigma(x)) = \sigma(\mathrm{Sq}^i(x))$ as claimed. □

Let us pause for a second and realize what we have already achieved:

6.17. **Corollary** *For all $n \geq 0$, $\Sigma^n(\eta)\colon S^{n+3} \to S^{n+2}$ is not null-homotopic. In particular, $\pi_n(S^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 4$.*

Proof. Recall that $\mathrm{C}(\eta) = \mathbb{CP}^2$. Hence, $\mathrm{C}(\Sigma^n(\eta)) = \Sigma^n(\mathrm{C}(\eta)) = \Sigma^n\mathbb{CP}^2$. Let $x \in H^2(\mathbb{CP}^2;\mathbb{F}_2)$ be the non-trivial element. Then $\sigma^n(x) \in H^{n+2}(\Sigma^n\mathbb{CP}^2;\mathbb{F}_2)$ is non-trivial. Moreover, we have $\mathrm{Sq}^2(\sigma^n(x)) = \sigma^n(\mathrm{Sq}^2(x)) = \sigma^n(x^2)$ and we know that $x^2 \neq 0$ in $H^4(\mathbb{CP}^2;\mathbb{F}_2)$ by Poincaré duality. Therefore, we conclude that $\mathrm{Sq}^2\colon H^{n+2}(\Sigma^n\mathbb{CP}^2;\mathbb{F}_2) \to H^{n+4}(\Sigma^n\mathbb{CP}^2;\mathbb{F}_2)$ is non-trivial. This contradicts that $\Sigma^n\eta$ is null-homotopic. Indeed, in this case, we would have $\mathrm{C}(\Sigma^n\eta) \simeq S^{n+2} \vee S^{n+4}$. A naturality argument then shows that Sq^2 is trivial on the cohomology of $S^{n+2} \vee S^{n+4}$. □

6.18. **Remark** The same arguments apply to the quaternionic and octonionic Hopf maps $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$. Indeed, their mapping cones are given by $\mathbb{HP}^2$ and $\mathbb{OP}^2$, respectively. Both are closed manifolds with $H^*(-;\mathbb{Z}) = \mathbb{Z}[u]/u^3$ with $|u| = 4$ and $|u| = 8$, respectively. Hence, we find that Sq^4 is non-trivial on $H^*(\Sigma^k\mathbb{HP}^2;\mathbb{F}_2)$ and that Sq^8 is non-trivial on $H^*(\Sigma^k\mathbb{OP}^2;\mathbb{F}_2)$ for any $k \geq 0$. Consequently, $\Sigma^k\nu$ and $\Sigma^k\sigma$ are not null-homotopic for any $k \geq 0$.

As a further consequence of stability and Lemma 6.13 we obtain:

6.19. **Corollary** (Unitality) *Let $x \in H^n(X;\mathbb{F}_2)$. Then $\mathrm{Sq}^0(x) = x$.*

Proof. If $n = 0$, then we find that $\mathrm{Sq}^0(x) = x^2 \in H^0(X;\mathbb{F}_2) = \prod \mathbb{F}_2$, and in this ring, we have $x^2 = x$ for all elements, simply because it is true for $\mathbb{F}_2$ itself. Therefore let us suppose that $n \geq 1$. By naturality, it suffices to treat the case $(X,x) = (K(\mathbb{F}_2,n),\iota_n)$. In this case $\mathrm{Sq}^0(\iota_n)$ is either ι_n or 0, since $H^n(K(\mathbb{F}_2,n);\mathbb{F}_2) \cong \mathbb{F}_2$. Hence, again by naturality, it suffices to find an arbitrary space Y with $0 \neq y \in H^n(Y;\mathbb{F}_2)$ and $\mathrm{Sq}^0(y) \neq 0$. We choose $Y = S^n$ and $y = i_n$ the unique choice. Then we find $\mathrm{Sq}^0(i_n) = \mathrm{Sq}^0(\sigma^{n-1}(i_1)) = \sigma^{n-1}(\mathrm{Sq}^0(i_1)) = i_n$. □

6.20. **Example** Let us fully describe the action of the Steenrod squares on the cohomlogy of $\mathbb{RP}^\infty$. To that end, let $t \in H^1(\mathbb{RP}^\infty;\mathbb{F}_2)$ be the non-trivial element. Then we have

$$\mathrm{Sq}^i(t^k) = \binom{k}{i} t^{k+i}.$$

Indeed, let us write $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \dots$ for the *total Steenrod square*. The Cartan formula is then equivalent to the formula $\mathrm{Sq}(x \cdot y) = \mathrm{Sq}(x) \cdot \mathrm{Sq}(y)$. Moreover, we have $\mathrm{Sq}(t) = t + t^2 = t(t+1)$ by unitality, square-likeness and triviality. Hence

$$\mathrm{Sq}(t^k) = \mathrm{Sq}(t)^k = t^k(t+1)^k = t^k \cdot \sum_{i=0}^{k} \binom{k}{i} t^i$$

as needed.

6.21. **Remark** Note that the above computation for the action of Sq^i on $H^*(\mathbb{RP}^\infty;\mathbb{F}_2)$ follows from the properties (2), (3), (4), and (6). That is, any other set of operations $\widetilde{\mathrm{Sq}}^i$ satisfying these properties agree with Sq^i when evaluated on the cohomology of $\mathbb{RP}^\infty$. Using the Cartan formula, it follows that $\widetilde{\mathrm{Sq}}^i$ also agrees with Sq^i on the cohomology of $(\mathbb{RP}^\infty)^{\times n}$, the n-fold cartesian product of $\mathbb{RP}^\infty$ for any $n \geq 1$. Now, let us try to show that $\widetilde{\mathrm{Sq}}^i = \mathrm{Sq}^i$ in general. To do so, let X be a space and $x \in H^n(X;\mathbb{F}_2)$ be a cohomology class for which we aim to show that $\widetilde{\mathrm{Sq}}^i(x) = \mathrm{Sq}^i(x)$. Note that $\widetilde{\mathrm{Sq}}^i$ are compatible with the suspension homomorphism as a formal consequence of the fact that $\widetilde{\mathrm{Sq}}^1(i_1) = i_1$ and of the Cartan formula for $\widetilde{\mathrm{Sq}}^i$. Hence, by passing to a suitably high suspension of X, we may assume that $|x| = n > i$, and in particular that $n+i < 2n$. Now, by naturality, it suffices to show that $\widetilde{\mathrm{Sq}}^i(\iota_n) = \mathrm{Sq}^i(\iota_n)$ where $\iota_n \in H^n(K_n;\mathbb{F}_2)$ is the tautological class. The decisive fact, which we are not yet able to prove, but which has nothing to do with the uniqueness of the Sq^i's, is the following: Considering the map $\iota_1^{\times n} \colon (\mathbb{RP}^\infty)^{\times n} \to K_n$, this map induces an injection on $H^k(-;\mathbb{F}_2)$ for $k < 2n$. Again by naturality, we then deduce the claim that $\widetilde{\mathrm{Sq}}^i(\iota_n) = \mathrm{Sq}^i(\iota_n)$ to the previously established fact that $\widetilde{\mathrm{Sq}}^i = \mathrm{Sq}^i$ on the cohomology of $(\mathbb{RP}^\infty)^{\times n}$.

The fact that the map $(\mathbb{RP}^\infty)^{\times n} \to K_n$ is injective on cohomology relies on an a priori computation of $H^*(K_n;\mathbb{F}_2)$ at least in the range of degrees $* \leq 2n$. We will perform this computation next term.

We finally come to the Adem relations. We will present a proof of these relations that we learned from Gijs Heuts who in turn learned it from Hopkins at Harvard; it seems to be based on a paper by Bullet–Macdonald [BM82]. The idea will be, somewhat similarly to the proof of the Cartan formula, to find a suitable relation for the function $P_t(-)$; deducing the Adem relations from this is then a "purely algebraic" argument.

The basic idea is the following. We recall that we may view P_t as a (ring) homomorphism

$$H^*(X;\mathbb{F}_2) \to H^*(X;\mathbb{F}_2)[t]$$

for any space X. In particular, we may apply this to the space $X \times \mathbb{RP}^\infty$ and obtain the composite

$$H^*(X;\mathbb{F}_2) \xrightarrow{P_s} H^*(X;\mathbb{F}_2)[s] \cong H^*(X \times \mathbb{RP}^\infty;\mathbb{F}_2) \xrightarrow{P_t} H^*(X \times \mathbb{RP}^\infty;\mathbb{F}_2)[t] = H^*(X)[s,t].$$

6.22. **Theorem** *Let X be a space and $x \in H^*(X;\mathbb{F}_2)$. Then there is the equality*

$$P_s(P_t(x)) = P_t(P_s(x)) \in H^*(X;\mathbb{F}_2)[s,t].$$

Proof. Considering the isomorphism $H^*(X;\mathbb{F}_2)[s,t] \cong H^*(X\times\mathbb{RP}^\infty\times\mathbb{RP}^\infty;\mathbb{F}_2)$ and the map P_sP_t as a ring homomorphism

$$H^*(X;\mathbb{F}_2) \to H^*(X\times\mathbb{RP}^\infty\times\mathbb{RP}^\infty;\mathbb{F}_2)$$

the statement of the theorem is equivalent to saying that this map has image contained in the fixed-points of the Σ_2-action on the codomain coming from the switch map $\tau\colon \mathbb{RP}^\infty\times\mathbb{RP}^\infty$. Now, recall that $P_t(x)\colon X_{h\Sigma_2}\to K_{2n}$ is given by the composite

$$X_{h\Sigma_2} \to (X\times X)_{h\Sigma_2} \to (K_n\times K_n)_{h\Sigma_2} \to K_{2n}.$$

Let us then consider the following diagram, the top row of which classifies the element $P_s(P_t(x))$:

$$\begin{tikzcd}
(X_{h\Sigma_2})_{h\Sigma_2} \arrow[r] \arrow[dd] & \left[(X_{h\Sigma_2})^{\times 2}\right]_{h\Sigma_2} \arrow[r] \arrow[d] & (K_{2n}\times K_{2n})_{h\Sigma_2} \arrow[r] & K_{4n} \\
 & \left[((X\times X)_{h\Sigma_2})^{\times 2}\right]_{h\Sigma_2} \arrow[r] \arrow[d, "\simeq"] & \left[((K_n\times K_n)_{h\Sigma_2})^{\times 2}\right]_{h\Sigma_2} \arrow[u] \arrow[d, "\simeq"] & \\
X_{hW} \arrow[r] \arrow[d] & (X\times X\times X\times X)_{hW} \arrow[r] \arrow[d] & (K_n\times K_n\times K_n\times K_n)_{hW} \arrow[d] & \\
X_{h\Sigma_4} \arrow[r] & (X\times X\times X\times X)_{h\Sigma_4} \arrow[r] & (K_n\times K_n\times K_n\times K_n)_{h\Sigma_4} \arrow[uuur, bend right] &
\end{tikzcd}$$

where $W = \Sigma_2 \wr \Sigma_2 = (\Sigma_2\times\Sigma_2)\rtimes\Sigma_2 \subseteq \Sigma_4$, the bent arrow comes from Lemma 6.2, and the top left most vertical map comes from the inclusion $\Sigma_2\times\Sigma_2\subseteq W$ induced by the diagonal $\Sigma_2\subseteq\Sigma_2\times\Sigma_2$: On this diagonal, the Σ_2 from the definition of the wreath product acts trivially, so the semi-direct product becomes simply a direct product. We note the the left vertical maps are concretely given by

$$X\times\mathbb{RP}^\infty\times\mathbb{RP}^\infty \to X\times BW \to X\times B\Sigma_4.$$

By the commutativity of the above diagram, it will be sufficient to show that this map is homotopic to the same map, precomposed with the map induced by the switch map on $\mathbb{RP}^\infty\times\mathbb{RP}^\infty$. For this, X plays no role and it suffices to prove that the diagram

$$\begin{tikzcd}
\mathbb{RP}^\infty\times\mathbb{RP}^\infty \arrow[r, "i"] \arrow[d, "\tau"] & B\Sigma_4 \\
\mathbb{RP}^\infty\times\mathbb{RP}^\infty \arrow[ur, "i"] &
\end{tikzcd}$$

commutes up to homotopy. By Lemma 4.35, this is implied by the statement that the two group homomorphisms

$$\begin{tikzcd}
\Sigma_2\times\Sigma_2 \arrow[r, "i"] \arrow[d, "\tau"] & \Sigma_4 \\
\Sigma_2\times\Sigma_2 \arrow[ur, "i"] &
\end{tikzcd}$$

are conjugated. To that end, let us make again explicit the inclusion i. It is given concretely by the pair of permutations (σ_1, σ_2) where σ_1 is the permuation $\tau_{1,2}\tau_{3,4}$, i.e. the permutation that switches 1 and 2 *and* switches 3 and 4, while σ_2 is the permutation that permutes the block $\{1,2\}$ and the block $\{3,4\}$, i.e. $\tau_{1,3}\tau_{2,4}$ which switches 1 and 3 and 2 and 4. It then suffices to see that $\tau_{2,3}\sigma_1\tau_{2,3} = \sigma_2$, so indeed i and $\tau \circ i$ are conjugated by the element $\tau_{2,3}$ of Σ_4. □

Next, we need to argue how the relation $P_t(P_s(x)) = P_s(P_t(x))$ implies the Adem relations. To do so, it will be convenient to introduce the following notation: We let Sq_i be the coefficient of t^i in P_t. That is, for $x \in H^n(X;\mathbb{F}_2)$, we have $\mathrm{Sq}_i(x) = \mathrm{Sq}^{n-i}(x)$. Then we have

$$\begin{aligned} P_t(P_s(x)) &= P_t(\sum_{i\geq 0} \mathrm{Sq}_i(x)s^i) \\ &= \sum_{i\geq 0} P_t(\mathrm{Sq}_i(x)) \cdot P_t(s)^i \\ &= \sum_{i\geq 0}(\sum_j \mathrm{Sq}_j(\mathrm{Sq}_i(x))t^j) \cdot (s^2+st)^i \\ &= \sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i(x) \cdot t^j \cdot (s^2+st)^i \end{aligned}$$

Here, we have used the Cartan formula, i.e. the statement that $P_t(-)$ is a ring homomorphism, and that for $s \in H^1(\mathbb{RP}^\infty;\mathbb{F}_2)$, that $P_t(s) = \mathrm{Sq}^1(s) + \mathrm{Sq}^0(s)t = s^2 + st$ by square-likeness and unitality. The formula $P_tP_s = P_sP_t$ then gives the formula

$$\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i(x) \cdot t^j \cdot (s^2+st)^i = \sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i(x) \cdot s^j \cdot (t^2+st)^i$$

Let us substitute and write $u = g(s) = (s^2 + st)$. Then we find that $\sum\limits_{n\geq 0} t^n\mathrm{Sq}_n\mathrm{Sq}_m$ is the coefficient of u^{-1} of the Laurent series in u given by:

$$\sum_{i,j\geq 0} t^j u^{i-m-1}\mathrm{Sq}_j\mathrm{Sq}_i.$$

We will now use some manipulations from complex analysis. For this recall that given a Laurent series $f(t) = \sum_{i>-\infty} a_i t^i$ that the residuum $\mathrm{Res}_{t=0}f(t)$ of f at $t = 0$ is given by the coefficient a_{-1}. In a complex analysis course, we learn the following formula. For $u = g(s)$ with $g(0) = 0$ and $f(u)$ Laurent series in u, we have

$$\mathrm{Res}_{u=0}f(u) = \mathrm{Res}_{s=0}[f(g(s)) \cdot g'(s)].$$

Together with the above, we therefore deduce

$$\begin{aligned} \sum_{n\geq 0} \mathrm{Sq}_n\mathrm{Sq}_m t^n &= \mathrm{Res}_{u=0}[\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i \cdot t^j u^{i-m-1}] \\ &= \mathrm{Res}_{s=0}[\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i \cdot t^j(s^2+st)^{i-m-1} \cdot g'(s)] \end{aligned}$$

where $g(s) = s^2 + st$ and therefore $g'(s) = 2s + t$ and from the first to the second line, we have used the above fact from complex analysis. The Adem relations are relations in characteristic

2, hence we may replace $g'(s) = 2s + t$ with t. Now, we compute

$$\begin{aligned}\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i \cdot t^j(s^2+st)^{i-m-1}t &= [\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i \cdot t^j(s^2+st)^i]t(s^2+st)^{-m-1} \\ &= [\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i \cdot s^j(t^2+st)^i]t(s^2+st)^{-m-1} \\ &= \sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i \cdot s^{j-m-1}t^{i+1}(s+t)^{i-m-1}\end{aligned}$$

Here, from line one to line two we have used the equation $P_tP_s = P_sP_t$. Hence, combined with the above we obtain

$$\sum_{n\geq 0} \mathrm{Sq}_n\mathrm{Sq}_m t^n = \mathrm{Res}_{s=0}[\sum_{i,j} \mathrm{Sq}_j\mathrm{Sq}_i \cdot s^{j-m-1}t^{i+1}(s+t)^{i-m-1}]$$

which is simply the coefficient of s^m in the Laurent series

$$\sum_{i,j\geq 0} \mathrm{Sq}_j\mathrm{Sq}_i s^j t^{i+1}(s+t)^{i-m-1}.$$

Writing out[30]

$$(s+t)^{i-m-1} = \sum_{k\geq 0} \binom{i-m-1}{k} s^k \cdot t^{i-m-1-k}$$

and calculating the coefficient of s^m in the Laurent series

$$\sum_{i,j,k\geq 0} \binom{i-m-1}{k} \cdot \mathrm{Sq}_j\mathrm{Sq}_i \cdot s^{j+k}t^{2i-m-k}$$

gives the equality

$$\sum_{n\geq 0} \mathrm{Sq}_n\mathrm{Sq}_m t^n = \sum_{\substack{i\geq 0\\ 0\leq j\leq m}} \binom{i-m-1}{m-j} \mathrm{Sq}_j\mathrm{Sq}_i \cdot t^{2i+j-2m}.$$

Setting $n = 2i + j - 2m$, we have $j = 2m + n - 2i$. Note that the condition $0 \leq j \leq m$ then translates to the condition $m + n \leq 2i \leq 2m + n$. Hence, we get

$$\mathrm{Sq}_n\mathrm{Sq}_m = \sum_{\substack{i\geq 0 \text{ s.t.}\\ m+n\leq 2i\leq 2m+n}} \binom{i-m-1}{2i-m-n} \mathrm{Sq}_{2m+n-2i}\mathrm{Sq}_i.$$

We now need to translate this back to the Steenrod squares Sq^a rather than the Sq_n's we have been using here. To do so, we pick an arbitrary cohomology class x of degree l. Then we have

$$\mathrm{Sq}_n\mathrm{Sq}_m(x) = \mathrm{Sq}_n(\mathrm{Sq}^{l-m}(x)) = \mathrm{Sq}^{2l-m-n}\mathrm{Sq}^{l-m}(x)$$

Likewise, we have

$$\mathrm{Sq}_{2m+n-2i}\mathrm{Sq}^i(x) = \mathrm{Sq}^{2l+i-2m-n}\mathrm{Sq}^{l-i}(x).$$

[30] We may also invert t and think about the above as Laurent series in both s and t, so the terms appearing indeed formally make sense.

Let us then substitute $j = l - i$, so that we obtain

$$\mathrm{Sq}^a\mathrm{Sq}^b(x) = \sum_{\substack{j \text{ s.t.} \\ m+n\leq 2(l-j)\leq 2m+n}} \binom{l-m-j-1}{2l-2j-m-n} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j(x).$$

Now, $l - m - j - 1 = b - j - 1$ and $2l - 2j - m - n = a - 2j$. Furthermore, the inequalities $m + n \leq 2(l - j) \leq 2m + n$ are equivalent to the inequalities $2l - 2m - n \leq j \leq 2l - m - n$ which in turn are equivalent to $a + b - l \leq 2j \leq a$. Hence we get

$$\mathrm{Sq}^a\mathrm{Sq}^b(x) = \sum_{\substack{j \text{ s.t.} \\ a+b-l\leq 2j\leq a}} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j(x).$$

Now observe that if $j \leq 0$, then $\mathrm{Sq}_j = 0$, so we may in addition assume in the sum that $j \geq 0$. Moreover, $2j < a+b-l$ is equivalent to $a+b-j > j+l$ in which case again $\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j(x) = 0$. Consequently, we obtain

$$\mathrm{Sq}^a\mathrm{Sq}^b(x) = \sum_{j=0}^{\lfloor a/2\rfloor} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j$$

giving the Adem relations, in fact for arbitrary a and b.

6.23. **Remark** Let us note that the condition $a < 2b$ is equivalent to the condition that for all $0 \leq j \leq \lfloor a/2 \rfloor$, we have $a + b - j \geq 2j$. A monomial $\mathrm{Sq}^k\mathrm{Sq}^l$ is called *admissible* if $k \geq 2l$; more generally, for a multi index $I = (i_1, \ldots, i_n)$, the monomial $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_n}$ is called admissible if $i_j \geq 2i_{j+1}$ for all j. Hence, the condition that $a < 2b$ gives that the above established Adem relation writes $\mathrm{Sq}^a\mathrm{Sq}^b$ as a sum of admissible monomials. It follows that *every* monomial in Steenrod squares can be written as a sum of admissible monomials.

Finally, we record the following property of the Steenrod squares:

6.24. **Lemma** *Let R be a commutative ring, $Y \xrightarrow{f} X$ a map and $\mathrm{C}(f)$ its mapping cone. Then $\widetilde{H}^*(\mathrm{C}(f); R)$ is an $H^*(X; R)$-module and for $x \in H^*(X; \mathbb{F}_2)$ and $a \in \widetilde{H}^*(\mathrm{C}(f); \mathbb{F}_2)$ and all $n \geq 0$, we have*

$$\mathrm{Sq}^n(a \cdot x) = \sum_{i=0}^{n} \mathrm{Sq}^i(a) \cdot \mathrm{Sq}^{n-i}(x).$$

Proof. The composite

$$Y \wedge X_+ \to X \wedge X_+ \to \mathrm{C}(f) \wedge X_+$$

is canonically null homotopic, since the functor $- \wedge T$ for any pointed space preserves pointed homotopies and $Y \to X \to \mathrm{C}(f)$ is pointed null homotopic. Then we consider the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \mathrm{C}(f) \arrow[d, dashed, "\bar{\Delta}"] \\
Y \wedge X_+ \arrow[r] & X \wedge X_+ \arrow[r] & \mathrm{C}(f) \wedge X_+
\end{tikzcd}$$

where the solid vertical maps are induced by the respective diagonal maps. Hence, the null homotopy of the lower composite induces the dashed arrow. This map has the property that the composite

$$\mathrm{C}(f) \xrightarrow{\bar{\Delta}} \mathrm{C}(f) \wedge X_+ \to \mathrm{C}(f) \wedge \mathrm{C}(f)_+$$

is itself induced by the diagonal of $\mathrm{C}(f)$. Then we define the module action by the composite

$$\widetilde{H}^*(\mathrm{C}(f); R) \otimes_R H^*(X; R) \to H^*(\mathrm{C}(f) \wedge X_+; R) \xrightarrow{\bar{\Delta}^*} H^*(\mathrm{C}(f); R).$$

Finally we find

$$\begin{aligned}
\mathrm{Sq}^n(a \cdot x) &= \mathrm{Sq}^n(\bar{\Delta}^*(a \wedge x)) \\
&= \bar{\Delta}^*(\mathrm{Sq}^n(a \wedge x)) \\
&= \bar{\Delta}^*[\sum_{i=0}^{n} \mathrm{Sq}^i(a) \wedge \mathrm{Sq}^{n-i}(x)] \\
&= \sum_{i=0}^{n} \bar{\Delta}^*(\mathrm{Sq}^i(a) \wedge \mathrm{Sq}^{n-i}(x)) \\
&= \sum_{i=0}^{n} \mathrm{Sq}^i(a) \cdot \mathrm{Sq}^{n-i}(x)
\end{aligned}$$

where we have used the naturality of Sq^n and the Cartan formula. □

Moving forward, it will be convenient to consider the following $\mathbb{F}_2$-algebra $\mathcal{A}^*$ commonly called the *Steenrod algebra*.

6.25. **Definition** The Steenrod algebra $\mathcal{A}^*$ is the free (non-commutative) algebra on the (graded) $\mathbb{F}_2$-module $\bigoplus_{n\geq 0} \mathbb{F}_2\{\mathrm{Sq}^n\}$, modulo the 2-sided ideal generated by the following elements:

- $1 + \mathrm{Sq}^0$, and
- For $i < 2j$, the element $R(i,j) = \mathrm{Sq}^i\mathrm{Sq}^j + \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$.

6.26. **Remark** By construction, $\mathcal{A}^*$ is a graded $\mathbb{F}_2$-algebra, the grading being induced from the grading $|\mathrm{Sq}^i| = i$ on the free algebra on the symbols Sq^i. Since the Adem relations preserve the grading, this indeed induces a grading of $\mathcal{A}^*$. We say that $\mathcal{A}$ is a graded connected algebra of finite type, which means that its degree 0 part is $\mathbb{F}_2$ and the degree n part is finite dimensional over $\mathbb{F}_2$ for each $n \geq 0$. Remark 6.23 says that the Steenrod algebra $\mathcal{A}^*$ is, as an $\mathbb{F}_2$-vector space, generated by admissible monomials. In fact, it turns out that the admissible monomials are also linearly independent[31], so that they form a basis of the $\mathbb{F}_2$-vector space $\mathcal{A}$. There are many more things that can be said about $\mathcal{A}^*$: For instance, it is a cocommutative (connected) graded Hopf algebra of finite type of $\mathbb{F}_2$ with the comultiplication $\psi\colon \mathcal{A}^* \to \mathcal{A}^* \otimes_{\mathbb{F}_2} \mathcal{A}^*$ determined by the formula

$$\psi(\mathrm{Sq}^n) = \sum_{i=1}^{n} \mathrm{Sq}^i \otimes \mathrm{Sq}^{n-i}.$$

The dual Steenrod algebra $\mathcal{A}_* = \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}^*, \mathbb{F}_2)$ is then a commutative (connected) graded Hopf algebra, and turns out to have underlying $\mathbb{F}_2$-algebra a polynomial algebra, much unlike

[31] A common way to prove this involves the computation of the cohomology of K_n.

$\mathcal{A}^*$ itself: as an $\mathbb{F}_2$-algebra, $\mathcal{A}^*$ is not polynomial. In fact, any element in $\mathcal{A}^*$ of positive degree is nilpotent (a result originally due to Milnor). For now, let us only hint at why this is.

For $n \geq 1$, let $\mathcal{A}(n)$ be the sub algebra of $\mathcal{A}^*$ generated by Sq^i for $i \leq 2^n$. Note that we have inclusion maps $\mathcal{A}^*(n) \to \mathcal{A}^*(n+1) \to \dots$ and that $\mathcal{A}^* = \mathrm{colim}_n \mathcal{A}^*(n) = \bigcup_n \mathcal{A}^*(n)$, i.e. the Steenrod algebra is the union of the sub algebras $\mathcal{A}^*(n)$. The decisive fact about these sub algebras $\mathcal{A}^*(n)$ is that they turn out to be *finite dimensional*, in fact of dimension $2^{\binom{n+2}{2}}$. It then also follows that every element of positive degree in $\mathcal{A}$ is nilpotent.

To move on, let us now record an algebraic lemma about binomial coefficients known as Lucas' theorem. We will use this theorem mainly for $p = 2$.

6.27. **Lemma** *Let $0 \leq k \leq n$ and p be a prime number. Let $k = \sum_{i=0}^l k_i p^i$ and $n = \sum_{i=0}^l n_i p^i$. Then*

$$\binom{n}{k} \equiv \prod_{i=0}^{l} \binom{n_i}{k_i} \mod p.$$

Let us also list some examples of (useful) Adem relations.

6.28. **Example** We have

$$\mathrm{Sq}^1\mathrm{Sq}^n = \begin{cases} \mathrm{Sq}^{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

Indeed, by the Adem relations, we have

$$\mathrm{Sq}^1\mathrm{Sq}^n = \sum_{j=0}^{\lfloor 1/2 \rfloor} \binom{n-j-1}{1-2j} \mathrm{Sq}^{n+1-j}\mathrm{Sq}^j = \binom{n-1}{1}\mathrm{Sq}^{n+1},$$

giving the desired result. Similarly, we have for $n \geq 2$

$$\mathrm{Sq}^2\mathrm{Sq}^n = \binom{n-1}{2}\mathrm{Sq}^{n+2} + \mathrm{Sq}^{n+1}\mathrm{Sq}^1$$

where concretely the binomial coefficient is non-zero if and only if $n - 1 \equiv 2(4)$. Again similarly, we have

$$\mathrm{Sq}^3\mathrm{Sq}^n = \binom{n-1}{3}\mathrm{Sq}^{n+3} + (n-1)\mathrm{Sq}^{n+2}\mathrm{Sq}^1.$$

Finally, we claim that for all $n \geq 1$ we have

$$\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0.$$

Indeed, this follows again from the Adem relations, using that the binomial coefficient

$$\binom{n-j-1}{2n-1-2j} = 0$$

for $j \leq n-1$ since then $2n-1-2j > n-1-j$.

6.29. **Terminology** Let us call Sq^k decomposable if it can be written as a sum of non-trivial products of Steenrod squares (necessarily of lower degree)[32]: $\mathrm{Sq}^k = \sum_i \mathrm{Sq}^{a_i}\mathrm{Sq}^{b_i}$ with $a_i, b_i > 0$. Sq^k is called indecomposable if it is not decomposable.

[32]This is the same as being indecomposable in the Steenrod algebra $\mathcal{A}^*$.

6.30. **Lemma** *Let $k \geq 1$. Then Sq^k is indecomposable if and only if $k = 2^n$ is a power of 2.*

Proof. Write $k = \sum_{i=0}^n a_i 2^i$ with $a_n \neq 0$. Let $b = 2^n$ and $a = k - b$. Then $a > 0$ if and only if k is not a power of 2, moreover $a < 2b$ and we have

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j.$$

From Lemma 6.27 above, we find that the binomial coefficient of the $j = 0$ summand $\binom{b-1}{a}$ is non-zero, so that $\mathrm{Sq}^k = \mathrm{Sq}^{a+b}$ is indeed decomposable.

Conversely, we claim Sq^{2^n} is indecomposable. To see this, consider $t \in H^1(\mathbb{RP}^\infty; \mathbb{F}_2)$. Then $\mathrm{Sq}^{2^n}(t^{2^n}) = t^{2^{n+1}} \neq 0$. However, for $0 < i < 2^n$, we have

$$\mathrm{Sq}^i(t^{2^n}) = \binom{2^n}{i} \cdot t^{2^n+i} = 0$$

where the first equality is shown in Example 6.20 and the second is for instance a consequence of Lemma 6.27, so Sq^{2^n} is indeed indecomposable. □

It follows that, as an $\mathbb{F}_2$-algebra, the Steenrod algebra $\mathcal{A}^*$ is generated by the elements Sq^{2^n} with $n \geq 0$. Further consequences of the above results are:

6.31. **Corollary** *Let X be a space with $H^*(X; \mathbb{F}_2) = \mathbb{F}_2[u]/u^m$ where $m \geq 3$[33]. Then $|u| = 2^n$ is a power of 2.*

Proof. By assumption, we have $\mathrm{Sq}^{|u|}(u) = u^2 \neq 0$. But since $H^k(X; \mathbb{F}_2) = 0$ for $|u| < k < 2|u|$, we find that $\mathrm{Sq}^{|u|}$ cannot be decomposable. Hence $|u| = 2^n$ by Lemma 6.30. □

6.32. **Corollary** *Assume that there exists a fibre bundle $S^{2n-1} \to S^n$ with typical fibre S^{n-1}. Then n is a power of 2.*

Proof. Such a fibre bundle is classified by a map $S^n \to B\mathrm{Homeo}(S^{n-1})$. There is a group homomorphism $\mathrm{Homeo}(S^{n-1}) \to \mathrm{Homeo}_0(D^n)$ obtained by thinking of $D^n = \mathrm{C}(S^{n-1})$ as the Cone on S^{n-1} and then simply coning off a homeomorphism. The resulting homeomorphism will in fact fix 0 as indicated. Since a homeomorphism preserves the boundary, there is also a group homomorphism $\mathrm{Homeo}_0(D^n) \to \mathrm{Homeo}(S^{n-1})$ obtained by restricting. This is a section to the map $\mathrm{Homeo}(S^{n-1}) \to \mathrm{Homeo}_0(D^n)$. Hence, under the given assumption, there is a D^n-fibre bundle $X \to S^n$ whose underlying sphere bundle is isomorphic to $S^{2n-1} \to S^n$. Note that X is a topological manifold with boundary given precisely by S^{2n-1} and that the projection $X \to S^n$ is a homotopy equivalence. Consider $M = D^{2n} \cup_{S^{2n-1}} X$, the space obtained from X by gluing in a disk.[34] Then M is a closed manifold of dimension $2n$ whose only non-trivial cohomology group outside of degrees 0 and $2n$ is given by $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$. By Poincaré duality, we find that $H^*(M; \mathbb{Z}) = \mathbb{Z}[u]/u^3$ with $|u| = n$. The claim hence follows from Corollary 6.31. □

6.33. **Remark** In fact, in both previous Corollaries one can strengthen the statement as follows: Not only is $|u|$ or n, respectively, a power of 2, in fact the only number that can appear are $1, 2, 4, 8$, and the statement even holds for fibrations, not necessarily fibre bundles. This is called the Hopf invariant one problem which was initially solved by Adams and later given a new proof using topological K-theory by Atiyah.

[33] $m = \infty$ is allowed, in which case we simply mean $\mathbb{F}_2[u]$.

[34] In particular, X is homotopy equivalent to the mapping cone of the projection map $S^{2n-1} \to S^n$.

Let us now prove some more results on unstable homotopy groups of spheres, extending our results from Corollary 6.17 and Remark 6.18.

6.34. **Proposition** *The Hopf maps η, ν, and σ satisfy that $\eta\Sigma\eta\colon S^4 \to S^2$, $\nu\Sigma^3\nu\colon S^{10} \to S^4$ and $\sigma\Sigma^7\sigma\colon S^{22} \to S^8$, and all their suspensions, are not null-homotopic.*

Proof. We prove the version for ν^2 here, the other cases are similar as we indicate at the end. To arrive at a contradiction let us assume that $\nu\Sigma^3\nu$ is null homotopic. This means that a dashed arrow in the following diagram exists

$$\begin{tikzcd} S^{10} \arrow[r, "\Sigma^3\nu"] & S^7 \arrow[r, "\nu"] \arrow[d, "i"] & S^4 \\ & \Sigma^3\mathbb{HP}^2 \arrow[ur, dashed, "\bar\nu"'] & \end{tikzcd}$$

such that $\nu \circ \Sigma^3\nu$ is homotopic to $\bar\nu \circ i \circ \Sigma^3\nu$. This is because $\Sigma^3\mathbb{HP}^2$ is the mapping cone of $\Sigma^3\nu$. Now let us consider the diagram of homotopy pushout squares:

$$\begin{tikzcd} S^7 \arrow[r] \arrow[d] & \Sigma^3\mathbb{HP}^2 \arrow[r] \arrow[d] & S^4 \arrow[d] \\ * \arrow[r] & S^{11} \arrow[r] \arrow[d] & \mathbb{HP}^2 \arrow[d] \\ & * \arrow[r] & X \end{tikzcd}$$

where X is defined as the homotopy pushout, that is, as the mapping cone of the map $S^{11} \to \mathbb{HP}^2$. This shows that the map $\mathbb{HP}^2 \to X$ induces an isomorphism on H^k for $k \leq 10$. The combined large right pushout square gives a homotopy cofibre sequence

$$S^4 \to X \to \Sigma^4\mathbb{HP}^2$$

showing that $X \to \Sigma^4\mathbb{HP}^2$ induces an isomorphism on $H^k(-;\mathbb{F}_2)$ for $k \geq 6$. Now we recall that $H^*(\mathbb{HP}^2;\mathbb{F}_2) = \mathbb{F}_2[u]/u^3$ with $|u| = 4$, and hence that $\mathrm{Sq}^4\colon H^4(\mathbb{HP}^2;\mathbb{F}_2) \to H^8(\mathbb{HP}^2;\mathbb{F}_2)$ is an isomorphism. Using naturality for the maps $\mathbb{HP}^2 \to X \to \Sigma^4\mathbb{HP}^2$, we obtain the following commutative diagram

$$\begin{tikzcd} & H^8(\Sigma^4\mathbb{HP}^2;\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^4"] \arrow[d, "\cong"] & H^{12}(\Sigma^4\mathbb{HP}^2;\mathbb{F}_2) \arrow[d, "\cong"] \\ H^4(X;\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^4"] \arrow[d, "\cong"] & H^8(X;\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^4"] \arrow[d, "\cong"] & H^{12}(X;\mathbb{F}_2) \\ H^4(\mathbb{HP}^2;\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^4"] & H^8(\mathbb{HP}^2;\mathbb{F}_2) & \end{tikzcd}$$

whose vertical maps are isomorphisms. This shows that $\mathrm{Sq}^4\mathrm{Sq}^4$ is non-trivial on $H^*(X;\mathbb{F}_2)$. However, the Adem relations give

$$\mathrm{Sq}^4\mathrm{Sq}^4 = \mathrm{Sq}^7\mathrm{Sq}^1 + \mathrm{Sq}^6\mathrm{Sq}^2.$$

Since $H^5(X;\mathbb{F}_2) = H^6(X;\mathbb{F}_2) = 0$, we find that

$$H^4(X;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^1} H^5(X;\mathbb{F}_2) \quad \text{and} \quad H^4(X;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^2} H^6(X;\mathbb{F}_2)$$

vanish. This contradicts that $\mathrm{Sq}^4\mathrm{Sq}^4$ is non-trivial on $H^*(X;\mathbb{F}_2)$. Hence, the assumption in the very beginning, that $\nu \circ \Sigma^3\nu$ is null-homotopic, leading to the existence of the space X, is false.

The same argument applies to any suspension of this map since we have used only the stable operations Sq^4. A similar argument applies to $\eta\Sigma\eta$ and its suspensions. Here one will use the Adem relation $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$. For $\sigma\Sigma^7\sigma$ and its suspensions one uses the Adem relation $\mathrm{Sq}^8\mathrm{Sq}^8 = \mathrm{Sq}^{15}\mathrm{Sq}^1 + \mathrm{Sq}^{14}\mathrm{Sq}^2 + \mathrm{Sq}^{12}\mathrm{Sq}^4$. □

6.35. **Remark** Note that $\eta \wedge \eta \colon S^6 \to S^4$ is homotopic to $\Sigma^2(\eta\Sigma\eta)$ and similarly $\nu \wedge \nu \simeq \Sigma^4(\nu\Sigma^3\nu) \colon S^{14} \to S^8$ and $\sigma \wedge \sigma \simeq \Sigma^8(\sigma\Sigma^7\sigma) \colon S^{30} \to S^{16}$. Hence we conclude that these maps are also not null homotopic.

As further example, we also consider some composites of different Hopf maps. These are not all cases one could consider, but are exemplary for the arguments.

6.36. **Proposition** *The maps $S^8 \xrightarrow{\Sigma^5\eta} S^7 \xrightarrow{\nu} S^4$ and $S^9 \xrightarrow{\Sigma^6\eta} S^8 \xrightarrow{\Sigma\nu} S^5$ are not null homotopic.*

Proof. As before, assume that the composite $S^8 \to S^7 \to S^4$ is null homotopic. Then ν factors (up to homotopy) as a composite $S^7 \to \Sigma^5\mathbb{CP}^2 \to S^4$. We may then compute the following interated homotopy pushout squares

$$\begin{tikzcd}
S^7 \arrow[r] \arrow[d] & \Sigma^5\mathbb{CP}^2 \arrow[r] \arrow[d] & S^4 \arrow[d] \\
* \arrow[r] & S^9 \arrow[r] \arrow[d] & \mathbb{HP}^2 \arrow[d] \\
 & * \arrow[r] & X
\end{tikzcd}$$

We therefore obtain two homotopy cofibre sequences:

$$S^9 \to \mathbb{HP}^2 \to X \to S^{10} \quad \text{and} \quad S^4 \to X \to \Sigma^6\mathbb{CP}^2 \to S^5$$

The first shows that $\mathbb{HP}^2 \to X$ induces an isomorphism on $H^k(-;\mathbb{F}_2)$ for $k \leq 8$ and the second that the map $X \to \Sigma^6\mathbb{CP}^2$ induces an isomorphism on $H^k(-;\mathbb{F}_2)$ for $k \geq 6$. It follows that the composite

$$H^4(X;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^4} H^8(X;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^2} H^{10}(X;\mathbb{F}_2)$$

is an isomorphism. But the Adem relations give $\mathrm{Sq}^2\mathrm{Sq}^4 = \mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1$. Now, $\mathrm{Sq}^5\mathrm{Sq}^1$ is trivial on X since X has trivial $H^5(-;\mathbb{F}_2)$. Moreover, Sq^6 is trivial on $H^4(-;\mathbb{F}_2)$ for degree reasons. The same argument applies to the composite $S^9 \to S^8 \to S^5$. □

6.37. **Remark** The above argument does not show that $\Sigma^2\nu\Sigma^7\eta \colon S^{10} \to S^6$ is not null homotopic, because there is no a priori reason that Sq^6 vanishes on the space X we build from a null homotopy. And indeed, by Freudenthal $\pi_{10}(S^6) \cong \pi_4(\mathbb{S})$ and this group turns out to vanish.

6.38. **Remark** The above argument does *not* show that the other composite $S^8 \xrightarrow{\Sigma\nu} S^5 \xrightarrow{\Sigma^2\eta} S^4$, and in fact similarly that $S^7 \xrightarrow{\nu} S^4 \xrightarrow{\Sigma\eta} S^3$ is not null homotopic. Indeed, running the same argument, assuming that the map is in fact null homotopic, one constructs a space X

on which $\mathrm{Sq}^4\mathrm{Sq}^2$ is non-zero. But the Adem relations don't allow to rewrite this in any other way which would lead to a contradiction.

It turns out, however, that $\Sigma(\eta)\nu$ is not null homotopic. Let us argue that this implies that also $(\Sigma^2\eta)\Sigma\nu$ is not null homotopic.[35] Indeed, this argument is much more simple (and in fact also proves that $\nu\Sigma^5\eta$ is not null-homotopic – but *not* that $\Sigma\nu\Sigma^6\eta$ is not null-homotopic as we have shown above. The map ν is part of a fibre sequence

$$S^3 \to S^7 \to S^4$$

and the fibre inclusion $S^3 \to S^7$ is of course null-homotopic (e.g. by cellular approximation). Hence the long exact sequence in homotopy groups splits into short exact sequences of the kind:

$$0 \to \pi_i(S^7) \xrightarrow{\nu_*} \pi_i(S^4) \xrightarrow{\partial} \pi_{i-1}(S^3) \to 0$$

This shows for instance that $\nu\Sigma^5(\eta)$ is not noll-homotopic, since we already know that $\Sigma^5\eta$ is not null homotopic. Now we observe that the map ∂ is induced by the map $\Omega S^4 \to S^3$ obtained from expanding the Hopf fibration once to the left. This map induces an isomorphism on π_3 by the long exact sequence. Now there is also a map $S^3 \to \Omega S^4$, the unit of the adjunction. By Freudenthal it also induces an isomorphism on π_3. This implies that the above short exact sequece in fact splits, i.e. that the map

$$\pi_i(S^7) \oplus \pi_{i-1}(S^3) \xrightarrow{\nu_*+\Sigma} \pi_i(S^4)$$

is an isomorphism and in particular that the suspension map $\pi_7(S^3) \to \pi_8(S^4)$ is injective. Hence, if $\nu\Sigma\eta$ is not null, then so is $\Sigma\nu\Sigma^2\eta$. We claim that $\pi_7(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $\Sigma\eta\nu$.[36] It follows that $\pi_8(S^4) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, generated by $\Sigma\nu\Sigma^2\eta$ and $(\Sigma^5\eta)\nu$.

6.39. **Remark** The argument from above really shows the following:

(1) $\pi_i(S^2) \cong \pi_i(S^3) \oplus \pi_{i-1}(S^1)$,
(2) $\pi_i(S^4) \cong \pi_i(S^7) \oplus \pi_{i-1}(S^3)$, and
(3) $\pi_i(S^8) \cong \pi_i(S^{15}) \oplus \pi_{i-1}(S^7)$

where the maps from right to left are given by composition with the appropriate Hopf maps plus the suspension homomorphism. In fact, there are equivalences of spaces

$$\Omega S^2 \simeq S^1 \times \Omega S^3, \quad \Omega S^4 \simeq S^3 \times \Omega S^7, \quad \Omega S^8 \simeq S^7 \times \Omega S^{15}.$$

which induce the above isomorphisms on homotopy groups (exercise).

There is a further construction which we will use later. Let us denote by $\widehat{A}^*$ the completed Steenrod algebra, where we complete at the ideal of positively graded elements in $\mathcal{A}^*$. In $\widehat{\mathcal{A}}^*$, the element

$$\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \mathrm{Sq}^3 + \dots$$

is well-defined. Moreover, it follows formally that this element is left and right invertible and hence invertible. Let us denote its inverse by Sq^{-1}. Indeed, one can inductively solve the equations $\mathrm{Sq} \cdot (1+x) = 1 = (1+x) \cdot \mathrm{Sq} = 1$ for $x \in \widehat{A}^*$.

[35] I learned these arguments from Achim Krause, but they are well-known and have all been worked out by Toda in his seminal computations of homotopy groups of spheres.

[36] Perhaps we will argue next term why that is. Essentially, there is a generalized Hopf invariant which can be computed to be non-zero on $\Sigma\eta\nu$, showing that $\Sigma\eta\nu$ is not null.

6.40. **Example** Writing out low degree terms gives

$$\mathrm{Sq}^{-1} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1 + \dots$$

One can be more explicit, however. To that end let us set $\chi(\mathrm{Sq}^0) = \mathrm{Sq}^0$ and inductively set for $n \geq 1$:

$$\chi(\mathrm{Sq}^n) = \sum_{i=1}^{n} \mathrm{Sq}^i \chi(\mathrm{Sq}^{n-i}).$$

6.41. **Lemma** *We have the equality* $\chi(\mathrm{Sq}) = \sum_{n\geq 0} \chi(\mathrm{Sq}^n) = \mathrm{Sq}^{-1}$ *in* $\widehat{\mathcal{A}}^*$.

Proof. Note that the definition of χ implies that for all $n \geq 1$, we have

$$\sum_{i=0}^{n} \mathrm{Sq}^i \chi(\mathrm{Sq}^{n-i}) = 0.$$

Moreover, since we already know Sq to be invertible in $\widehat{\mathcal{A}}^*$, it suffices to show that $\chi(\mathrm{Sq})$ is a one-sided inverse of Sq. To that end, we compute

$$\begin{aligned}
\mathrm{Sq} \cdot \chi(\mathrm{Sq}) &= (\sum_{i\geq 0} \mathrm{Sq}^i) \cdot (\sum_{j \geq 0} \chi(\mathrm{Sq}^j)) \\
&= \sum_{i,j\geq 0} \mathrm{Sq}^i \chi(\mathrm{Sq}^j) \\
&= \sum_{n\geq 0} \sum_{i=0}^{n} \mathrm{Sq}^i \chi(\mathrm{Sq}^{n-i}) \\
&= \mathrm{Sq}^0 \chi(\mathrm{Sq}^0) = 1
\end{aligned}$$

as needed. □

6.42. **Remark** Note that the invertibility of Sq implies that also $\chi(\mathrm{Sq}) \cdot \mathrm{Sq} = 1$. This is perhaps less obvious from the definition and implies that $\chi(\mathrm{Sq}^n) = \sum_{i=1}^{n} \chi(\mathrm{Sq}^{n-i})\mathrm{Sq}^i$.

6.43. **Remark** Part of the definition of a Hopf algebra is a map called the antipode. The above definition of χ extends in at most one way to an map $\chi(\mathcal{A}^*)^{\mathrm{op}} \to \mathcal{A}^*$ of algebras and it turns out that this map is the antipode of the Hopf algebra $\mathcal{A}^*$.

7. The theorem of Leray–Hirsch

We will phrase the entire section in terms of fibrations. We will indicate the non-trivial statements we use here to prove our main results. If we restrict ourselves to fibre bundles rather than fibrations, many of the statements we use for fibrations become much simpler to prove (similarly to as we have seen in the classification of bundles: we have given a full argument for the classification of fibre bundles, but not for that of fibrations) and most of the standard textbooks provide references where these arguments are carried out in detail.

We begin with the theorem of Leray–Hirsch. The setup we want to put this in is as follows. Consider a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r, "\pi"] \arrow[d] & B \arrow[d, equal] \\ \overline{F} \arrow[r] & \overline{E} \arrow[r, "\overline{\pi}"] & B \end{tikzcd}$$

consisting of horizontal fibre sequences, that is π and $\overline{\pi}$ are fibrations with typical fibre F and $\overline{F}$ and assume that B is connected for simplicity. Further, let R be a commutative ring.

7.1. **Theorem** *Assume that there are homogenuous classes $\{x_i\}_{i\in I} \subseteq H^*(\overline{E}, E; R)$ such that there are only finitely many x_i of the same degree and such that the images y_i of the x_i under the map $H^*(\overline{E}, E; R) \to H^*(\overline{F}, F; R)$ form a basis of the graded R-module $H^*(\overline{F}, F; R)$. Then sending y_i to x_i induces a unique $H^*(B; R)$-module map*

$$\Phi_\pi \colon H^*(\overline{F}, F; R) \otimes_R H^*(B; R) \to H^*(\overline{E}, E; R).$$

This map is an isomorphism of $H^*(B; R)$-modules.

7.2. **Remark** Note that the relative cohomologies appearing above are isomorphic to the cohomologies of the respective mapping cones. The map Φ_π then uses that by Lemma 6.24, $H^*(\overline{E}, E; R)$ is an $H^*(\overline{E}; R)$ and hence via $\overline{\pi}^*$ also an $H^*(B; R)$-module. Moreover, note that F and E may also well be chosen to be empty.

7.3. **Remark** Warning: The map Φ_π is not a map of R-algebras, at least not in any evident way. And in fact, in many instances where the above theorem applies, the multiplications on source and target of Φ_π indeed do not agree. We will come back to this later when discussing examples.

Proof of Theorem 7.1. We first observe that the map Φ_π induces the following maps. For every CW complex X equipped with a map $f\colon X \to B$, let us denote by $E_{|X}$ the pullback $f^*(E)$. Doing so, we obtain a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E_{|X} \arrow[r, "\pi_X"] \arrow[d] & X \arrow[d, equal] \\ \overline{F} \arrow[r] & \overline{E}_{|X} \arrow[r, "\overline{\pi}_X"] & X \end{tikzcd}$$

consisting of horizontal fibre sequences and elements $f^*(x_i) \in H^*(\overline{E}_{|X}, E_{|X}; R)$; here we (by abuse of notation) denote the map $H^*(\overline{E}, E; R) \to H^*(\overline{E}_{|X}, E_{|X}; R)$ which is induced by f by f^*. The restrictions of the classes to the fibres are simply given by the elements y_i, and so by assumption again form a basis. Hence we obtain by the same reasoning as above the map

$$\Phi_{\pi_X} \colon H^*(\overline{F}, F; R) \otimes_R H^*(X; R) \to H^*(\overline{E}_{|X}, E_{|X}; R).$$

Moreover, both the source and the target of this map form a functor on the category of CW complexes equipped with a map to B, and the maps Φ_{π_X} are readily checked to form the components of a natural transformation. We will now show that this map is a natural isomorphism. Once this is shown, we may choose a weak equivalence $X \to B$ with X a CW complex. Then the claim of the theorem follows from the fact that singular cohomology sends weak equivalences to isomorphisms as follows from Theorem 2.1.

We now aim to prove that Φ_{π_X} is an isomorphism essentially by induction over the cells. To that end, let us first consider the case where X is a disjoint union of points. Then the result is true by assumption (this uses the assumption that $H^*(\overline{F}, F; R)$ is degree-wise *finite* free over R). Let us now prove by induction over the dimension of X that the map Φ_{π_X} is an isomorphism for finite dimensional X. To do so, in the inductive step, we consider a pushout

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod D^n & \longrightarrow & X \end{array}$$

which exhibits X as n-cells attached to its $(n-1)$-skeleton. We will now use the following fact, namely that the diagrams

$$\begin{array}{ccc} E_{|\coprod S^{n-1}} & \longrightarrow & E_{|X_{n-1}} \\ \downarrow & & \downarrow \\ E_{|\coprod D^n} & \longrightarrow & E_{|X} \end{array} \qquad \begin{array}{ccc} \overline{E}_{|\coprod S^{n-1}} & \longrightarrow & \overline{E}_{|X_{n-1}} \\ \downarrow & & \downarrow \\ \overline{E}_{|\coprod D^n} & \longrightarrow & \overline{E}_{|X} \end{array}$$

are homotopy pushouts.[37] In particular, upon passing to homotopy cofibres (i.e. mapping cones) of the maps from the left square to the right square, we again obtain a homotopy pushout square. Let us now denote the source of Φ by $F(-)$ and the target of Φ by $G(-)$. Then, the above diagrams together with the naturality of the maps Φ give rise to the commutative diagram of horizontal exact sequences

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & F(X) & \longrightarrow & F(X_{n-1}) \oplus F(\coprod D^n) & \longrightarrow & F(\coprod S^{n-1}) & \longrightarrow & F(X)[1] & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & G(X) & \longrightarrow & G(X_{n-1}) \oplus G(\coprod D^n) & \longrightarrow & G(\coprod S^{n-1}) & \longrightarrow & G(X)[1] & \longrightarrow & \dots \end{array}$$

here the $(-)[1]$ denotes the appropriate shifting operation on graded R-modules (which is where the functors F and G naturally take values). The upper sequence comes from the long exact sequence of the pushout describing X; note that $H^*(\overline{F}, F; R)$ is R-free which implies that the tensored sequence is indeed again exact. The lower sequence comes from the homotopy pushout square of mapping cones described above. The two squares from the left commute by naturality of the maps Φ. The right square in fact also commutes, this is a direct computation. Now note that the vertical maps are isomorphisms at the places involving X_{n-1} and $\coprod S^{n-1}$ by the inductive hypothesis, and at the place involving $\coprod D^n$ since D^n is contractible, and hence the map under investigation is the isomorphic to the one for a disjoint union of points which we have already observed to be an isomorphism (essentially by assumption, in fact). By the 5-lemma, we conclude that the map left vertical displayed map above is also an isomorphism. This shows that the components of Φ_π on finite dimensional CW complexes are isomorphisms. So finally assume that X is a general CW complex. We may write $X = \operatorname{colim}_n X_n$ where X_n is the n-skeleton. By another application of the fact that

[37] This is a result called "fibrations satisfy descent" or "fibrations glue" and is again a consequence of the straightening-unstraightening equivalence we have indicated when talking about the classification of fibrations. See Remark 7.4 for a workaround in the given situation.

fibrations glue, we find that $E_{|X} = \mathrm{hocolim}_n E_{|X_n}$ and $\overline{E}_{|X} = \mathrm{hocolim}_n \overline{E}_{|X_n}$.[38] Now, since the maps $X_n \to X$ as well as the maps $E_{|X_n} \to E_{|X}$ and $\overline{E}_{|X_n} \to \overline{E}_{|X}$ become more and more connected as n grows, it follows that the canonical maps

$$H^*(X;R) \to \lim_n H^*(X_n;R) \quad \text{and} \quad H^*(\overline{E}_{|X}, E_{|X};R) \to \lim_n H^*(\overline{E}_{|X_n}, E_{|X_n};R)$$

are isomorphisms, and in fact in each cohomlogical degree, the inverse limit is eventually constant. Since $H^*(\overline{F}, F;R)$ is a degree-wise finite R-free module, this implies that the map

$$H^*(\overline{F}, F;R) \otimes_R H^*(X;R) \to \lim_n \left[H^*(\overline{F}, F;R) \otimes_R H^*(X_n;R)\right]$$

is also an isomorphism. Hence, the theorem finally follows from the commutative square

$$\begin{tikzcd}
H^*(\overline{F}, F;R) \otimes_R H^*(X;R) \arrow[r, "\Phi_{\pi_X}"] \arrow[d, "\cong"'] & H^*(\overline{E}_{|X}, E_{|X};R) \arrow[d, "\cong"] \\
\lim_n \left[H^*(\overline{F}, F;R) \otimes_R H^*(X;R)\right] \arrow[r, "\lim_n \Phi_{\pi_{X_n}}", "\cong"'] & \lim_n H^*(\overline{E}_{|X_n}, E_{|X_n};R)
\end{tikzcd}$$

since the vertical maps are isomorphisms as we have just explained, and the lower horizontal map is an inverse limit of isomorphisms by the previous step, and hence also an isomorphism. □

7.4. **Remark** We briefly comment on the use of "fibrations satisfy descent" in the above argument, as we in fact can work around it using the following trick having to do with the fact that the homotopy pushouts we consider are quite special ones. Indeed, note that when X is obtained from X_{n-1} by attaching n-cells, we can form an open cover $X = U \cup V$ by letting U be a thickening of X_{n-1} in X and V to be the images of the open cells under the attaching maps. This yields a pushout description of X which is homotopy equivalent to the cell-attachment pushout for X. Now, the pullback of an open cover is again an open cover, so we see that $E_{|X}$ admits an open cover of the form $E_{|U} \cup E_{|V}$. Moreover, the $E_{|U} \simeq E_{|X_n}$ and $E_{|V} \simeq E_{|\coprod D^n}$, yielding the claimed homotopy pushouts.

Similarly, there are open sets $X_n \subseteq U_n \subseteq X$ such that the inclusion $X_n \to U_n$ is a deformation retraction (in particular a homotopy equivalence) and such that the U_n's form an increasing sequence of opens in X. Then $E_{|X} = \bigcup_n E_{|U_n}$. As before, the maps $E_{|U_n} \to E_{|X}$ are become more and more connected as n grows, so we may now use the interaction of cohomology with increasing open filtrations which have this connectivity properties.

7.5. **Example** Suppose $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration with homotopy retraction $r \colon E \to F$, i.e. so that $ri \simeq \mathrm{id}_F$[39] and that $H^*(F;R)$ is R-free of finite type with basis $y_{i_{i \in I}}$. Considering the classes $x_i = r^*(y_i)$ we see that we can apply Leray–Hirsch and in addition that the isomorphism

$$H^*(F;R) \otimes_R H^*(B;R) \to H^*(E;R)$$

is a map of $H^*(B;R)$-algebras and hence an isomorphism of $H^*(B;R)$-algebras. Indeed, this follows simply from the fact that it is the extension of scalars (along $R \to H^*(B;R)$) of the R-algebra homomorphism $r^* \colon H^*(F;R) \to H^*(E;R)$. In particular, we deduce a cohomological Künneth result for products $F \times B$ in case the R-cohomology of F is R-free of finite type.

[38] Here, the hocolim is the construction dual to holim which we have briefly discussed earlier, a so called mapping telescope. It behaves formally very similar to the skeletal filtration of a CW complex.

[39] Exercise: Show that the map $(i,p) \colon E \to F \times B$ is an equivalence.

Let us use the theorem of Leray–Hirsch to compute some further cohomology groups.

7.6. **Theorem** *Let $n \geq 1$. Then there are isomorphisms of rings*
(1) $H^*(\mathrm{U}(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[e_1, e_3, \dots, e_{2n-1}]$,
(2) $H^*(\mathrm{SU}(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[e_3, \dots, e_{2n-1}]$, *and*
(3) $H^*(\mathrm{Sp}(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[f_3, f_7, \dots, f_{4n-1}]$.

Proof. In all cases one argues inductively and in all cases the induction starts at $n = 1$ where we have $\mathrm{U}(1) \cong S^1$, $\mathrm{SU}(1) = \mathrm{Sp}(1) = S^3$ which evidently have the claimed cohomology rings. We discuss now the inductive step of case (1); the other cases are done in the exact same way. To that end, we recall that there is a fibre sequence

$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to S^{2n-1}$$

from which, using the relative Hurewicz theorem and the universal coefficient theorem, we deduce that $H^k(\mathrm{U}(n);\mathbb{Z}) \to H^k(\mathrm{U}(n-1);\mathbb{Z})$ is an isomorphism for $k \leq 2n-3$. In particular, we can lift all exterior generators of $H^*(\mathrm{U}(n-1);\mathbb{Z})$ to elements which we again call e_{2i-1} in the cohomology of $\mathrm{U}(n)$. It follows that we can lift a basis of the $\mathbb{F}_2$-vector space $H^*(\mathrm{U}(n-1);\mathbb{F}_2)$ to $H^*(\mathrm{U}(n);\mathbb{F}_2)$. Using Leray–Hirsch and the inductive hypothesis, we obtain an isomorphism of $[\Lambda_{\mathbb{Z}}[e_{2n-1}] = H^*(S^{2n-1};\mathbb{Z})]$-modules

$$\Lambda_{\mathbb{Z}}[e_1, \dots, e_{2n-3}] \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}[e_{2n-1}] \to H^*(\mathrm{U}(n);\mathbb{Z})$$

which shows the claimed statement additively. Since the exterior algebra over $\mathbb{Z}$ is torsion free and since the e_{2i-1} have odd degree, we deduce that, in $H^*(\mathrm{U}(n);\mathbb{Z})$, we have $e_{2i-1}^2 = 0$. This implies that the map $\Lambda_{\mathbb{Z}}[e_1, \dots, e_{2n-3}] \to H^*(\mathrm{U}(n);\mathbb{Z})$ obtained by sending e_i to e_i is a *ring* homomorphism. Consequently, as discussed in Example 7.5 we deduce that the isomorphism of $\Lambda_{\mathbb{Z}}[e_{2n-1}]$-modules is in fact one of $\Lambda_{\mathbb{Z}}[e_{2n-1}]$-algebras, and we obtain the claimed statement. □

7.7. **Remark** The case of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are less direct. Let us indicate why that is. Running the same inductive argument as above, let us try to prove that (additively) we have $H^*(\mathrm{SO}(n);\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}[a_1, \dots, a_{n-1}]$.[40] So let us consider the fibre sequence

$$\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}$$

and note that it implies that the map $H^k(\mathrm{SO}(n);\mathbb{F}_2) \to H^k(\mathrm{SO}(n-1);\mathbb{F}_2)$ is an isomorphism for $k < n-2$; indeed by the relative Hurewicz and UCT, there is an exact sequence

$$H^{n-2}(\mathrm{SO}(n);\mathbb{F}_2) \to H^{n-2}(\mathrm{SO}(n-1);\mathbb{F}_2) \to H^{n-1}(S^{n-1};\mathbb{F}_2) \cong \mathbb{F}_2 \to H^{n-1}(\mathrm{SO}(n);\mathbb{F}_2)$$

and we need to come up with an argument that the elements $a_{n-2} \in H^{n-2}(\mathrm{SO}(n-1);\mathbb{F}_2)$ can in fact be lifted.

Moreover, the ring structure on $H^*(\mathrm{SO}(n);\mathbb{F}_2)$ surely is not exterior: Consider the case $\mathrm{SO}(3) \cong \mathbb{RP}^3$ for instance.

We will now discuss another application of the theorem of Leray–Hirsch, namely the splitting principle. To state it, let us recall that there is a canonical continuous group homomorphism $\mathrm{GL}_{n+1}(\mathbb{K}) \to \mathrm{Homeo}(\mathbb{KP}^n)$. Hence any $\mathbb{K}$-vector bundle $\pi\colon E \to B$ admits a fibrewise projectivization $p\colon \mathbb{P}(E) \to B$, a fibre bundle with typical fibre $\mathbb{KP}^n$, which is classified by the composite

$$B \to \mathrm{BGL}_{n+1}(\mathbb{K}) \to \mathrm{BHomeo}(\mathbb{KP}^n).$$

[40] This turns out to be correct, after all.

7.8. **Proposition** *Let $\pi\colon E \to B$ be a $\mathbb{K}$-vector bundle of $\mathbb{K}$-rank n and $p\colon \mathbb{P}(E) \to B$ be its projectivization. Let R be a commutative ring, which is an $\mathbb{F}_2$-algebra if $\mathbb{K} = \mathbb{R}$. Then $p^*(E)$ splits as a sum of vector bundles $L \oplus Q$, where L is a $\mathbb{K}$-line bundle. Moreover, the map $H^*(B;R) \to H^*(\mathbb{P}(E);R)$ is injective.*

Proof. Pick an open cover $\{U_i\}_{i\in I}$ which trivializes π. Then $\mathbb{P}(E) \to B$ is also trivial over $\{U_i\}$ and $p^*(E)_{|p^{-1}(U_i)} \cong U_i \times \mathbb{KP}^{n-1} \times \mathbb{K}^n$. Recall that the tautological bundle $\gamma_{\mathbb{K}}$ on $\mathbb{KP}^{n-1}$ is the sub bundle of the trivial bundle $\mathbb{KP}^{n-1} \times \mathbb{K}^n$ consisting of the pairs (L,x) where $x \in L$. We then define $L \subseteq p^*(E)$ to satisfy $L_{|p^{-1}(U_i)} = U_i \times \gamma_{\mathbb{K}} \subseteq U_i \times \mathbb{KP}^{n-1} \times \mathbb{K}^n$. It is readily checked that L indeed defines a subset of $p^*(E)$ which is a sub $\mathbb{K}$-line bundle. That is, $L \to \mathbb{P}(E)$ is a $\mathbb{K}$-line bundle, and the inclusion $L \to p^*(E)$ is fibrewise $\mathbb{K}$-linear. Then we may use an appropriate choice of (fibrewise) scalar product[41] on E to split $p^*(E)$ as $L \oplus L^\perp$ so that $Q = L^\perp$ is again a $\mathbb{K}$-vector bundle.

Now, by construction, the $\mathbb{K}$-line bundle $L \to \mathbb{P}(E)$ becomes isomorphic to the tautological $\mathbb{K}$-line bundle $\gamma_{\mathbb{K}}$ over $\mathbb{KP}^n$ when pulled back along the fibre inclusion $\mathbb{KP}^n \to \mathbb{P}(E)$. Let x be $w_1(L)$ if $\mathbb{K} = \mathbb{R}$, $c_1(L)$ is $\mathbb{K} = \mathbb{C}$ or $q_1(L)$ if $\mathbb{K} = \mathbb{H}$; see Examples 4.38, 4.37, and 4.39. Then the restricion of x to the fibre $\mathbb{KP}^n$ becomes a polynomial generator of the cohomology ring of $\mathbb{KP}^n$ which in all three cases is polynomial over R; here, we use that R is an $\mathbb{F}_2$-algebra if $\mathbb{K} = \mathbb{R}$. Hence we deduce that the elements $1 = x^0, x, x^2, \dots, x^n$ form elements in $H^*(\mathbb{P}(E);R)$ whose restriction to $H^*(\mathbb{KP}^n;R)$ form an R-basis. Theorem 7.1 then implies in particular that the map $H^*(B;R) \to H^*(\mathbb{P}(E);R)$ is injective, in fact a direct summand inclusion. $\square$

The following statement is often referred to as the *splitting principle*.

7.9. **Corollary** *Let $\pi\colon E \to B$ be a $\mathbb{K}$-vector bundle and R be a commutative ring which is a $\mathbb{F}_2$-algebra if $\mathbb{K} = \mathbb{R}$. Then there exists a map $f\colon X \to B$ such that*

(1) *the map $f^*\colon H^*(B;R) \to H^*(X;R)$ is injective, and*
(2) *the vector bundle $f^*(E)$ is isomorphic to a sum of $\mathbb{K}$-line bundles.*

Proof. Apply Proposition 7.8 to π and then to $Q \to \mathbb{P}(E)$ and so on. $\square$

7.10. **Remark** Corollary 7.9 implies the following. Let $E \to B$ be a $\mathbb{K}$-vector bundle of rank n. Then there exists a (homotopy) commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathrm{B}(\mathrm{GL}_1(\mathbb{K}))^{\times n} \arrow[d] \\ B \arrow[r] & \mathrm{BGL}_n(\mathbb{K}) \end{tikzcd}$$

with f^* injective. Let us denote by B' the homotopy pullback of the diagram obtained by removing X. Then we have a canonical map $X \to B'$ whose composite with $B' \to B$ is f. It follows that the map $H^*(B;R) \to H^*(B';R)$ is also injective, so we may think of B' as the minimal choice for X appearing in the splitting principle.

7.11. **Remark** In this remark, we want to take the above perspective more seriously and think about a variant of the above observation for general compact connected Lie groups G. Indeed, for such, let $T \subseteq G$ be a sub Lie group, for instance a maximal torus.[42] Recall that the

[41] That is, we choose a reduction of the structure group of E to the isometry group of $\mathbb{K}^n$.

[42] That is, a Lie subgroup of the form $\mathrm{U}(1)^{\times n}$ for a maximal number n.

homotopy fibre of $BT \to BG$ identifies with G/T, see Example 4.33. In case T is a maximal torus, it turns out that $H^*(G/T;\mathbb{Z})$ is degree-wise finitely generated free and concentrated in even degrees; this is a theorem of Bott–Samelson [BS55]. Moreover, for appropriate rings R, the map $H^*(BT;R) \to H^*(G/T;R)$ is surjective, e.g. when $G = \mathrm{U}(n), \mathrm{Sp}(n)$ the ring $\mathbb{Z}$ works.[43] The same results turn out to be true when $G = \mathrm{O}(n)$, $T = \mathrm{O}(1)^{\times n}$, and R is an $\mathbb{F}_2$-algebra or when $G = \mathrm{Sp}(n)$, $T = \mathrm{Sp}(1)^{\times n}$ and $R = \mathbb{Z}$. Let us from now on assume that this is the case.

Then let $B \to BG$ be a map classifying a principle G-bundle over B and consider the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & BT \arrow[d] \\ B \arrow[r] & BG \end{tikzcd}$$

so that $X \to B$ again has homotopy fibre G/T. It is perfectly fine to consider the case $B = BG$ and the map being the identity. From the composite

$$H^*(BT;R) \to H^*(X;R) \to H^*(G/T;R)$$

we deduce that the latter map is again surjective and hence we may apply the theorem of Leray–Hirsch to deduce that the map

$$H^*(G/T;R) \otimes_R H^*(B;R) \to H^*(X;R)$$

is an isomorphism. In particular, we find that $H^*(B;R) \to H^*(X;R)$ is injective. In the case where $G = \mathrm{U}(n)$, its maximal torus is $\mathrm{U}(1)^{\times n}$ and we obtain a direct argument for the splitting principle. In case $G = \mathrm{Sp}(n)$ it turns out that its maximal torus is again $\mathrm{U}(1)^{\times n}$, so we obtain the following strengthening of the splitting principle for quaternionic bundles: Not only is the pullback $f^*(E)$ appearing in Corollary 7.9 a sum of $\mathbb{H}$-line bundles, in fact, after suitable choice of X, it is a sum of $\mathbb{H}$-line bundles which are obtained from $\mathbb{C}$-line bundles via $- \otimes_{\mathbb{C}} \mathbb{H}$.[44] Exercise: Prove that such an X can be found just from Corollary 7.9.

7.12. **Remark** One can also use the theorem of Leray–Hirsch to compute the cohomology rings of the classifying spaces $\mathrm{BU}(n)$, $\mathrm{BSp}(n)$ and $\mathrm{BO}(n)$; the latter with $\mathbb{F}_2$-coefficients, but the argument involves several applications of Leray–Hirsch in inductive manners. We will instead compute these cohomologies via the Serre spectral sequence next term and perhaps briefly discuss the relation to the approach using Leray–Hirsch.

We end this section with some further applications of what we proven this term.

7.13. **Proposition** *Suppose that there is a fibration sequence $\mathbb{CP}^n \to \mathbb{CP}^\infty \to X$, where the first map is the canonical map. Then n is either 0 or 1. Conversely, in these cases, a fibration as indicated exists.*

Proof. The fibre of $\mathrm{id}\colon \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ is indeed $\mathbb{CP}^0$. Moreover, the fibre of $\mathbb{CP}^\infty \to \mathbb{HP}^\infty$ is given by $S^3/S^1 \cong \mathbb{CP}^1$, by **??**. So we need to argue the converse and assume that there is

[43] In case T is a maximal torus, one sufficient criterion for such R is that $H^*(BG;R)$ is concentrated in even degrees – this turns out to be true for $\mathbb{R} = \mathbb{Z}$ and $G = \mathrm{U}(n), \mathrm{Sp}(n)$ and can for instance be deduced from explicit cell structures on the Grassmannians from Example 4.41, see e.g. [Gha19]

[44] Note that $\mathbb{H}$ is indeed a $(\mathbb{C},\mathbb{H})$-bimodule, so the indicated symbol provides a canonical functor from $\mathbb{C}$-vector spaces to $\mathbb{H}$-vector spaces.

a fibration as written in the proposition. The map $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ is surjective, so we may apply the Leray–Hirsch thereom. We deduce that the ring map $H^*(X;\mathbb{Z}) \to H^*(\mathbb{CP}^\infty;\mathbb{Z})$ is additively the inclusion of a direct summand, and writing out the Leray–Hirsch isomorphism, we find that $H^*(X;\mathbb{Z}) = \mathbb{Z}[u]$ with $|u| = 2n$. Steenrod operations show that this can only be the case when $n = 0, 1$. □

7.14. **Proposition** *Suppose that there is a fibration sequence $\mathbb{CP}^1 \to \mathbb{CP}^3 \to X$. Then $X \simeq S^4$. Conversely, a fibration sequence $\mathbb{CP}^1 \to \mathbb{CP}^3 \to S^4$ exists.*

Proof. We first show that a fibration as claimed exists. We write $S^4 = S^7/S^3$, we deduce from the canonical inclusion $S^1 \subseteq S^3$[45] that there is a tautological map $\mathbb{CP}^3 \to \mathbb{HP}^1$. Its homotopy fibre F is then again given by $S^3/S^1 = \mathbb{CP}^1$ as needed. Indeed, this follows from the diagram consisting of horizontal and vertical fibre sequences:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & S^3/S^1 \arrow[d] \\
S^7 \arrow[r] \arrow[d, equal] & \mathbb{CP}^3 \arrow[r] \arrow[d] & \mathbb{CP}^\infty \arrow[d] \\
S^7 \arrow[r] & \mathbb{HP}^1 \arrow[r] & \mathbb{HP}^\infty
\end{tikzcd}$$

Conversely, applying again Leray–Hirsch reveals that $H^*(X;\mathbb{Z}) \simeq H^*(S^4;\mathbb{Z})$, and the long exact sequence in homotopy groups shows that $\pi_1(X) = 1$. It follows (though this is somewhat non-trivial unless we assume X to have finitely generated homology to begin with) that $H_*(X;\mathbb{Z}) \cong H_*(S^4;\mathbb{Z})$, so the Hurewicz theorem implies that $X \simeq S^4$ as needed. □

7.15. **Proposition** *Let M be the pullback $\mathbb{CP}^3 \to S^4 \leftarrow S^4$ with right hand map being a map of degree k and the left hand map being the fibration from Proposition 7.14. Compute the cohomology ring $H^*(M;\mathbb{Z})$.*

Proof. We consider the (pullback) diagram of fibration sequences

$$\begin{tikzcd}
\mathbb{CP}^1 \arrow[r] \arrow[d, equal] & \mathbb{CP}^3 \arrow[r, "q"] \arrow[d, "f"] & S^4 \arrow[d, "\cdot k"] \\
\mathbb{CP}^1 \arrow[r] & M \arrow[r, "p"] & S^4.
\end{tikzcd}$$

Since the bottom sequence is pulled back from the top one, and the top one satisfies the assumptions of Leray–Hirsch, so does the bottom. Specifically, let $x = f^*(c_1) \in H^2(M;\mathbb{Z})$ be pullback of the tautological class in $H^2(\mathbb{CP}^3;\mathbb{Z})$. Then we obtain an induced isomorphism of $H^*(S^4;\mathbb{Z})$-modules

$$H^*(\mathbb{CP}^1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^4;\mathbb{Z}) \to H^*(M;\mathbb{Z}).$$

In particular, the cohomology of M, additively, looks like that of $\mathbb{CP}^3$ and we can name generators of all non-trivial cohomology groups: We have $1 \in H^0(M;\mathbb{Z})$, $x \in H^2(M;\mathbb{Z})$, $p^*(i_4) \in H^4(M;\mathbb{Z})$ and $x \cdot p^*(i_4) \in H^6(M;\mathbb{Z})$ all constitute generators. Since $p^*(i_4)^2 = 0$, we have in fact already computed all products of generators, except $x^2 \in H^4(M;\mathbb{Z})$, so to compute the cohomology ring of M it suffices to do that. To that end, we write

$$x^2 = f^*(c_1)^2 = f^*(c_1^2) = f^*(q^*(i_4)) = kp^*(i_4).$$

[45] or equivalently $\mathrm{U}(1) \subseteq \mathrm{Sp}(1)$.

□

7.16. **Proposition** *Let G be a finite p-group. Then $H^*(B^nG; R) = R$ if $p \in R^\times$.*

Proof. As in the case above, inductively, it suffices to show the case $n = 1$. It is a classical fact from algebra that a finite p-group has non-trivial center, and hence there is a fibre sequence

$$BC(G) \to BG \to B(G/C(G)).$$

If we can show that $H^*(BC(G); R) = R$, then we may apply Leray–Hirsch and obtain an isomorphism

$$H^*(B(G/C(G); R) \xrightarrow{\cong} H^*(BG; R).$$

Then we may use induction over the number of elements of G to deduce $R = H^*(B(G/C(G)); R)$ and hence the corollary. Hence, it suffices to treat the case of finite abelian p-groups A. In this case, there is a short exact sequence

$$1 \to C_p \to A \to A/C_p \to 1$$

so running the same argument as above, we reduce to showing that

$$H^*(BC_p; R) = R$$

this follows the from the fact that a model for BC_p is given by an infinite dimensional lens space, whose integral cohomology is $\mathbb{F}_p[u]$ with $|u| = 2$. □

Appendix A. Homotopy Pullbacks

We give a brief interlude on homotopy pullback diagrams here. To do so, let us fix a diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "t"] & D \end{tikzcd} \tag{5}$$

which commutes up to specified homotopy h. We say that two such diagrams are weakly equivalent if one maps to the other (this includes a homotopy witnessing the homotopy commutativity of the resulting cube) via a map which is a weak equivalence in each corner.

A.1. **Lemma** *Associated to the data* (5) *there is a canonically induced map* $\mathrm{hofib}_c(f) \to \mathrm{hofib}_{t(c)}(g)$.

Proof. We recall that $\mathrm{hofib}_c(f)$ consists of pairs (a, γ) with $a \in A$ and $\gamma\colon [0,1] \to C$ with $\gamma(0) = c$ and $\gamma(1) = f(a)$. The map we are looking for assigns to such a pair (a, γ) the pair $(s(a), h(a, 0) \star t(\gamma))$. □

A.2. **Definition** A diagram (5) is called a *homotopy pullback* if for all $c \in C$, the induced map $\mathrm{hofib}_c(f) \to \mathrm{hofib}_{t(c)}(g)$ is a weak equivalence.

A.3. **Example** This example illustrates how important the datum of the homotopy in the above notion is. Consider the following square.

$$\begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ * \arrow[r, "x"] & X \end{tikzcd}$$

There are two somewhat canonical homotopies between the two composites (both of which are the map which is constant at x) to consider:

(1) Since the diagram commutes strictly, we may choose the constant homotopy.
(2) We may also consider the homotopy $\mathrm{ev}\colon \Omega_x X \times [0,1] \to X$ given by $\mathrm{ev}(\gamma, t) = \gamma(t)$.

Now, the vertical homotopy fibres of the above diagram are both given by $\Omega_x X$. The constant homotopy from (1) induces the map $\Omega_x X \to \Omega_x X$ sending every based loop to the constant loop at x. However, the homotopy from (2) induces a map which is homotopic to the identity. (Exercise)

A.4. **Example** Suppose the diagram (5) is a pullback diagram, in particular strictly commutative and equipped with the constant homotopy witnessing commutativity, and that g is a fibration. We claim that such squares are homotopy pullbacks. Indeed, in this case, f is also a fibration and for every $c \in C$ we may consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{fib}_c(f) & \longrightarrow & \mathrm{fib}_{t(c)}(g) \\ \downarrow & & \downarrow \\ \mathrm{hofib}_c(f) & \longrightarrow & \mathrm{hofib}_{t(c)}(g) \end{array}$$

in which the vertical maps are the canonical inclusions, which are weak equivalences since f and g are fibrations. Since the top horizontal map is a homeomorphism (recall that we assumed the original square to be a pullback square), the bottom horizontal map is a weak equivalence, showing the claim.

A.5. **Remark** Suppose any of the maps appearing in diagram (5) is replaced by a homotopic map. Then there is a canonically induced homotopy h witnessing that the diagram commutes up to homotopy. With these associated data, the new diagram is a homotopy pullback if and only if the old diagram is a homotopy pullback. Indeed, if for instance f is replaced by a homotopic map f', one proves that this homotopy induces a homotopy equivalence $\mathrm{hofib}_c(f) \simeq \mathrm{hofib}_c(f')$ fitting in a commutative triangle with the respective maps to $\mathrm{hofib}_{t(c)}(g)$. Similarly if g is replaced by g'. If t is replaced by a homotopic map t', then again there is a homotopy equivalence $\mathrm{hofib}_{t(c)}(g) \simeq \mathrm{hofib}_{t'(c)}(g)$ compatible with the map from $\mathrm{hofib}_c(f)$. Finally, if s is replaced by a homotopic map s', then the two induced maps $\mathrm{hofib}_c(f) \to \mathrm{hofib}_{t(c)}(g)$ are homotopic.

A.6. **Lemma** *Consider a diagram*

$$\begin{array}{ccccc} A & \xrightarrow{s} & B & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{t} & D & \xrightarrow{v} & Y \end{array}$$

both of which commute up to specified homotopy. Then the big diagram also commutes up to specified homotopy. Moreover, assuming that the right square is a homotopy pullback, then the left square is a homotopy pullback if and only if the big square is a homotopy pullback.

Proof. The main point to observe is that for $c \in C$, the composite

$$\mathrm{hofib}_c(f) \to \mathrm{hofib}_{t(c)}(g) \to \mathrm{hofib}ut(c)(k)$$

is the map associated to the homotopy commutative big diagram. Then we may use the 3-for-2 property for weak equivalences. □

A.7. **Remark** One might think that the 3-for-2 property for weak equivalences implies a 3-for-2 property for homotopy pullbacks in the above situation. Let us see why this is not the case and assume that the big square and the left square are homotopy pullback diagrams. To show that the right square is a homotopy pullback diagram, we need to show that the map $\mathrm{hofib}_d(g) \to \mathrm{hofib}_{v(d)}(k)$ is a weak equivalence for all $d \in D$. If there exists $c \in C$ with $t(c) = d$ then we may in fact conclude what we want. But in general, this need not be the case, the extreme case is $A = C = \varnothing$.

A.8. **Lemma** *Suppose* (5) *is a homotopy pullback. Choose a factorization of g with $B \to B'$ a homotopy equivalence and $B' \to D$ a fibration. Then the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B' \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pullback diagram.

Proof. This follows from Lemma A.6 once we acknowledge that the square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d] \\ D \arrow[r, equal] & D \end{tikzcd}$$

is a homotopy pullback diagram, which in turn follows from the fact that $B \to B'$ is a weak equivalence. □

A.9. **Lemma** *Suppose* (5) *is a homotopy pullback and g is a fibration. Then s is homotopic to a map s' making the diagram commute strictly. In particular, there is a map $A \to P$ where P denotes the pullback of t and g and this map is a weak equivalence.*

Proof. Consider the lifting problem

$$\begin{tikzcd} A \times \{0\} \arrow[r, "s"] \arrow[d] & B \arrow[d] \\ A \times [0,1] \arrow[r, "h"] \arrow[ur, dashed] & D \end{tikzcd}$$

where h is the homotopy between gs and tf. As g is a fibration, we can find a solution, H and $s' = H(-,1)$ does the job. We then obtain a map $A \to P$ participating in a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r, equal] & C \arrow[r] & D \end{tikzcd}$$

in which the big diagram is a homotopy pullback and the right diagram is a homotopy pullback, since for fibrations, the homotopy fibre and the fibre are homotopy equivalent. By Lemma A.6 we conclude that the left square is a homotopy pullback and then that $A \to P$ is a weak equivalence. □

A.10. **Lemma** *Consider a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & C \arrow[r] & D \end{tikzcd}$$

both whose small squares commute up to specified homotopy. Assume that the left square is a homotopy pullback and that X and Y are CW complexes. Then a dotted lift up to compatible homotopy induces a dashed lift up to compatible homotopy.

Proof. The conclusion of the lemma is stable under replacing the right hand square by a weakly equivalent one. Arguing as above, we then replace the right square by a weakly equivalent square which is a strict pullback and whose vertical maps are fibrations. We may also replace the map $X \to Y$ by a subcomplex inclusion. Then a dotted lift up to compatible homotopy can be strictified by Lemma 5.1 and therefore induces a dashed strict lift since the right square is now a pullback. □

Appendix B. Odd primary Steenrod operations

In this appendix, we will sketch how the odd primary analog of the construction and properties of the Steenrod squares are obtained. The very first thing we will need is the following:

B.1. **Lemma** *Let p be an odd prime. Then $H^*(B\Sigma_p;\mathbb{F}_p) = \mathbb{F}_p[u] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\epsilon] = \mathbb{F}_p[u,\epsilon]/\epsilon^2$ where $|u| = 2p-2$ and $|\epsilon| = 2p-3$ and $\epsilon^2 = 0$.*

Proof sketch. First, one shows that $H^*(BC_p;\mathbb{F}_p) = \mathbb{F}_p[x,e]/e^2$ with $|e| = 1$ and $|x| = 2$. Then we observe that $C_p \subseteq \Sigma_p$ is a p-Sylow subgroup. It follows from a transfer argument that therefore $H^*(B\Sigma_p;\mathbb{F}_p) \to H^*(BC_p;\mathbb{F}_p)$ is the inclusion of an (additive) direct summand. Moreover, we observe that the normalizer of C_p in Σ_p is given by the semi-direct product $C_p \rtimes \mathbb{F}_p^\times$, from which it follows that the map $H^*(B\Sigma_p;\mathbb{F}_p) \to H^*(BC_p;\mathbb{F}_p)$ factors through the inclusion $H^*(BC_p;\mathbb{F}_p)^{\mathbb{F}_p^\times} \subseteq H^*(BC_p;\mathbb{F}_p)$. In fact, the former term turns out to be $H^*(B(C_p \rtimes \mathbb{F}_p^\times);\mathbb{F}_p)$. Now, recall that $\mathbb{F}_p^\times$ is a cyclic group, i.e. isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$. This group acts in the tautological way on $H^k(BC_p;\mathbb{F}_p) = \mathbb{F}_p$ for $k = 1, 2$. In particular, we find that x^{p-1} is fixed under the $\mathbb{F}_p^\times$-action and it then turns out that $H^*(BC_p;\mathbb{F}_p)^{\mathbb{F}_p^\times} = \mathbb{F}_p[x^{p-1}, ex^{p-1}]/(ex^{p-1})$ which is what we claim the $\mathbb{F}_p$-cohomology of Σ_p to be. Hence, one is reduced to showing that the map $H^*(B\Sigma_p;\mathbb{F}_p) \to H^*(B(C_p \rtimes \mathbb{F}_p);\mathbb{F}_p)$ is also surjective. This follows from a computation with the double coset formula. □

B.2. **Lemma** *Let X be a pointed space with base-point preserving G-action. If X is $(n-1)$-connected, then the map $X \to (X,x)_{hG}$ induces the inclusion $H^n(X;M)^G \subseteq H^n(X;M)$ upon applying $H^n(-;M)$.*

Proof. The proof of this result is essentially given in the proof of Lemma 6.2. □

In particular, we deduce:

B.3. **Corollary** *For $k \geq 1$, there is a, unique up to homotopy, extension of the map $\iota_n^{\times k}\colon \prod_k K(\mathbb{F}_p,n) \to K(\mathbb{F}_p,nk)$ to a map $(\bigwedge_k K(\mathbb{F}_p,n),*)_{h\Sigma_k} \to K(\mathbb{F}_p,nk)$.*

B.4. **Construction** We may then perform the same construction as in the $p = 2$ case: For a space X with $x \in H^n(X; \mathbb{F}_p)$, consider the composite

$$X \to X^{\times p} \to K(\mathbb{F}_p, n)^{\times} p \to K(\mathbb{F}_p, np)$$

classifying the element x^p. By the above lemma, it induces the following map

$$X_{h\Sigma_p} \to (X^{\times p})_{h\Sigma_p} \to (\bigwedge_k K(\mathbb{F}_p, n), *)_{h\Sigma_p} \to K(\mathbb{F}_p, np)$$

classifying an element $P(x) \in H^{np}(X \times B\Sigma_p; \mathbb{F}_p) \cong [H^*(X; \mathbb{F}_p)[u, \epsilon]/\epsilon^2]_{np}$. Denote then by $P^i(x)$ the coefficient of t^{n-2i}. With this, we find that P^i is a natural operation

$$P^i \colon H^n(-; \mathbb{F}_p) \to H^{n+2i(p-1)}(-; \mathbb{F}_p).$$

We can compose P^i with the Bockstein operator $H^k(-; \mathbb{F}_p) \to H^{k+1}(-; \mathbb{F}_p)$ to obtain further operations

$$\beta P^i \colon H^n(-; \mathbb{F}_p) \to H^{n+2i(p-1)+1}(-; \mathbb{F}_p).$$

B.5. **Remark** In case $p = 2$, we have by construction that $P^i = \mathrm{Sq}^{2i}$, and $\beta P^i = \mathrm{Sq}^1\mathrm{Sq}^{2i} = \mathrm{Sq}^{2i+1}$, see Example 6.28.

B.6. **Theorem** *The operations P^i and βP^i satisfy the following relations.*

1. *Naturality: That is, if $x = f^*(y)$, then $P^i(x) = f^*(P^i(y))$; same for βP^i.*
2. *Triviality: We have $P^i(x) = 0$ for $i > 2n$ and for $i < 0$.*
3. *Square-likeness: We have $P^n(x) = x^p$ if $|x| = 2n$.*
4. *Cartan formula: $P^i(x \times y) = \sum_{k+l=i} P^k(x) \times P^l(y)$ in $H^{n+2i(p-1)}(X \times Y; \mathbb{F}_2)$; same for $\wedge$ in place of $\times$.*
5. *Stability: We have $P^i(\sigma(x)) = \sigma(P^i(x))$ where σ is the suspension isomorphism.*
6. *Unitality: We have $P^0(x) = x$.*
7. *Adem relations: For $i < pj$, we have*

$$P^iP^j(x) = \sum_k (-1)^{k+i} \binom{(p-1)(j-k)-1}{i-pk} P^{j+i-k}P^k(x)$$

and for $i \leq pj$, we have

$$\begin{aligned} P^i\beta P^j(x) &= \sum_k (-1)^{k+i} \binom{(p-1)(j-k)}{i-pk} \beta P^{i+j-k}P^k(x) \\ &+ \sum_k (-1)^{k+i+1} \binom{(p-1)(j-k)-1}{i-pk-1} P^{i+j-1}\beta P^k(x). \end{aligned}$$

where in all cases, the binomial coefficient is to be interpreted in $\mathbb{F}_p$.

B.7. **Remark** In this remark, we briefly indicate what arguments are similar to the case $p = 2$ and which ones differ slightly. In principle, the proofs of these results follow the same line of thought as in the $p = 2$ case. E.g. naturality, triviality, square-likeness, and the Cartan-formula are really proven in the same way. To get at stability and unitality, we again need to prove by hand that $P^0(i_1) = i_1$, where $i_1 \in H^1(S^1; \mathbb{F}_p)$ is the generator. This amounts to proving that the map

$$(S^1, *)_{h\Sigma_p} \to [(S^1, *)^{\wedge p}]_{h\Sigma_p}$$

induced by the diagonal is an isomorphism on $H^p(-;\mathbb{F}_p)$. Here, let us denote by $\mathbb{R}^\rho$ the permutation Σ_p-representation on $\mathbb{R}^p$. The diagonal is a direct summand in this representation, let us denote an (orthogonal) complement by $\mathbb{R}^{\bar\rho}$, so that the diagonal map is Σ_p-equivariantly equivalent to the inclusion $\mathbb{R} \to \mathbb{R} \oplus \mathbb{R}^{\bar\rho}$.[46] Consequently, the map we care about is the map induced on one-point compactifications and further applying $(-)_{h\Sigma_p}$. Hence, we want to investigate the map $S^0 \to S^{\bar\rho}$ again before applying Σ_p-orbits, but as a map of Σ_p-spaces (and finally suspend everything once). This is more involved that the case $p = 2$ where we could use the simple cofibre sequence $(C_2)_+ \to S^0 \to S^\sigma$;

The proof of the Adem relations then follow the same strategy as in the case $p = 2$. One proves the analog of Theorem 6.22 and then performs a similar coefficient-comparison argument with power series; see again [BM82].

We then have the following result analogous to Lemma 6.30.

B.8. **Lemma** *The operation P^i is indecomposable if and only if $i = p^n$ is a power of p.*

Proof. Here, we say that P^j is *decomposable* if it can be written as a non-trivial sum over terms of the form P^kP^l. □

B.9. **Proposition** *Let p be an odd prime and X be a space with $H^*(X;\mathbb{F}_p) = \mathbb{F}_p[u]/u^m$ with $p < m \leq \infty$. Then $|u| = p^k \cdot l$ where l is an even divisor of $2(p-1)$.*

Proof. Since $0 \neq u^2$, by graded commutativity of the cohomology ring, we find that $|u|$ is of the form $2n$. Then $P^n(u) = u^p \neq 0$. Now, P^n can be written as a sum of products of terms of the form P^{p^i}, we find that there exists some i such that $P^{p^i}(u) \neq 0$ in $H^{2n+2p^i(p-1)}(X;\mathbb{F}_p)$. Since the cohomology of X is concentrated in degrees divisible by $2n$, this simply means that $2p^i(p-1)$ is divisible by $2n$. Hence, $2n$ is of the form $2p^kl$ with $k < i$ and $2l$ is a divisor of $2(p-1)$ as claimed. □

B.10. **Corollary** *Let X be a space with $H^*(X;\mathbb{Z}) = \mathbb{Z}[u]/u^m$ for $m > 3$. Then $|u|$ is 2 or 4.*

Proof. Reducing modulo 2, we find that $|u| = 2^n$ is a power of 2. Note that then even divisors of $2(3-1) = 4$ are precisely 2 and 4, so by reducing modulo 3 we get

$$2^n = |u| = 3^k\alpha$$

where α is either 2 or 4. It follows that $k = 0$ and hence the result. □

References

[BM82] S. R. Bullett and I. G. Macdonald, *On the Adem relations*, Topology **21** (1982), no. 3, 329–332.

[BS55] R. Bott and H. Samelson, *The cohomology ring of G/T*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 490–493.

[Dol63] A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. (2) **78** (1963), 223–255.

[Gha19] M. Ghazel, *On the cell structure of flag manifolds*, arXiv:1904.03967 (2019).

[Hsi63] W. C. Hsiang, *On Wu's formula of Steenrod squares on Stiefel-Whitney classes*, Bol. Soc. Mat. Mexicana (2) **8** (1963), 20–25.

[Lan23] M. Land, *Topology 1; lecture notes*, available upon request, 2023.

[Mil56a] J. Milnor, *Construction of universal bundles. I*, Ann. of Math. (2) **63** (1956), 272–284.

[Mil56b] ———, *Construction of universal bundles. II*, Ann. of Math. (2) **63** (1956), 430–436.

[Seg68] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 105–112.

[46] For $p = 2$, $\mathbb{R}^{\bar\rho}$ is the sign representation $\mathbb{R}^\sigma$.

[Str66] A. Strøm, *Note on cofibrations*, Math. Scand. **19** (1966), 11–14.
[Str68] ———, *Note on cofibrations. II*, Math. Scand. **22** (1968), 130–142.
[tD08] T. tom Dieck, *Algebraic topology*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.
[War76] R. B. Warfield, Jr., *Nilpotent groups*, Lecture Notes in Mathematics, vol. 513, Springer-Verlag, Berlin-New York, 1976.
[Win24] C. Winges, *Topology 2; lecture notes*, available upon request, 2024.
[Wu50] W. T. Wu, *Les i-carrés dans une variété grassmannienne*, C. R. Acad. Sci. Paris **230** (1950), 918–920.

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